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Generalization of Some Fractional Integral Operator Inequalities for Convex Functions via Unified Mittag–Leffler Function

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Abstract: This paper aims to obtain the bounds of a class of integral operators containing Mittag–Leffler functions in their kernels. A recently defined unified Mittag–Leffler function plays a vital role in connecting the results of this paper with the well-known bounds of fractional integral operators published in the recent past. The symmetry of a function about a line is a fascinating property that plays an important role in mathematical inequalities. A variant of the Hermite–Hadamard inequality is established using the closely symmetric property for (α, m)-convex functions.

Keywords: integral operators; fractional integral operators; bounds; (α, m)-convex function; symmetry

MSC: 26D10; 31A10; 26A33

Citation: Nonlaopon, K.; Farid, G.; Yasmeen, H.; Shah, F.A.; Jung, C.Y. Generalization of Some Fractional Integral Operator Inequalities for Convex Functions via Unified Mittag–Leffler Function. *Symmetry* **2022**, *14*, 922. https://doi.org/10.3390/sym14050922

Academic Editors: Praveen Agarwal, Carlo Cattani and Shilpi Jain

Received: 6 April 2022
Accepted: 27 April 2022
Published: 30 April 2022

1. Introduction

A Swedish mathematician, Magnus Gösta Mittag–Leffler, introduced the following function, named the Mittag–Leffler function [1]:

$$E_\alpha(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\alpha l + 1)}, \tag{1}$$

where $\alpha, z \in \mathbb{C}$ and $\Re(\alpha) > 0$.

The above function is the natural extension of many exponential, hyperbolic, and trigonometric functions. Due to its adverse use in various branches of mathematics, many scholars have published several generalizations and extensions of this function. Moreover, they have also proved several integral transforms of Mittag–Leffler functions and expressed them in the form of some famous special functions [2,3]. The integral operators containing Mittag–Leffler functions are frequently used to prove several eminent inequalities such as the Hadamard, Ostrowski, Minkowski, Opial, and Chebyshev inequalities.

Convex functions have an important role in the further study of well-known classical inequalities. A convex function $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ is defined by $\varphi(\tau\vartheta + (1-\tau)y) \leq \tau\varphi(\vartheta) + (1-\tau)\varphi(y)$, where $\tau \in [0,1]$ and $\vartheta, y \in [\sigma_1, \sigma_2]$. This definition leads to give a lot of new definitions such as m-convex [4], quasi-convex [5], strongly convex [6], preinvex functions [7], and invex functions [8], etc. More or less all classical integral inequalities for the aforementioned convexities have been studied. For this, one can see [9–12].

Motivated and inspired by the ongoing research in the field of integral inequalities, this paper aims to establish the boundedness of fractional integral operators containing an extended Mittag–Leffler function [13] called the unified Mittag–Leffler function. A Hadamard-type integral inequality is established using a symmetry-like function. For this purpose, we have utilized a well-known convexity named the (α, m)-convexity of a function. The results of this article are the generalizations of some fractional inequalities already proven in different papers. The classical Riemann–Liouville fractional integrals are defined as follows:

Definition 1. *Let $\varphi \in L_1[\sigma_1, \sigma_2]$. Then, left-sided and right-sided Riemann–Liouville fractional integrals of a function f of order μ where $\Re(\mu) > 0$ are defined as follows:*

$$I^{\mu}_{\sigma_1^+}\varphi(\vartheta) = \frac{1}{\Gamma(\mu)}\int_{\sigma_1}^{\vartheta}(\vartheta - \tau)^{\mu-1}\varphi(\tau)d\tau, \quad \vartheta > \sigma_1, \tag{2}$$

and

$$I^{\mu}_{\sigma_2^-}\varphi(\vartheta) = \frac{1}{\Gamma(\mu)}\int_{\vartheta}^{\sigma_2}(\tau - \vartheta)^{\mu-1}\varphi(\tau)d\tau, \quad \vartheta < \sigma_2, \tag{3}$$

where $\Re(\mu)$ is real value of μ and $\Gamma(\mu) = \int_0^\infty e^{-z}z^{\mu-1}dz$.

The k-analogue of the Riemann–Liouville fractional integral is defined as follows:

Definition 2 ([14])**.** *Let $\varphi \in L[\sigma_1, \sigma_2]$. Then, k-fractional Riemann–Liouville integrals of order μ, where $\Re(\mu) > 0$, $k > 0$, are defined as:*

$${}_kI^{\mu}_{\sigma_1^+}\varphi(\vartheta) = \frac{1}{k\Gamma_k(\mu)}\int_{\sigma_1}^{\vartheta}(\vartheta - \tau)^{\frac{\mu}{k}-1}\varphi(\tau)d\tau, \quad \vartheta > \sigma_1, \tag{4}$$

and

$${}_kI^{\mu}_{\sigma_2^-}\varphi(\vartheta) = \frac{1}{k\Gamma_k(\mu)}\int_{\vartheta}^{\sigma_2}(\tau - \vartheta)^{\frac{\mu}{k}-1}\varphi(\tau)d\tau, \quad \vartheta < \sigma_2, \tag{5}$$

where $\Gamma_k(.)$ is defined as follows:

$$\Gamma_k(\mu) = \int_0^\infty t^{\mu-1}e^{-\frac{t^k}{k}}dt, \quad \Re(\mu) > 0.$$

The definition of generalized Riemann–Liouville fractional integrals by a monotonically increasing function is defined as follows:

Definition 3 ([15])**.** *Let $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be an integrable function. Let ψ be an increasing and positive function on $(\sigma_1, \sigma_2]$, having a continuous derivative ψ' on (σ_1, σ_2). The left-sided and right-sided fractional integrals of a function φ with respect to another function ψ on $[\sigma_1, \sigma_2]$ of order μ where $\Re(\mu) > 0$ are defined by*

$$I^{\mu,\psi}_{\sigma_1^+}\varphi(\vartheta) = \frac{1}{\Gamma(\mu)}\int_{a}^{\vartheta}\psi'(\tau)(\psi(\vartheta) - \psi(\tau))^{\mu-1}\varphi(\tau)d\tau, \quad \vartheta > \sigma_1 \tag{6}$$

and

$$I^{\mu,\psi}_{\sigma_2^-}\varphi(\vartheta) = \frac{1}{\Gamma(\mu)}\int_{\vartheta}^{\sigma_2}\psi'(\tau)(\psi(\tau) - \psi(\vartheta))^{\mu-1}\varphi(\tau)d\tau, \quad \vartheta < \sigma_2. \tag{7}$$

The k-analogue of a generalized Riemann–Liouville fractional integral is defined as follows:

Definition 4 ([16])**.** *Let $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be an integrable function. Let ψ be an increasing and positive function on $(\sigma_1, \sigma_2]$, having a continuous derivative ψ' on (σ_1, σ_2). The left-sided and right-sided fractional integrals of a function φ with respect to another function ψ on $[\sigma_1, \sigma_2]$ of order μ where $\Re(\mu) > 0$, $k > 0$ are defined by*

$${}_kI^{\mu,\psi}_{\sigma_1^+}\varphi(\vartheta) = \frac{1}{k\Gamma_k(\mu)}\int_{a}^{\vartheta}\psi'(\tau)(\psi(\vartheta) - \psi(\tau))^{\frac{\mu}{k}-1}\varphi(\tau)d\tau, \quad \vartheta > \sigma_1, \tag{8}$$

and

$$ {}_k I^{\mu,\psi}_{\sigma_2^-}\varphi(\vartheta) = \frac{1}{k\Gamma_k(\mu)} \int_\vartheta^b \psi'(\tau)(\psi(\tau) - \psi(\vartheta))^{\frac{\mu}{k}-1}\varphi(\tau)d\tau, \quad \vartheta < \sigma_2. \tag{9} $$

The left-sided and right-sided generalized fractional integral operators containing an extended generalized Mittag–Leffler function is defined as follows:

Definition 5 ([17]). *Let $\omega, \alpha, \beta, \gamma, \theta, \lambda \in \mathbb{C}, \Re(\alpha), \Re(\beta), \Re(\gamma) > 0, \Re(\theta) > \Re(\lambda) > 0$ with $p \geq 0, \delta > 0$ and $0 < k \leq r + \Re(\alpha)$. Let $\varphi \in L_1[\sigma_1, \sigma_2]$ and $\vartheta \in [\sigma_1, \sigma_2]$. Then, the generalized fractional integral operators $\varepsilon^{\omega,\lambda,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma}\varphi$ and $\varepsilon^{\omega,\gamma,\delta,k,c}_{\sigma_2^-,\alpha,\beta,\gamma}\varphi$ are defined by*

$$ \left(\varepsilon^{\omega,\lambda,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma}\varphi\right)(\vartheta; s) = \int_{\sigma_1}^{\vartheta} (\vartheta - \tau)^{\alpha-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(\vartheta - \tau)^{\mu}; s)\varphi(\tau)d\tau \tag{10} $$

and

$$ \left(\varepsilon^{\omega,\lambda,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma}\varphi\right)(\vartheta; s) = \int_{\vartheta}^{\sigma_2} (\tau - \vartheta)^{\alpha-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(\tau - \vartheta)^{\mu}; s)\varphi(\tau)d\tau, \tag{11} $$

where

$$ E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma,c}(z; s) = \sum_{l=0}^{\infty} \frac{\beta_s(\lambda + lk, \theta - \lambda)(\theta)_{lk} z^l}{\beta(\lambda, \theta - \lambda)(\gamma)_{lr}\Gamma(\alpha l + \beta)} \tag{12} $$

is the extended generalized Mittag–Leffler function.

Next, the unified integral operator, which unifies several classes of fractional integral operators, is defined as follows:

Definition 6 ([18]). *Let φ, $\psi : [\sigma_1, \sigma_2] \to \mathbb{R}, 0 < \sigma_1 < \sigma_2$ be the functions such that φ is positive and $\varphi \in L_1[\sigma_1, \sigma_2]$ and ψ are differentiable and strictly increasing. Let $\frac{\Upsilon}{\vartheta}$ be an increasing function on $[\sigma_1, \infty)$ and $\omega, \alpha, \gamma, \lambda, \theta \in \mathbb{C}, \Re(\alpha), \Re(\gamma) > 0, \Re(\theta) > \Re(\lambda) > 0, \alpha, \delta > 0, s \geq 0$ and $0 < k \leq \delta + \alpha$. Then, for $\vartheta \in [\sigma_1, \sigma_2]$, the left and right integral operators are defined by*

$$ ({}^{\Upsilon}_{\psi}F^{\omega,\lambda,\rho,\theta,k,\delta}_{\sigma_1^+,\alpha,\beta,\gamma}\varphi)(\vartheta; s) = \int_{\sigma_1}^{\vartheta} K^{\tau}_{\vartheta}(E^{\lambda,\theta,k,\delta}_{\alpha,\beta,\gamma}\psi; \Upsilon)\varphi(\tau)d(\psi(\tau)) \tag{13} $$

and

$$ ({}^{\Upsilon}_{\psi}F^{\omega,\lambda,\theta,k,\delta}_{\sigma_2^-,\alpha,\beta,\gamma}\varphi)(\vartheta; s) = \int_{\vartheta}^{\sigma_2} K^{\vartheta}_{\tau}(E^{\lambda,\theta,k,\delta}_{\alpha,\beta,\gamma}\psi; \Upsilon)\varphi(\tau)d(\psi(\tau)), \tag{14} $$

where

$$ K^{\tau}_{\vartheta}(E^{\lambda,\theta,k,\delta}_{\alpha,\beta,\gamma}\psi; \Upsilon) = \frac{\Upsilon(\psi(\vartheta) - \psi(\tau))}{\psi(\vartheta) - \psi(\tau)} E^{\lambda,\theta,k,\delta}_{\alpha,\beta,\gamma}(\omega(\psi(\vartheta) - \psi(\tau))^{\mu}; s). $$

The unified Mittag–Leffler function is defined as follows:

Definition 7 ([13]). *For $\underline{a} = (a_1, a_2, \ldots, a_n), \underline{b} = (b_1, b_2, \ldots, b_n), \underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i, \in \mathbb{C}; i = 1, 2, 3, \ldots, n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0$ for all i. Let $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{C}, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\theta)\} > 0$ and $k \in (0, 1) \cup \mathbb{N}$ with $s \geq 0$. Let $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$ with $Im\ (\rho) = Im(\delta + \nu + \alpha)$; then, the unified Mittag–Leffler function is defined by*

$$ M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, s) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{r} \beta_s(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{r} \beta(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)} \tag{15} $$

where β_s is the extension of the beta function and it is defined as follows:

$$ \beta_s(\vartheta, y) = \int_0^1 \tau^{\vartheta-1}(1 - \tau)^{y-1} e^{-\left(\frac{s}{\tau(1-\tau)}\right)} d\tau. \tag{16} $$

Next, the generalized fractional integral operator containing the unified Mittag–Leffler function is defined as follows:

Definition 8 ([13]). *Let $\varphi \in L_1[\sigma_1, \sigma_2]$. Then $\forall\ \vartheta \in [\sigma_1, \sigma_2]$, the fractional integral operator containing the unified Mittag–Leffler function $M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, s)$ is defined by*

$$\left(\Omega^{\omega,\lambda,\rho,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; \underline{a}, \underline{b}, \underline{c}, s) = \int_{\sigma_1}^{\vartheta} (\vartheta-\tau)^{\alpha-1} M^{\lambda,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\vartheta-\tau)^{\mu}; \underline{a}, \underline{b}, \underline{c}, s)\varphi(\tau)d\tau, \tag{17}$$

$$\left(\Omega^{\omega,\lambda,\rho,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; \underline{a}, \underline{b}, \underline{c}, s) = \int_{\vartheta}^{\sigma_2} (\tau-\vartheta)^{\alpha-1} M^{\lambda,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\vartheta-\tau)^{\mu}; \underline{a}, \underline{b}, \underline{c}, s)\varphi(\tau)d\tau, \tag{18}$$

By setting $a_i = l, s = 0$ and $\Re(s) > 0$ in (8), we get the fractional integral operator associated with the generalized Q function as follows:

$$\left({}_{Q}\Omega^{\omega,\lambda,\rho,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; \underline{a}, \underline{b}) = \int_{\sigma_1}^{\vartheta} (\vartheta-\tau)^{\alpha-1} Q^{\lambda,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\vartheta-\tau)^{\mu}; \underline{a}, \underline{b})\varphi(\tau)d\tau, \tag{19}$$

$$\left({}_{Q}\Omega^{\omega,\lambda,\rho,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; \underline{a}, \underline{b}) = \int_{\vartheta}^{b} (\tau-\vartheta)^{\alpha-1} Q^{\lambda,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\tau-\vartheta)^{\mu}; \underline{a}, \underline{b})\varphi(\tau)d\tau, \tag{20}$$

where

$$Q^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{r} \beta(b_i, l)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{r} \beta(a_i, l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}$$

is a generalized Q function defined in [19].

Recently, Gao et al. [20] gave the further generalization and extension of the above integral operator as follows:

Definition 9. *Let $\Upsilon \in L_1[\sigma_1, \sigma_2], 0 < \sigma_1, \sigma_2 < \infty$ be a positive function and let $\psi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable and strictly increasing function. Let $\frac{\zeta}{\vartheta}$ be an increasing function on $[\sigma_1, \infty)$ for all $\vartheta \in [\sigma_1, \sigma_2]$. Then, the unified integral operator in its generalized form satisfying all the convergence conditions of the unified Mittag–Leffler function is defined by*

$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; s) = \int_{\sigma_1}^{\vartheta} \Lambda^{\tau}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi; \Upsilon\right)\varphi(\tau)d(\psi(\tau)), \tag{21}$$

$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta; s) = \int_{\vartheta}^{\sigma_2} \Lambda^{\vartheta}_{\tau}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi; \Upsilon\right)\varphi(\tau)d(\psi(\tau)), \tag{22}$$

where

$$\Lambda^{\tau}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}, \psi; \Upsilon\right) = \frac{\Upsilon(\psi(\vartheta)-\psi(\tau))}{\psi(\vartheta)-\psi(\tau)} M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\tau))^{\mu}, \underline{a}, \underline{b}, \underline{c}; s).$$

Remark 1.

(i) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = 0$ in (21) and (22), the unified integral operator given in Definition6 is obtained;*

(ii) *If $\Upsilon(\vartheta) = \vartheta^{\alpha}$ and $\psi(\vartheta) = \vartheta$, the operators given in Definition 8 are obtained;*

(iii) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = 0, \Upsilon(\vartheta) = \vartheta^{\alpha}$ and $\psi(\vartheta) = \vartheta$ in (21) and (22), the operator given in Definition 5 is obtained;*

(iv) *If $\Upsilon(\vartheta) = \vartheta^{\alpha}$ for $\alpha > 0$, in (21), the operator given in [20] is obtained.*

Definition 10. *By setting $a_i = l, s = 0$ and $\Re(\rho) > 0$ in (21) and (22), we get the fractional integral operator associated with the generalized Q function:*

$$\left({}^{\psi}_{Q}\Omega^{\Upsilon,\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta;\underline{a},\underline{b}) = \int_{\sigma_1}^{\vartheta} \Lambda_{\vartheta}^{\tau}\left(Q^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)\varphi(\tau)d(\psi(\tau)), \tag{23}$$

$$\left({}^{\psi}_{Q}\Omega^{\Upsilon,\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\vartheta;\underline{a},\underline{b}) = \int_{\vartheta}^{\sigma_2} \Lambda_{\tau}^{\vartheta}\left(Q^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)\varphi(\tau)d(\psi(\tau)), \tag{24}$$

where

$$\Lambda_{\vartheta}^{\tau}\left(Q^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right) = \frac{\Upsilon(\psi(\vartheta)-\psi(\tau))}{\psi(\vartheta)-\psi(\tau)} Q^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\tau))^{\mu};\underline{a},\underline{b}).$$

Convexity is an essential notion often used in mathematics, mathematical statistics, graph theory, etc. In particular, a convex function contributes a lot to the formulation of new inequalities which behave as generalizations of classical inequalities. The Hermite–Hadamard inequality is a straight sequel of convex functions. It describes the lower as well as upper bound of an integral mean of a convex function over an interval $[\sigma_1, \sigma_2]$. The Hermite–Hadamard inequality was first generalized by Fejér [21] with the help of the symmetric function about the midpoint of the interval $[\sigma_1, \sigma_2]$ on which the convex function is defined. In [22], Farid introduced an inequality of Hermite–Hadamard type by using symmetric convex functions. Presently, the Hermite–Hadamard inequality is generalized by defining new classes of convex functions which are clearly related to convex functions. This paper gives the Hermite–Hadamard-type inequality by using convex functions close to symmetric functions. The definition of (α, m)-convexity is defined as follows:

Definition 11 ([23])**.** *A function $\varphi : [0, \sigma_2] \to \mathbb{R}, \sigma_2 > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if*

$$\varphi(\tau\vartheta + m(1-\tau)y) \le \tau^{\alpha}\varphi(\vartheta) + m(1-\tau^{\alpha})\varphi(y) \tag{25}$$

holds for all $\vartheta, y \in [0, \sigma_2]$ and $\tau \in [0, 1]$.

Example 1.

(i) *A $(1, 1)$-convex function is an example of convex function;*
(ii) *A $(\alpha, 1)$-convex function is an example of an α-convex function;*
(iii) *A $(1, m)$-convex function is an example of an m-convex function.*

In the upcoming section, we have investigated the bounds of integral operators given in (21) and (22) with the help of (α, m)-convexity. Moreover, we have also provided the Hermite–Hadamard-type inequality by using the condition close to symmetry about the interval's midpoint. The established results give many fractional and conformable integral inequalities. From here onward, we consider the parameters of Mittag–Leffler functions as real numbers.

2. Main Results

Theorem 1. *Let $\varphi \in L_1[\sigma_1, \sigma_2]$ be an integrable (α, m)-convex function. Let $\frac{\Upsilon}{\vartheta}$ be an increasing function on $[\sigma_1, \sigma_2]$ and let $\psi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable and strictly increasing function; also, let $\frac{\Upsilon}{\vartheta}$ be an increasing function on $[\sigma_1, \sigma_2]$. Then, for all $\vartheta \in [\sigma_1, \sigma_2]$, we have the following inequalities containing the unified Mittag–Leffler function $M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b},\underline{c},s)$ satisfying all the convergence conditions:*

$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)-\varphi(\sigma_1)\psi(\sigma_1)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\varphi\left(\frac{\vartheta}{m}\right)-\varphi(\sigma_1)\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right], \tag{26}$$
$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) \leq \Lambda_{\sigma_2}^{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\vartheta)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\right){}^{\alpha}I_{\sigma_2^-}\psi(\vartheta)\right]. \tag{27}$$

and hence

$$\begin{aligned}
&\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s)\\
&\leq \Lambda_{\vartheta}^{\sigma_1}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)-\varphi(\sigma_1)\psi(\sigma_1)\right)\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\varphi\left(\frac{\vartheta}{m}\right)-\varphi(\sigma_1)\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right]\\
&\quad+\Lambda_{\sigma_2}^{\vartheta}\left(M^{\omega,\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)\right)\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\vartheta)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\right){}^{\alpha}I_{\sigma_2^-}\psi(\vartheta)\right].
\end{aligned} \tag{28}$$

Proof. One can have the following inequality under the assumptions of Υ and ψ

$$\frac{\Upsilon(\psi(\vartheta)-\psi(\tau))}{\psi(\vartheta)-\psi(\tau)} \leq \frac{\Upsilon(\psi(\vartheta)-\psi(\sigma_1))}{\psi(\vartheta)-\psi(\sigma_1)} \tag{29}$$

for all $\tau\in[\sigma_1,\vartheta]$ and $\vartheta\in[\sigma_1,\sigma_2]$. Multiplying with

$$M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\tau))^{\mu};\underline{a},\underline{b},\underline{c},s)\psi'(\tau),$$

we obtain

$$K_{\vartheta}^{\tau}(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\psi;\Upsilon)\psi'(\tau) \leq \frac{\Upsilon(\psi(\vartheta)-\psi(\sigma_1))}{\psi(\vartheta)-\psi(\sigma_1)}M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\tau))^{\mu};\underline{a},\underline{b},\underline{c},s)\psi'(\tau). \tag{30}$$

By using

$$M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\tau))^{\mu};\underline{a},\underline{b},\underline{c},s) \leq M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\sigma_1))^{\mu};\underline{a},\underline{b},\underline{c},s),$$

the following inequality is obtained:

$$\Lambda_{\vartheta}^{\tau}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\psi;\Upsilon\right)\psi'(\tau) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\psi;\Upsilon\right)\psi'(\tau). \tag{31}$$

Using the definition of (α,m)-convexity for φ, the following inequality is valid:

$$\varphi(\tau) \leq \left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\varphi(\sigma_1)+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\varphi\left(\frac{\vartheta}{m}\right). \tag{32}$$

Integrating over $[\sigma_1,\vartheta]$ after multiplying (31) and (32), we obtain

$$\int_{\sigma_1}^{\vartheta} \Lambda_{\vartheta}^{\tau}\left(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\varphi(\tau)d(\psi(\tau)) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left(\varphi(\sigma_1)\int_{\sigma_1}^{\vartheta}\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha} d(\psi(\tau)) + m\varphi\left(\frac{\vartheta}{m}\right)\int_{\sigma_1}^{\vartheta}\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)d(\psi(\tau)).$$

Using Definition 9 and integrating by parts, we obtain

$$\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,r}\ \varphi\right)(\vartheta;s) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,r}\psi;\Upsilon\right)\left[m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)-\varphi(\sigma_1)\psi(\sigma_1) - \frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\varphi\left(\frac{\vartheta}{m}\right)-\varphi(\sigma_1)\right)^{\alpha} I_{\sigma_1^+}\psi(\vartheta)\right]. \tag{33}$$

Now, for $\tau \in (\vartheta,\sigma_2]$ and $\vartheta \in [\sigma_1,\sigma_2]$, the following inequality holds:

$$\Lambda_{\vartheta}^{\tau}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\psi'(\tau) \leq \Lambda_{\sigma_2}^{\vartheta}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\psi;\Upsilon)\psi'(\tau). \tag{34}$$

Using the (α,m)-convexity of φ, we have

$$\varphi(\tau) \leq \left(\frac{\tau-\vartheta}{\sigma_2-\vartheta}\right)^{\alpha}\varphi(\sigma_2) + m\left(1-\left(\frac{\tau-\vartheta}{\sigma_2-\vartheta}\right)^{\alpha}\right)\varphi\left(\frac{\vartheta}{m}\right). \tag{35}$$

Using the same technique as for (31) and (32), the following inequality from (34) and (35) can be obtained:

$$\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,r}\ \varphi\right)(\vartheta;s) \leq \Lambda_{\sigma_2}^{\vartheta}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \varphi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)\right) - \frac{\Gamma(\alpha+1)}{(\sigma_2-\vartheta)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{\vartheta}{m}\right)\right)^{\alpha} I_{\sigma_2^-}\psi(\vartheta)\right]. \tag{36}$$

By adding (33) and (36), (28) can be obtained. □

Remark 2.

(i) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda$ and $\rho=\nu=0$, in (28), then [24] (Theorem 1) is obtained;*

(ii) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda, \rho=\nu=0$ and $(\alpha,m)=(1,1)$ in (28), then [25] (Theorem 1) is obtained;*

(iii) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda, \rho=\nu=\omega=0, \Upsilon(\tau)=\frac{\Gamma(\mu)\tau^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ for a left-hand integral and $\Upsilon(\tau)=\frac{\Gamma(\nu)\tau^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ for a right-hand integral in (28), then [26] (Theorem 1) can be obtained;*

(iv) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda, \rho=\nu=\omega=0, \Upsilon(\tau)=\Gamma(\mu)\tau^{\mu}$ and $(\alpha,m)=(1,1)$ in (28), then [27] (Theorem 1) is obtained;*

(v) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda, \rho=\nu=\omega=0, \Upsilon(\tau)=\frac{\Gamma(\mu)\tau^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ for a left-hand integral and $\Upsilon(\tau)=\frac{\Gamma(\nu)\tau^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ for a right-hand integral, $(\alpha,m)=(1,1)$ and $\psi(\vartheta)=\vartheta$ in (28), then [28] (Theorem 1) can be obtained;*

(vi) *If $r=1, b_1=\lambda+lk, a_1=\theta-\lambda, c_1=\lambda, \rho=\nu=\omega=0, \Upsilon(\tau)=\Gamma(\mu)\tau^{\mu}$ for a left-hand integral and $\Upsilon(\tau)=\Gamma(\nu)\tau^{\nu}$ for a right-hand integral, $\psi(\vartheta)=\vartheta$ and $(\alpha,m)=(1,1)$ in (28), then [22] (Theorem 1) is obtained.*

Example 2. *If $\alpha = 1$, then the following inequality holds for an m-convex function:*

$$\begin{aligned}
&\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma^{+},\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^{-},\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) \\
&\leq \Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta) - \varphi(\sigma_1)\psi(\sigma_1)\right)\right. \\
&\left.-\frac{1}{(\vartheta-\sigma_1)}\left(m\varphi\left(\frac{\vartheta}{m}\right) - \varphi(\sigma_1)\right)\int_{\sigma_1}^{\vartheta} d(\psi(\tau))\right] \\
&+\Lambda^{\vartheta}_{\sigma_2}\left(M^{\omega,\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2) - m\varphi\left(\frac{\vartheta}{m}\right)\psi(\vartheta)\right)\right. \\
&\left.-\frac{1}{(\sigma_2-\vartheta)}\left(\varphi(\sigma_2) - m\varphi\left(\frac{\vartheta}{m}\right)\right)\int_{\vartheta}^{\sigma_2}\psi(\tau)d\tau\right]. \qquad (37)
\end{aligned}$$

Example 3. *If $(\alpha, m) = (1,1)$, then the following inequality holds for a convex function:*

$$\begin{aligned}
&\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^{+},\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^{-},\alpha,\beta,\gamma,\mu,\nu}\,\varphi\right)(\vartheta;s) \\
&\leq M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\sigma_1))^{\alpha};\underline{a},\underline{b},\underline{c},s))(\Upsilon(\psi(\vartheta)-\psi(\sigma_1))(\varphi(\vartheta)+\varphi(\sigma_1)) \\
&\quad + M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\sigma_2)-\psi(\vartheta))^{\alpha};\underline{a},\underline{b},\underline{c},s))(\Upsilon(\psi(\sigma_2)-\psi(\vartheta))(\varphi(\vartheta)+\varphi(\sigma_2)). \qquad (38)
\end{aligned}$$

The following lemma is required to establish the next result.

Lemma 1 ([26]). *Let $\varphi : [0,\infty] \longrightarrow \mathbb{R}$ be an (α, m)-convex function. If $\varphi(\vartheta) = \varphi(\frac{\sigma_1+\sigma_2-\vartheta}{m})$ for $0 < \sigma_1 < \sigma_2$, then the following inequality holds:*

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right) \leq \frac{1}{2^{\alpha}}(1+m(2^{\alpha}-1))\varphi(\vartheta) \qquad (39)$$

for all $\vartheta \in [\sigma_1, \sigma_2]$ and $m \in (0,1]$.

The following result provides the upper and lower bounds of the sum of operators (21) and (22) in the form of a Hadamard inequality.

Theorem 2. *Under the assumptions of Theorem 1, in addition, if $\varphi(\vartheta) = \varphi\left(\frac{\sigma_1+\sigma_2-\vartheta}{m}\right)$, then*

$$\begin{aligned}
&\frac{2^{\alpha}\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)}{(1+m(2^{\alpha}-1))}\left(\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^{-},\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_1;s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^{+},\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_2;s)\right) \\
&\leq \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^{-},\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_1;s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^{+},\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s) \\
&\leq 2\Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\,\psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2) - m\varphi\left(\frac{\sigma_1}{m}\right)\psi(\sigma_1)\right)\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}\left(\varphi(\sigma_2) - m\varphi\left(\frac{\sigma_1}{m}\right)\right)^{\alpha} I_{\sigma_2^{-}}\psi(\sigma_1)\right]. \qquad (40)
\end{aligned}$$

Proof. Under the assumptions of Υ and ψ, we have

$$\frac{\Upsilon(\psi(\vartheta)-\psi(\sigma_1))}{\psi(\vartheta)-\psi(\sigma_1)} \leq \frac{\Upsilon(\psi(\sigma_2)-\psi(\sigma_1))}{\psi(\sigma_2)-\psi(\sigma_1)}. \qquad (41)$$

Multiplying with

$$M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\sigma_1))^{\mu};\underline{a},\underline{b},\underline{c},s)\psi'(\vartheta),$$

we can obtain from (41) the following inequality:

$$\Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\psi'(\vartheta)\leq\frac{\Upsilon(\psi(\sigma_2)-\psi(\sigma_1))}{\psi(\sigma_2)-\psi(\sigma_1)}M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\sigma_1))^{\mu};\underline{a},\underline{b},\underline{c},s)\psi'(\vartheta). \tag{42}$$

By using

$$M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\vartheta)-\psi(\sigma_1))^{\mu};\underline{a},\underline{b},\underline{c},s)\leq M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\sigma_2)-\psi(\sigma_1))^{\mu};\underline{a},\underline{b},\underline{c},s),$$

the following inequality is obtained:

$$\Lambda^{\sigma_1}_{\vartheta}(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon)\psi'(\vartheta)\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon)\psi'(\vartheta). \tag{43}$$

Using the (α,m)-convexity of φ for $\vartheta\in[\sigma_1,\sigma_2]$, we have

$$\varphi(\vartheta)\leq\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\varphi(\sigma_2)+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\varphi\left(\frac{\sigma_1}{m}\right). \tag{44}$$

Multiplying (43) and (44) and integrating the resulting inequality over $[\sigma_1,\sigma_2]$, one can obtain

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\varphi(\vartheta)d(\psi(\vartheta))\\&\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon)\left(m\varphi\left(\frac{\sigma_1}{m}\right)\int_{\sigma_1}^{\sigma_2}\left(1-\left(\frac{\vartheta-\sigma_1}{\sigma_2-\sigma_1}\right)^{\alpha}\right)d(\psi(\vartheta))\right.\\&\left.\quad+\varphi(\sigma_2)\int_a^b\left(\frac{\vartheta-\sigma_1}{\sigma_2-\sigma_1}\right)^{\alpha}d(\psi(\vartheta))\right).\end{aligned}$$

By using Definition 9 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\ \varphi\right)(\sigma_1;s)\leq{}&\Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\sigma_1}{m}\right)\psi(\sigma_1)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{\sigma_1}{m}\right)\right){}^{\alpha}I_{\sigma_2^-}\psi(\sigma_1)\right].\end{aligned} \tag{45}$$

Now, for $\vartheta\in[\sigma_1,\sigma_2]$, the following inequality holds true:

$$\Lambda^{\vartheta}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\psi'(\vartheta)\leq\Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\psi'(\vartheta). \tag{46}$$

Using the same technique that we did for (43) and (44), the following inequality can be observed from (44) and (46):

$$\begin{aligned}\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\ \varphi\right)(\sigma_2;s)\leq{}&\Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{a}{m}\right)\psi(\sigma_1)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{a}{m}\right)\right){}^{\alpha}I_{\sigma_2^-}\psi(\sigma_1)\right].\end{aligned} \tag{47}$$

By adding (45) and (47), the following inequality can be obtained:

$$\begin{aligned}&\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s)\\&\leq \Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\sigma_1}{m}\right)\psi(\sigma_1)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}\left(\varphi(\sigma_2)-m\varphi\left(\frac{a}{m}\right)\right){}^{\alpha}I_{\sigma_2^-}\psi(\sigma_1)\right].\end{aligned}\tag{48}$$

Multiplying both sides of (39) by $\Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)d(\psi(\vartheta))$, and integrating over $[\sigma_1,\sigma_2]$, we have

$$\begin{aligned}&\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)d(\psi(\vartheta))\\&\leq\frac{1+m(2^{\alpha}-1)}{2^{\alpha}}\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon\right)\varphi(\vartheta)d(\psi(\vartheta)).\end{aligned}$$

From Definition 9, the following inequality is obtained:

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_1;s)\leq\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_1;s).\tag{49}$$

Similarly, multiplying both sides of (39) by $\Lambda^{\vartheta}_{\sigma_2}(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\psi;\Upsilon)d(\psi(\vartheta))$, and integrating over $[\sigma_1,\sigma_2]$, we have

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_2;s)\leq\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s).\tag{50}$$

By adding (49) and (50), the following inequality is obtained:

$$\begin{aligned}&\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left(\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_2;s)\right)\\&\leq\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s).\end{aligned}\tag{51}$$

Combining (48) and (51), inequality (40) can be achieved. □

Remark 3.

(i) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda$ *and* $\rho=\nu=0$, *in* (40), [24] *(Theorem 2) is obtained;*

(ii) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda,\rho=\nu=0$ *and* $(\alpha,m)=(1,1)$ *in* (40), [25] *(Theorem 2) is obtained;*

(iii) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda,\rho=\nu=\omega=0,\Upsilon(\tau)=\Gamma(\mu)\tau^{\frac{\mu}{k}+1}$ *for a left-hand integral and* $\Upsilon(\tau)=\Gamma(\nu)\tau^{\frac{\nu}{k}+1}$ *in* (40), *then* [26] *(Theorem 3) is obtained;*

(iv) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda,\rho=\nu=\omega=0,\Upsilon(\tau)=\Gamma(\mu)\tau^{\mu+1}$ *for a left-hand integral and* $\Upsilon(\tau)=\Gamma(\nu)\tau^{\nu+1}$ *for a right-hand integral in* (40), $(\alpha,m)=(1,1)$ *in* (40), [27] *(Theorem 3) is obtained;*

(v) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda,\rho=\nu=\omega=0,\Upsilon(\tau)=\Gamma(\mu)\tau^{\frac{\mu}{k}+1}$ *for a left-hand integral and* $\Upsilon(\tau)=\Gamma(\nu)\tau^{\frac{\nu}{k}+1}$ *for a right-hand integral,* $(\alpha,m)=(1,1)$ *and* $\psi(\vartheta)=\vartheta$ *in* (40), *then* [28] *(Theorem 3) is obtained.*

(vi) *If* $r=1,b_1=\lambda+lk,a_1=\theta-\lambda,c_1=\lambda,\rho=\nu=\omega=0,\Upsilon(\tau)=\Gamma(\mu)\tau^{\mu+1}$ *for a left-hand integral and* $\Upsilon(\tau)=\Gamma(\nu)\tau^{\nu+1}$ *for a right-hand integral,* $(\alpha,m)=1$ *and* $\psi(\tau)=t$ *in* (40), [22] *(Theorem 3) is obtained.*

Example 4. *If $\alpha = 1$, then the following inequality holds for an m-convex function:*

$$\begin{aligned}
&\frac{2^{\alpha}\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)}{(1+m(2^{\alpha}-1))}\left(\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_2;s)\right)\\
&\le\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s)\\
&\le 2\Lambda^{\sigma_1}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}\ \psi;\Upsilon\right)\left[\left(\varphi(\sigma_2)\psi(\sigma_2)-m\varphi\left(\frac{\sigma_1}{m}\right)\psi(\sigma_1)\right)\right.\\
&\quad\left.-\frac{1}{(\sigma_2-\sigma_1)}\left(\varphi(\sigma_2)-m\varphi\left(\frac{\sigma_1}{m}\right)\right)\int_{\sigma_1}^{\sigma_2}\psi(\tau)d\tau\right].
\end{aligned}\tag{52}$$

Example 5. *If $(\alpha, m) = (1,1)$, then the following inequality holds for a convex function:*

$$\begin{aligned}
&\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\left(\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\sigma_2;s)\right)\\
&\le\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_1;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}\varphi\right)(\sigma_2;s)\\
&\le 2M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\sigma_2)-\psi(\sigma_1))^{\alpha};\underline{a},\underline{b},\underline{c},s))(\Upsilon(\psi(\sigma_2)-\psi(\sigma_1))(\varphi(\sigma_1)+\varphi(\sigma_2)).
\end{aligned}\tag{53}$$

Theorem 3. *Let $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable function. If $|\varphi'|$ is (α, m)-convex and we let $\psi : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable and strictly increasing function, then we also let $\frac{\Upsilon}{\vartheta}$ be an increasing function on $[\sigma_1, \sigma_2]$. Then, for all $\vartheta \in [\sigma_1, \sigma_2]$, we have the following inequality containing the unified Mittag–Leffler function $M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, s)$ satisfying all the convergence conditions:*

$$\begin{aligned}
&\left|\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi*\psi\right)(\vartheta;s)+\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi*\psi\right)(\vartheta;s)\right|\\
&\le\Lambda^{\sigma_1}_{\vartheta}\left(M^{\omega,\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \psi;\Upsilon\right)\left[\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)-|\varphi'(\sigma_1)|\psi(\sigma_1)\right)\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|-|\varphi'(\sigma_1)|\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right]\\
&\quad+\Lambda^{\vartheta}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\psi;\Upsilon\right)\left[\left(|\varphi'(\sigma_2)|\psi(\sigma_2)-m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)\right)\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\vartheta)^{\alpha}}\left(|\varphi'(\sigma_2)|-m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\right){}^{\alpha}I_{\sigma_2^-}\psi(\vartheta)\right],
\end{aligned}\tag{54}$$

where

$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi*\psi\right)(\vartheta;s):=\int_{\sigma_1}^{\vartheta}\Lambda^{\tau}_{\vartheta}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\psi;\Upsilon\right)\varphi'(\tau)d(\psi(\tau))$$

and

$$\left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi*\psi\right)(\vartheta;s):=\int_{\vartheta}^{\sigma_2}\Lambda^{\vartheta}_{\tau}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \psi;\Upsilon\right)\varphi'(\tau)d(\psi(\tau)).$$

Proof. Let $\vartheta \in [\sigma_1, \sigma_2]$ and $\tau \in [\sigma_1, \vartheta]$. Then, using the (α, m)-convexity of $|\varphi'|$, we have

$$|\varphi'(\tau)|\le\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}|\varphi'(\sigma_1)|+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|.\tag{55}$$

The inequality (55) can be written as follows:

$$-\left(\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}|\varphi'(\sigma_1)|+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\right)$$
$$\leq \varphi'(\tau) \leq \left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}|\varphi'(\sigma_1)|+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|. \quad (56)$$

Let us consider the second inequality of (56):

$$\varphi'(\tau) \leq \left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}|\varphi'(\sigma_1)|+m\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|. \quad (57)$$

Multiplying (31) and (57) and integrating over $[\sigma_1, \vartheta]$, we obtain

$$\int_{\sigma_1}^{\vartheta} \Lambda_{\vartheta}^{\tau}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\varphi'(\tau)d(\psi(\tau)) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left(|\varphi(\sigma_1)|\int_{\sigma_1}^{\vartheta}\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}d(\psi(\tau))\right.$$
$$\left.+m\left|\varphi\left(\frac{\vartheta}{m}\right)\right|\int_{\sigma_1}^{\vartheta}\left(1-\left(\frac{\vartheta-\tau}{\vartheta-\sigma_1}\right)^{\alpha}\right)d(\psi(\tau))\right).$$

By using (21) of Definition 9 and integrating by parts, the following inequality is obtained:

$$\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,r}\ \varphi*\psi\right)(\vartheta;s) \leq \Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left[\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)-|\varphi'(\sigma_1)|\psi(\sigma_1)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|-|\varphi'(\sigma_1)|\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right]. \quad (58)$$

If we consider the left-hand side of the inequality (56) and adopt the same pattern as for the right-hand side inequality, then

$$\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,r}(\varphi*\psi)\right)(\vartheta;s) \geq -\Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left[\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)-|\varphi'(\sigma_1)|\psi(\sigma_1)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|-|\varphi'(\sigma_1)|\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right]. \quad (59)$$

From (58) and ((59), the following inequality is observed:

$$\left|\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,r}(\varphi*\psi)\right)(\vartheta;s)\right| \leq \Lambda_{\vartheta}^{\sigma_1}\left(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left[\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)-|\varphi'(\sigma_1)|\psi(\sigma_1)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\vartheta-\sigma_1)^{\alpha}}\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|-|\varphi'(\sigma_1)|\right){}^{\alpha}I_{\sigma_1^+}\psi(\vartheta)\right]. \quad (60)$$

Now, using the (α, m)-convexity of $|\varphi'|$ on $(\vartheta, \sigma_2]$ for $\vartheta \in [\sigma_1, \sigma_2]$ we have

$$|\varphi'(\tau)| \leq \left(\frac{\tau-\vartheta}{\sigma_2-\vartheta}\right)^{\alpha}|\varphi'(\sigma_2)|+m\left(1-\left(\frac{\tau-\vartheta}{\sigma_2-\vartheta}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|. \quad (61)$$

Using the same procedure as we did for (31) and (55), one can obtain the following inequality from (34) and (61):

$$\left|\left({}_{\psi}^{\Upsilon}\Omega_{\sigma_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\gamma,\delta,k,c}(\varphi*\psi)\right)(\vartheta;s)\right| \leq \Lambda_{\sigma_2}^{\vartheta}\left(M_{\alpha,\beta,\gamma,\delta,\mu,\nu'}^{\lambda,\rho,\theta,k,r}\ \psi;\Upsilon\right)\left[\left(|\varphi'(\sigma_2)|\psi(\sigma_2)-m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\vartheta)^{\alpha}}\left(|\varphi'(\sigma_2)|-m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\right){}^{\alpha}I_{\sigma_2^-}\psi(\vartheta)\right]. \quad (62)$$

By adding (60) and (62), inequality (54) can be achieved. □

Remark 4. *(i)* *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda$ and $\rho = \nu = 0$ in (54), [24] (Theorem 3) is obtained;*

(ii) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = 0$ and $(\alpha, m) = (1, 1)$ in (54), then [25] (Theorem 3) is obtained;*

(iii) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = \omega = 0, \Upsilon(\tau) = \Gamma(\mu)\tau^{\frac{\mu}{k}+1}$ for a left-hand integral and $\Upsilon(\tau) = \Gamma(\nu)\tau^{\frac{\nu}{k}+1}$ for a right-hand integral in (54), then [26] (Theorem 2) is obtained;*

(iv) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = \omega = 0, \Upsilon(\tau) = \Gamma(\mu)\tau^{\mu+1}$ for a left-hand integral and $\Upsilon(\tau) = \Gamma(\nu)\tau^{\nu+1}$ for a right-hand integral, $(\alpha, m) = (1, 1)$ in (54), then [27] (Theorem 2) is obtained;*

(v) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = \omega = 0, \Upsilon(\tau) = \Gamma(\mu)\tau^{\frac{\mu}{k}+1}$ for a left-hand integral and $\Upsilon(\tau) = \Gamma(\nu)\tau^{\frac{\nu}{k}+1}$ for a right-hand integral, $(\alpha, m) = (1, 1)$ and $\psi(\vartheta) = \vartheta$ in (54), then [28] (Theorem 2) is obtained;*

(vi) *If $r = 1, b_1 = \lambda + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = \omega = 0, \Upsilon(\tau) = \Gamma(\mu)\tau^{\mu+1}$ for a left-hand integral and $\Upsilon(\tau) = \Gamma(\nu)\tau^{\nu+1}$ for a right-hand integral, $\psi(\vartheta) = \vartheta$ and $(\alpha, m) = (1, 1)$ in (54), then [22] (Theorem 2) is obtained.*

Example 6. *If $\alpha = 1$, then the following inequality holds for an m-convex function*

$$\begin{aligned}
&\left| \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi * \psi \right)(\vartheta; s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi * \psi \right)(\vartheta; s) \right| \\
&\leq \Lambda^{\sigma_1}_{\vartheta}\left(M^{\omega,\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \psi; \Upsilon \right) \left[\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta) - |\varphi'(\sigma_1)|\psi(\sigma_1) \right) \right. \\
&\quad \left. - \frac{1}{(\vartheta - \sigma_1)}\left(m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right| - |\varphi'(\sigma_1)| \right) \int_{\sigma_1}^{\vartheta} \psi(\tau)d\tau \right] \\
&\quad + \Lambda^{\vartheta}_{\sigma_2}\left(M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu'}\ \psi; \Upsilon \right) \left[\left(|\varphi'(\sigma_2)|\psi(\sigma_2) - m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right|\psi(\vartheta) \right) \right. \\
&\quad \left. - \frac{1}{(\sigma_2 - \vartheta)}\left(|\varphi'(\sigma_2)| - m\left|\varphi'\left(\frac{\vartheta}{m}\right)\right| \right) \int_{\vartheta}^{\sigma_2} \psi(\tau)d\tau \right].
\end{aligned} \tag{63}$$

Example 7. *If $(\alpha, m) = (1, 1)$, then the following inequality holds for a convex function*

$$\begin{aligned}
&\left| \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi \right)(\vartheta; s) + \left({}^{\Upsilon}_{\psi}\Omega^{\omega,\lambda,\rho,\theta,k,r}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi \right)(\vartheta; s) \right| \\
&\leq M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\vartheta) - \psi(\sigma_1))^{\alpha}; \underline{a}, \underline{b}, \underline{c}, s))(\Upsilon(\psi(\vartheta) - \psi(\sigma_1))(|\varphi'(\vartheta)| + |\varphi'(\sigma_1)|) \\
&\quad + M^{\lambda,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,\nu}(\omega(\psi(\sigma_2) - \psi(\vartheta))^{\alpha}; \underline{a}, \underline{b}, \underline{c}, s))(\Upsilon(\psi(\sigma_2) - \psi(\vartheta))(\varphi'(\vartheta)| + |\varphi'(\sigma_2)|).
\end{aligned} \tag{64}$$

3. Conclusions

We have investigated the bounds of fractional integral operators containing the Mittag–Leffler function in their kernels. The established bounds are compact formulas that generate fractional integral inequalities for various well-known integral operators. All the presented and deduced results hold for convex, *m*-convex and star-shaped functions. This work can be further extended for different kinds of classes of functions that already exist in the literature. For example, classes of strongly convex and refined convex functions can be applied to improve such bounds of fractional integral operators.

Author Contributions: Conceptualization, G.F. and K.N.; investigation, K.N., G.F., H.Y., F.A.S. and C.Y.J.; methodology, K.N., G.F., H.Y., F.A.S. and C.Y.J.; validation, K.N., G.F., H.Y., F.A.S. and C.Y.J.; visualization, K.N., G.F., H.Y., F.A.S. and C.Y.J.; writing—original draft, G.F. and K.N.; writing—review and editing, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research received funding support from the National Science, Research, and Innovation Fund (NSRF), Thailand.

Conflicts of Interest: The authors declare no conflict of interest.

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