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A Boundedly Rational Decision-Making Model Based on Weakly Consistent Preference Relations

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Abstract: Completeness is one of the basic assumptions about the rational preference relation in classical decision theory. Strongly and weakly consistent preferences are presented by abandoning the completeness of the rational preference relation. Some expansion and contraction conditions are proposed and the relationships between these conditions of rationality are discussed. The relationships between the conditions of rationality and boundedly rational choice behavior based on strongly and weakly consistent preferences are analyzed and discussed. Furthermore, an example about the choices of chocolates with interval ordinal numbers is given to explain some of the main conclusions in this paper. The results can be used as references for the study of boundedly rational decisions.

Keywords: bounded rationality; weakly consistent preference; interval ordinal number

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1. Introduction

Fully rational decisions are built on several important internal consistency conditions, such as the Weak Axiom of Revealed Preference [1], the Strong Axiom of Revealed Preference [2], the Congruence Axiom [3], the Weak Congruence Axiom [4], and so on. These conditions are treated as properties of rational decisions, as well as testable hypotheses regarding the decision maker's (DM) choice behavior. As for a rational DM, her preferences should be stable and consistent under full rationality. However, individual preferences are unstable and variable in reality. Lichtenstein and Slovic [5] first found the preference reversal phenomenon through an experiment about gambling choices. Results showed that individual preferences are affected by judgement, the background and procedure of choice. Moreover, the experiments conducted by Kahneman and Tversky [6] also showed that individual preferences are inconsistent usually in risk decisions. They indicated that there exist systematic biases between individual behaviors and the assumptions of rational preferences. Therefore, the full rationality assumption about individual preferences is questioned.

The most systematic and consistent attack on the full rationality assumption is directed by Simon [7]. Simon criticizes many aspects of classical decision theory and points out that in reality, individuals do not have infinite computational capabilities and access to information like the paradigm of rationality assumes. In the early 1950s, Simon put forward the theory of bounded rationality [7]. In the spirit of Simon's idea of bounded rationality, some studies proposed relevant boundedly rational models by abandoning the completeness assumption of the preference relation. For example, Bewley [8] proposed a theory of choice under uncertainty that removes the completeness assumption from the Anscombe–Aumann formulation of Savage's theory and introduced an inertia assumption. Faro [9] further characterized variational Bewley preferences over Anscombe and Aumann acts based on Bewley [8]. Barokas [10] provided a taxonomy of four known models of rationalization by incomplete preferences. Cettolin and Riedl [11] designed several experiments to test incomplete preferences due to uncertainty. Results showed that about half of the participants' behaviors were consistent with incomplete preferences. Gerasimou [12] characterized partially dominant choice with acyclic underlying preferences. Inspired by the model of Gerasimou [12], Qin [13] assumed that the underlying preference is transitive and provided an axiomatic basis for partially dominant choice with transitive underlying preferences. Gerasimou [14] showed that the DM in the Bewley model [8] is different between distinct monetary acts whenever the set of priors is fully dimensional under general conditions of preferences. Other studies characterized boundedly rational choice behaviors by relaxing the theory of revealed preference. For example, Ok et al. [15] relaxed the weak axiom of revealed preference and developed a revealed preference theory of reference-dependent behavior. García-Sanz and Alcantud [16] studied the theory of rational choice under two different criteria and characterized the rationality of choice correspondences by two sequential criteria. Cantone et al. [17] described an axiomatic approach to revealed preference theory, and the rationalization of multivalued choice was studied by maximizing the revealed preference relation. Pal [18] postulated a number of choice consistency conditions that are proven to be necessary and sufficient for the rationalization of choice functions. Armouti-Hansen and Kops [19] extended two boundedly rational models, i.e., the categorize-then-choose heuristic and the rational shortlist method, and axiomatic characterizations are provided for these two models. Yang [20] obtained a sufficient and necessary condition for the rationalizability of choice functions, and an algorithm was proposed to verify the rationalizability of a given choice function.

The above-mentioned study mainly focused on axiomatic characterizations of boundedly rational models, and the psychological factors of the DM and external environmental factors are less considered. The DM's choices are constrained by internal factors (such as attention, memory, etc.) and external factors (such as time, environment, information collection and processing, etc.), and hence it is impossible for them to make completely rational decisions. The axiomatic modeling of bounded rationality combined with psychological factors of the DM has become hot research. For example, Tyson [21] offered an axiomatic basis for satisficing the decision-making of Simon based on the two factors of cognitive constraints and environmental complexity. He considered that the DM's cognitive constraints can prevent her from perceiving her own preferences among the available alternatives. The DM's perception decreases with respect to the complexity of the choice problem. Tyson [22] further characterized rationalizability of choice functions by the conditions of the Menu Order and Cover Dominance, and proved how the rationalizability of menu preferences parallels the rationalizability of choice functions. Cerreia-Vioglio et al. [23] studied the DM's mental preferences and behavioral preferences, and provided axiomatic characterizations to these two preferences. Stewart [24,25] generalized the rationalizability of choice functions by acyclic binary relations and characterized weakly pseudo-rationalizable choice functions in terms of hyper-relations satisfying certain properties.

The complexity of the choice problem is aligned with set inclusion in the satisficing model of Tyson [21]. We consider that the set inclusion is a special case between sets, and it is only applied to a few cases. On one hand, if there is large number of alternatives in a set of alternatives, a DM cannot perceive all preferences among the available alternatives because of cognitive constraints and environmental complexity. The increasing number of alternatives increases the difficulty of decision-making on the set inclusion. On the other hand, if there are only partial common alternatives between two sets of alternatives and all alternatives in the two sets of alternatives are of the same type, the "nestedness" hypothesis on the set inclusion of Tyson [21] can be extended to the intersection set, and therefore the hypothesis is more universal. Inspired by Tyson [21], we extend the set inclusion to the intersection set, and the strongly and weakly consistent preferences are presented. Moreover, the relationships between the conditions of rationality and boundedly rational choice behavior based on strongly and weakly consistent preferences are analyzed and discussed in this paper. The rest of this paper is organized as follows. Section 2 presents the concepts of strongly and weakly consistent preferences. Section 3 mainly discusses the rationality properties of boundedly rational choice based on weakly consistent preferences. Section 4 gives an example about the choices of chocolates combined with interval ordinal numbers to explain some of the main conclusions. The concluding Section 5 summarizes related work in this paper.

2. Strongly and Weakly Consistent Preferences

Traditional economic and psychological theories suggest that having more options is beneficial to DMs [26]. However, too many choices can sometimes lead to contradictory results. For example, consumers make "no choice" or experience more regret. The results of a series of experiments by Iyengar and Lepper [27] provided some preliminary research evidence that having a large number of alternatives to choose from can be overwhelming. An experiment was conducted in a supermarket. Two food stands were set up by the staff in a supermarket; there were six and 24 different jams in two stands. Results showed that there were more customers at the stand with 24 different jams, 60% of 242 customers stopped for a taste at the stand with 24 different jams, and 40% of 260 customers stopped for a taste at the stand with 6 different jams. However, the ultimate results were unexpected: 30% of customers who stopped for a taste at the stand with six different jams bought at least one jam, whereas only 3% of customers who stopped for a taste at the stand with 24 different jams bought jams. In another experiment, a group of participants was required to choose between six chocolates and another group of participants was required to choose between 30 chocolates. As a result, participants who had chosen from six alternatives reported that they were more satisfied with the taste of their chocolates than the participants who chose from among 30 alternatives.

The results of the two experiments above show that it is not easy for DMs to make decisions when facing more choices. It is difficult for the DM to discern the differences among more choices because of extrinsic factors (such as environmental complexity) and intrinsic factors (such as incomplete preferences). Here, incomplete preferences refers to the DM not always being able to compare all alternatives in the set of alternatives in reality. Moreover, too many choices can weaken the perception of the DM and lead to the inconsistency in the DM's preferences. To extend the concepts of inconsistent preferences, strongly and weakly consistent preferences are presented to describe the choice overload phenomenon in this paper.

A binary relation R on the set X is a subset of $X \times X$; we can abbreviate $(x, y) \in \mathrm{R}$ as $x\mathrm{R}y$. Given a relation R, we can define its converse $\mathrm{R}' = \{(x, y) | y\mathrm{R}x, x \in X, y \in X\}$, complement $\bar{\mathrm{R}} = \{(x, y) | \neg(x\mathrm{R}y), x \in X, y \in X\}$, and symmetric residue $\mathrm{R}^0 = \bar{\mathrm{R}} \cap \bar{\mathrm{R}}'$. We can also informally refer to the binary relation R as a preference. Usually, $x\mathrm{R}y$ represents that x "is at least as good as" y. Strict preference P and indifference I are, respectively, the asymmetrical and symmetrical parts of binary relation R. Table 1 lists 10 properties to describe characteristics of the binary relation R [21].

Table 1. Ten properties of the binary relation R.

Properties	Expressions	Properties	Expressions
Reflexivity	$x\mathrm{R}x$	Transitivity	If $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$
Irreflexivity	$x\bar{\mathrm{R}}x$	Residual transitivity	If $x\mathrm{R}^0y$ and $y\mathrm{R}^0z$,then $x\mathrm{R}^0z$
Symmetry	If $x\mathrm{R}y$, then $y\mathrm{R}x$	Cross transitivity	If $x\mathrm{R}y$ and $y\ \mathrm{R}^0z$ (or $x\ \mathrm{R}^0y$ and $y\mathrm{R}z$), then $x\mathrm{R}z$
Asymmetry	If $x\mathrm{R}y$, then $y\bar{\mathrm{R}}x$	Negative transitivity	If $x\bar{\mathrm{R}}y$ and $y\bar{\mathrm{R}}z$, then $x\bar{\mathrm{R}}z$
Acyclicity	If $x_1\mathrm{R}\ x_2\ \mathrm{R}\ x_3 \ldots \mathrm{R}\ x_n$, then $x_n\bar{\mathrm{R}}x_1$	Completeness	If $x\bar{\mathrm{R}}y$, then $y\mathrm{R}x$

Individual preferences are assumed to be rational in classical decision theory. A preference relation is rational whenever it is complete and transitive. A complete preference relation refers to a DM always being able compare all alternatives in the set of alternatives. However, when facing more choices, it is difficult for a DM to compare all alternatives because of extrinsic factors (such as environmental complexity) and intrinsic factors (such as incomplete preferences) in reality. Moreover, too many choices can weaken the perception of the DM and lead to inconsistency in the DM's preferences. Tyson [21] proposed the "nestedness" hypothesis with set inclusion to address this issue. The "nestedness" hypothesis refers to any (strict) preference perceived in a given problem B also being perceived in each simpler problem $A \subset B$. We consider that set inclusion is a special case of set relations. In this paper, without loss of generality, we extend set inclusion to the intersection set $A \cap B$. We assume that the sets A and B are two intersectant sets of alternatives of the same type. Strongly and weakly consistent preferences are defined in terms of relationships between preferences in the two mutually intersectant sets of alternatives by abandoning the completeness assumption about the preference relation.

Definition 1. *A binary relation R is said to be a* ***strongly consistent preference*** *if* $\forall x, y \in A \cap B \neq \phi$, $|A| \leq |B|$, *and we have* $x\mathrm{R}_B y$ *if and only if* $x\mathrm{R}_A y$. *A binary relation R is said to be a* ***weakly consistent preference*** *if* $\forall x, y \in A \cap B \neq \phi$, $|A| \leq |B|$, *and we have* $x\mathrm{R}_B y$ *only if* $x\mathrm{R}_A y$.

Here, $|A|$ and $|B|$ denote the number of elements in sets A and B, respectively. Let us write R_A and R_B for the relation with A and B, respectively. Obviously, the preference $x\mathrm{R}_A y$ is a sufficient and necessary condition for $x\mathrm{R}_B y$ when R is a strongly consistent preference relation. Namely, a preference is perceived in set B with large cardinality and it is also perceived in set A with small cardinality by the DM and vice versa. The preference $x\mathrm{R}_A y$ is a necessary condition for $x\mathrm{R}_B y$ when R is a weakly consistent preference relation. Namely, a preference is perceived in set B with large cardinality only if it is also perceived in set A with small cardinality by the DM. This is because the increase in alternatives may weaken the DM's perception ability.

3. Boundedly Rational Choices Based on Strongly and Weakly Consistent Preferences

Other than complete rationality, there have been no authoritative definitions about bounded rationality until now [28–30]. However, some properties of individual bounded rationality [31,32] can be concluded as:

(1) The DM is unable to perceive alternatives.
(2) The DM is unable to rank all alternatives.
(3) The DM chooses the alternative still according to the "optimization" principle within bounds of perceptibility and decidability.

In this paper, individual bounded rationality means that the DM cannot perceive all preferences among alternatives and can only perceive parts of her actual preferences when she faces more complicated choices. For example, a consumer intends to buy goods in a supermarket. If there are only six kinds of goods, she can easily rank them and make choices. However, there are always dozens of goods in a supermarket. Even if the consumer refers to a handbook, it is also difficult for her to perceive differences among all goods and form her complete preferences in time. Therefore, the consumer always makes choices only based on parts of her actual preferences according to the "satisfaction" principle. In this section, individual boundedly rational decision-making combined with strongly and weakly consistent preferences will be discussed.

3.1. Choice Function and Conditions of Rationality

Fix a **nonempty finite** set X and a domain $\Omega = \{A \subset X : A \neq \phi\}$. X denotes all of the available alternatives of the finite set and Ω denotes the set of nonempty subsets of X. A choice function on X is a mapping $C : \Omega \to \Omega$. $C(A)$ of each $A \in \Omega$ denotes the set of alternatives that in *A* cannot be ruled out. A reasonable choice function C should satisfy the following postulates [19]:

(1) **Availability**: $C(A) \subset A$.

(2) **Decisiveness**: $A \neq \phi \Rightarrow |C(A)| \geq 1$.

Here, $|C(A)|$ denotes the number of elements in the set $C(A)$. Obviously, availability gives the upper bound of the set $C(A)$. Decisiveness indicates that the choice function $C(A)$ always exists for each non-empty subset.

A relation R is said to generate a choice function C when each choice set contains those and only those available alternatives that are maximal with respect to R, i.e., when $\forall A$ we have Equation (1):

$$C(A, \mathrm{R}) = \{x \in A \mid (\forall y \in A) y \overline{\mathrm{R}} x\} \tag{1}$$

Inspired by various expansion and contraction conditions suggested by Sen [4] and Tyson [21], we introduce and propose the following Conditions 1–4 for describing some observable features of rational decisions.

Condition 1. *If* $\forall x$ *and* A, B *such that* $x \in C(A) \cap C(B)$, *we have* $x \in C(A \cup B)$.

Condition 1 is Sen's property γ [4], which indicates that if an alternative is chosen in each subset, then it must also be chosen in the union of all subsets.

Condition 2. *If* $\forall x \in A \cap B$, $|A| \leq |B|$ *such that both* $x \in C(A)$ *and* $C(B) \subset A$, *we have* $x \in C(B)$.

Condition 2 is an extension of Weak Expansion suggested by Tyson [21]. Condition 2 shows that for two intersectant sets of alternatives, if an alternative is chosen in the small set (with small cardinality), it must be chosen in the big set (with large cardinality) as long as the chosen alternatives of the big set are contained in the small set.

Condition 3. *If* $\forall x, y \in A \cap B$, $|A| \leq |B|$ *such that both* $x \in C(A)$ *and* $y \in C(B)$, *we have* $x \in C(B)$.

Condition 3 is an extension of Strong Expansion suggested by Tyson [21]. Condition 3 requires that for two intersectant sets of alternatives, the chosen alternatives of the small set must be chosen in the big set as long as the chosen alternatives of the big set are partly contained in the small set.

Condition 4. *If* $\forall x \in A \cap B$, $|A| \leq |B|$ *such that* $x \in C(B)$, *we have* $x \in C(A)$.

Condition 4 is an extension of Contraction suggested by Sen [4]. Condition 4 requires that for two intersectant sets of alternatives, the chosen alternatives of the big set must be chosen in the small set.

Proposition 1. *Condition 3 implies both Condition 1 and Condition 2, but the converse does not hold.*

Proof. Let $x, y \in A$ such that $x \in C(A) \cap C(B)$, but $x \notin C(A \cup B)$. Since $C(A \cup B)$ is not empty, we assume that $y \in C(A \cup B)$, and clearly $y \in A \cup B$. Since $|A| \leq |A \cup B|$ and $|B| \leq |A \cup B|$, we have $x \in C(A \cup B)$ by Condition 3, contradicting $x \notin C(A \cup B)$. This shows that Condition 1 is implied by Condition 3. To show that Condition 3 implies Condition 2, let $x \in A \cap B$, $|A| \leq |B|$ such that both $x \in C(A)$ and $C(B) \subset A$, but $x \notin C(B)$. Since $C(B)$ is not empty, there exists $y \in C(B)$, so we have $x \in C(B)$ by Condition 3. This contradicts $x \notin C(B)$. Regarding the converse, the following counterexample is adequate: $C(\{wx\}) = \{w\}$, $C(\{xy\}) = \{x\}$, $C(\{yz\}) = \{y\}$, $C(\{xz\}) = \{z\}$, $C(\{xyz\}) = \{yz\}$, and $C(\{wxy\}) = \{wx\}$. Moreover, this function satisfies both Condition 1 and Condition 2, and violates Condition 3. □.

Proposition 1 indicates that Condition 3 is a stronger rationality condition than Condition 1 and Condition 2. Obviously, the decision-making that satisfies Condition 3 is more rational than both Condition 1 and Condition 2. Moreover, Condition 3 is a necessary condition for rational decision-making.

3.2. The Relationship between Choice Function and Conditions of Rationality

In this subsection, the relationship between the choice function and conditions of rationality above combined with strongly and weakly consistent preferences will be studied.

Definition 2. *Let R be a binary relation on* X. *Let* $x \in A \cap B$ *and* $|A| \leq |B|$; x *is said to be* ***relation dominant*** *in* A *in terms of R if and only if* $y \overline{\mathrm{R}}_A x$, $\forall y \in A \cap \overline{B}$.

From the definition of relation dominance, the scope of relation dominance is confined in two intersectant sets of alternatives. It can be regarded as a local relation dominance.

Proposition 2 *For any choice function* $C(A, \mathrm{R})$ *that is generated by a strongly consistent preference* R, *we have that* $C(A, \mathrm{R})$ *satisfies Condition 4 if each alternative in* $A \cap B$ *is relation dominant in* A *in terms of* R.

Proof. If Condition 4 fails, then there must exist an alternative $x \in A \cap B$, $|A| \leq |B|$ such that $x \in C(B, \mathrm{R})$, but $x \notin C(A, \mathrm{R})$. $x \notin C(A, \mathrm{R})$ implies that there exists a $y' \in A$ such that $y' \mathrm{R}_A x$. If $y' \in B$, then $y' \mathrm{R}_B x$ since R is strongly consistent, contradicting $x \in C(B, \mathrm{R})$. Alternatively, if $y' \in A$ and $y' \notin B$ such that $y' \mathrm{R}_A x$, this contradicts the fact that x is relation dominant in A. □.

Proposition 2 gives a sufficient condition for the choice function to satisfy Condition 4 when a choice function is generated by a strongly consistent preference relation. Proposition 2 gives a condition for the rationalization of the choice function that is generated by a strongly consistent preference relation. The following Proposition 3 gives a necessary condition for the choice function to satisfy Condition 4 when a choice function is generated by a binary relation.

Proposition 3 *For any choice function* $C(A, \mathrm{R})$ *that is generated by a binary relation* R, *we have that the chosen alternatives in* $A \cap B$ *are relation dominant in* A *(or* B *) in terms of* R *if* $C(A, \mathrm{R})$ *satisfies Condition 4.*

Proof. The conclusion is immediate. □.

Here, the choice function is generated by a common binary relation R, and R is not a strongly consistent preference relation. Obviously, the choice function is more general in Proposition 3. This implies that Proposition 3 is more general than Proposition 2.

Proposition 4. *For any choice function* $C(A,\mathrm{R})$ *that is generated by a weakly consistent preference* R, *we have that* $C(A,\mathrm{R})$ *satisfies Condition 3 if each alternative in* $A \cap B$ *is relation dominant in* B *in terms of* R.

Proof. If Condition 3 fails, then there must exist alternatives $x, y \in A \cap B$ and $|A| \leq |B|$ such that $x \in C(A,\mathrm{R})$ and $y \in C(B,\mathrm{R})$, but $x \notin C(B,\mathrm{R})$. $x \notin C(B,\mathrm{R})$ implies that there exists a $y' \in B$ such that $y' \mathrm{R}_B x$. If $y' \in A$, then $y' \mathrm{R}_A x$ since R is weakly consistent, contradicting $x \in C(A,\mathrm{R})$. Alternatively, if $y' \in B$ and $y' \notin A$ such that $y' \mathrm{R}_B x$, this contradicts the fact that x is relation dominant in B. □.

Proposition 4 gives a sufficient condition for the choice function to satisfy Condition 3 when a choice function is generated by a weakly consistent preference relation. Proposition 4 gives a condition for the rationalization of the choice function that is generated by a weakly consistent preference relation. Compared with Proposition 2, Proposition 4 indicates that Condition 3 is more rational than Condition 4. The following Corollary 1 is obtained combined with Propositions 1 and 4.

Corollary 1. *For any choice function* $C(A,\mathrm{R})$ *that is generated by a weakly consistent preference* R, *we have that* $C(A,\mathrm{R})$ *satisfies Condition 1 and Condition 2 if each alternative in* $A \cap B$ *is relation dominant in* B *in terms of* R.

Proposition 5. *For any choice function* $C(A,\mathrm{R})$ *that is generated by a binary relation* R, *(1) we have that* R *is acyclic. (2) We have that* $C(A,\mathrm{R})$ *satisfies Condition 1 if* R *is weakly consistent.*

Proof. (1) If R is not acyclic, then there exists a set $A = \{x_1, x_2 \cdots, x_n\}$ such that $x_1 \mathrm{R} x_2 \mathrm{R} \cdots x_n \mathrm{R} x_1$, and hence $C(A,\mathrm{R}) = \phi$, contradicting $|C(A,\mathrm{R})| \geq 1$. So, R is acyclic. (2) If Condition 1 fails, then there must exist both an alternative x and sets A, B such that $x \in C(A,\mathrm{R}) \cap C(B,\mathrm{R})$, but $x \notin C(A \cup B,\mathrm{R})$. $x \notin C(A \cup B,\mathrm{R})$ implies that there exists a $y \in A \cup B$ such that $y \mathrm{R}_{A \cup B} x$. Since R is a weakly consistent preference, if $y \in A$, then $y \mathrm{R}_A x$, contradicting $x \in C(A,\mathrm{R})$. Alternatively, if $y \in B$, then $y \mathrm{R}_B x$, contradicting $x \in C(B,\mathrm{R})$. Namely, C satisfies Condition 1. □.

Proposition 5 indicates that the acyclicity condition is a minimum requirement for the existence of a choice function generated by a binary relation R. Proposition 5 shows that the choice function that is generated by a weakly consistent relation satisfies Condition 1. Moreover, Condition 1 is a weak rationality condition by Proposition 5. Proposition 5 can be considered a decision criterion for boundedly rational decision-making.

Proposition 6. *For any choice function* $C(A,\mathrm{R})$ *that is generated by a weakly consistent preference* R, *we have that* $C(A,\mathrm{R})$ *satisfies Condition 1 and Condition 2 if* R *is transitive.*

Proof. The choice function satisfies Condition 1 by Proposition 5. If Condition 2 fails, then there must exist sets $C(B,\mathrm{R}) \subset A$, $|A| \le |B|$, and an alternative $x \in A \cap B$ such that $x \in C(A,\mathrm{R})$, but $x \notin C(B,\mathrm{R})$. $x \notin C(B,\mathrm{R})$ implies that there exists a $y_1 \in B$ such that $y_1 \mathrm{R}_B x$. If $y_1 \in A$, then $y_1 \mathrm{R}_A x$ since R is a weakly consistent preference, contradicting $x \in C(A,\mathrm{R})$. Alternatively, if $y_1 \in B$ and $y_1 \notin A$, then $y_1 \notin C(B,\mathrm{R})$ since $C(B,\mathrm{R}) \subset A$. Since $y_1 \notin C(B,\mathrm{R})$, there exists a $y_2 \in B$ such that $y_2 \mathrm{R}_B y_1$. Therefore, $y_2 \mathrm{R}_B x$ since R is transitive. Using induction, there must exist a $y_k \in B$ such that $y_k \mathrm{R}_B y_{k-1} \mathrm{R}_B y_{k-2} \cdots \mathrm{R}_B x$ since B is a finite set of alternatives. Therefore, $y_k \in C(B,\mathrm{R})$ and $y_k \notin A$, contradicting $C(B,\mathrm{R}) \subset A$. □.

Proposition 6 gives a rationality property of a choice function generated by a weakly consistent preference relation R when R is transitive. Proposition 6 can also be considered a decision criterion for boundedly rational decision-making. Proposition 6 indicates that the transitivity of weakly consistent preference relation R can guarantee Condition 2, but the converse does not hold. For example, consider the set of alternatives $X = \{wxyz\}$, the choice function generated by a weakly consistent preference R, taking $C(\{wx\},\mathrm{R}) = \{w\}$, $C(\{wy\},\mathrm{R}) = \{y\}$, $C(\{wz\},\mathrm{R}) = \{w\}$, $C(\{xy\},\mathrm{R}) = \{x\}$, $C(\{xz\},\mathrm{R}) = \{x\}$, $C(\{yz\},\mathrm{R}) = \{yz\}$, $C(\{wxy\},\mathrm{R}) = \{xy\}$, $C(\{wxz\},\mathrm{R}) = \{w\}$, $C(\{wyz\},\mathrm{R}) = \{yz\}$, $C(\{xyz\},\mathrm{R}) = \{x\}$, and $C(\{wxyz\},\mathrm{R}) = \{xy\}$. Moreover, this function satisfies both Condition 1 and Condition 2. A group of preference relations corresponding to these choices is given as follows: $w\mathrm{R}_{wx}x$, $y\mathrm{R}_{wy}w$, $w\mathrm{R}_{wz}z$, $x\mathrm{R}_{xy}y$, $x\mathrm{R}_{xz}z$, $y\mathrm{R}_{wxy}w$, $w\mathrm{R}_{wxz}x$, $x\mathrm{R}_{wxz}z$, $y\mathrm{R}_{wyz}w$, $x\mathrm{R}_{xyz}y$, $x\mathrm{R}_{xyz}z$, $x\mathrm{R}_{wxyz}z$, and $y\mathrm{R}_{wxyz}w$. It is easy to check that R is not transitive.

Proposition 7. *For any choice function $C(A,\mathrm{R})$ that is generated by a weakly consistent preference R, we have that $C(A,\mathrm{R})$ satisfies Condition 3 if R is negatively transitive.*

Proof. If Condition 3 fails, then there must exist alternatives $x, y \in A \cap B$ and $|A| \le |B|$ such that $x \in C(A,\mathrm{R})$, $y \in C(B,\mathrm{R})$, but $x \notin C(B,\mathrm{R})$. This implies that $x \bar{\mathrm{R}}_B y$ and $y \bar{\mathrm{R}}_A x$. R is a weakly consistent preference; therefore $y \bar{\mathrm{R}}_B x$. We have $x \bar{\mathrm{R}}_B y$ and $y \bar{\mathrm{R}}_B x$, and thus $x \mathrm{R}_B^0 y$. Moreover, there must also exist an alternative $z \in B$ such that $z\mathrm{R}_B x \mathrm{R}_B^0 y$. Since R is negatively transitive, R is also cross transitive. This implies that $z\mathrm{R}_B y$, and hence $y \notin C(B,\mathrm{R})$, contradicting $y \in C(B,\mathrm{R})$. □.

Proposition 7 gives a rationality property of a choice function generated by a weakly consistent preference relation R when R is negatively transitive. Proposition 7 can be considered a decision criterion for rational decision-making. Proposition 7 indicates that the negative transitivity of weakly consistent preference relation R can guarantee Condition 3, but the converse does not hold. For example, consider the set of alternatives $X = \{wxyz\}$, the choice function generated by a weakly consistent preference R, taking $C(\{wx\},\mathrm{R}) = \{x\}$, $C(\{wy\},\mathrm{R}) = \{y\}$, $C(\{wz\},\mathrm{R}) = \{z\}$, $C(\{xy\},\mathrm{R}) = \{x\}$, $C(\{xz\},\mathrm{R}) = \{z\}$, $C(\{yz\},\mathrm{R}) = \{y\}$, $C(\{wxy\},\mathrm{R}) = \{xy\}$, $C(\{wxz\},\mathrm{R}) = \{xz\}$, $C(\{wyz\},\mathrm{R}) = \{yz\}$, $C(\{xyz\},\mathrm{R}) = \{xyz\}$, and $C(\{wxyz\},\mathrm{R}) = \{xyz\}$. Moreover, this choice function satisfies Condition 3. A group of preference relations corresponding to these choices is given as follows: $x\mathrm{R}_{wx}w$, $y\mathrm{R}_{wy}w$, $z\mathrm{R}_{wz}w$, $x\mathrm{R}_{xy}y$, $z\mathrm{R}_{xz}x$, $y\mathrm{R}_{yz}z$,

$y\mathrm{R}_{wxy}w$, $x\mathrm{R}_{wxz}w$, $y\mathrm{R}_{wyz}w$, and $y\mathrm{R}_{wxyz}w$. It is easy to check that R is not negatively transitive.

4. An Example of the Choices of Chocolates Combined with Interval Ordinal Numbers

The ordinal preference mentioned above reflects the preference order of the DM; it does not contain the degree of preference. Cardinal utility is proposed to reflect the degree of preference about the DM in classical decision theory. In classical decision theory, a group of corresponding relations are defined to integrate the DM's ordinal preferences and cardinal utilities or values: For $\forall x, y \in A \in \Omega$, we have $x\mathrm{R}y \Leftrightarrow V(x) \geq V(y)$, $x\mathrm{P}y \Leftrightarrow V(x) > V(y)$, and $x\mathrm{I}y \Leftrightarrow V(x) = V(y)$, in which *V*: *A*→[0,+∞) is a utility or value function of alternatives. Specially, for $\forall x, y \in A \in \Omega$, we have $x\mathrm{R}y$ or $y\mathrm{R}x$ when the DM's preference relation R is complete. It implies that for $\forall x, y \in A \in \Omega$, we have $V(x) \geq V(y)$ or $V(y) \geq V(x)$. However, the DM's preferences are incomplete usually under bounded rationality. The DM only perceives parts of preferences among alternatives so that she can only give intervals of utilities or values *V*(*x*) about alternatives [32].

Specifically, **an interval ordinal number** is defined as follows: For $\forall x \in A$, the utility or value of *x* is denoted as $V_A(x) \in \left[\underline{V_A(x)}, \overline{V_A(x)}\right] \subseteq [0,1]$, where $\underline{V_A(x)}$ and $\overline{V_A(x)}$ represent the lower bound and the upper bound of $V_A(x)$, respectively. Meanwhile, the corresponding relations between perceived preferences and the intervals of $V_A(x)$ are as follows [32]: for $\forall x, y \in A$, $x\mathrm{R}_A y \Rightarrow \underline{V_A(x)} \geq \overline{V_A(y)}$ and $x\mathrm{I}_A y \Rightarrow \underline{V_A(x)} = \overline{V_A(x)} = \underline{V_A(y)} = \overline{V_A(y)}$ or the alternatives *x* and *y* are incomparable. In addition, the intervals of utilities or values are varied about the same alternatives, which the DM perceives in different sets of alternatives. So, it is assumed that $\left[\underline{V_A(x)}, \overline{V_A(x)}\right] \subseteq \left[\underline{V_B(x)}, \overline{V_B(x)}\right]$ for $\forall x \in A \cap B, |A| \leq |B|$.

Next, we take choices about chocolates (*v*, *w*, *x*, *y*, and *z*) as an example for further explanations and analyses combined with interval ordinal numbers. Usually, a DM only gives the intervals of utilities or values about alternatives from her intuition or experiences, and makes choices according to a certain decision rule and threshold $\mu(A)$. Then, $C(A) = \{x \in A | V(x) \geq \mu(A)\}$ can be used to describe the decision-making in which the threshold $\mu(A)$ is given subjectively by the DM. We assume that $\mu(A) = \underset{x \in A}{Max} \underline{V_A(x)}$ in this paper. Table 2 gives a DM's choices and perceived intervals of values about chocolates while her weakly consistent preferences are acyclic and transitive. Table 3 gives another DM's choices and perceived intervals while her weakly consistent preferences are acyclic and negatively transitive.

Table 2. Perceived intervals and choices of the DM with acyclic and transitive preferences.

The Set of Chocolates	*v*	*w*	*x*	*y*	*z*	*C*(·)	$\mu(A)$
{*wx*}	→	[0.4, 0.45]	[0.35, 0.4]	→	→	{*w*}	0.4
{*vx*}	[0.25, 0.3]	→	[0.35, 0.4]	→	→	{*x*}	0.35
{*xy*}	→	→	[0.35, 0.4]	[0.25, 0.3]	→	{*x*}	0.35
{*xz*}	→	→	[0.35, 0.4]	→	[0.25, 0.3]	{*x*}	0.35
{*wxy*}	→	[0. 4, 0.55]	[0.3, 0.4]	[0.2, 0.3]	→	{*w*}	0.4
{*wxz*}	→	[0. 4, 0.55]	[0.3, 0.4]	→	[0.2, 0. 3]	{*w*}	0.4
{*xyz*}	→	→	[0.3, 0.4]	[0.2, 0.35]	[0.2, 0.35]	{*xyz*}	0.3
{*wxyz*}	→	[0. 4, 0.6]	[0.25, 0.4]	[0.2, 0.4]	[0.2, 0.35]	{*w*}	0.4

$\{vwxy\}$	[0.2, 0.3]	[0. 4, 0.6]	[0.25, 0.4]	[0.2, 0.4]	→	$\{w\}$	0.4
$\{vwxz\}$	[0.2, 0.3]	[0. 4, 0.6]	[0.25, 0.4]	→	[0.2, 0.35]	$\{w\}$	0.4
$\{vxyz\}$	[0.2, 0.3]	→	[0.25, 0.4]	[0.2, 0.4]	[0.2, 0.35]	$\{vxyz\}$	0.25

Table 3. Perceived intervals and choices of the DM with acyclic and negatively **transitive preferences.**

The Set of Chocolates	v	w	x	y	z	$C(\cdot)$	$\mu(A)$
$\{wx\}$	→	[0.4, 0.45]	[0.2, 0.3]	→	→	$\{w\}$	0.4
$\{vx\}$	[0.4, 0.45]	→	[0.2, 0.3]	→	→	$\{v\}$	0.4
$\{xy\}$	→	→	[0.2, 0.3]	[0.45, 0.5]	→	$\{y\}$	0.45
$\{xz\}$	→	→	[0.2, 0.3]	→	[0.35, 0.5]	$\{z\}$	0.35
$\{wxy\}$	→	[0. 4, 0.55]	[0.2, 0.35]	[0.35, 0.5]	→	$\{wy\}$	0.4
$\{wxz\}$	→	[0. 4, 0.55]	[0.2, 0.35]	→	[0.35, 0.55]	$\{wz\}$	0.4
$\{xyz\}$	→	→	[0.2, 0.4]	[0.35, 0.5]	[0.35, 0.55]	$\{xyz\}$	0.35
$\{wxyz\}$	→	[0. 4, 0.6]	[0.15, 0.4]	[0.35, 0.55]	[0.25, 0.55]	$\{wyz\}$	0.4
$\{vwxy\}$	[0.4, 0.55]	[0. 4, 0.6]	[0.15, 0.4]	[0.35, 0.55]	→	$\{vwy\}$	0.4
$\{vwxz\}$	[0.4, 0.55]	[0. 4, 0.6]	[0.15, 0.4]	→	[0.25, 0.55]	$\{vwz\}$	0.4
$\{vxyz\}$	[0.4, 0.6]	→	[0.15, 0.4]	[0.35, 0.55]	[0.25, 0.55]	$\{vyz\}$	0.4

The choices in Table 2 are consistent with Proposition 6. It is easy to check that the DM's preferences are weakly consistent; e.g., the preference $w\mathrm{R}_{\{wxyz\}}x$ is perceived in the set of chocolates $\{wxyz\}$ and hence also in the subset of chocolates $\{wxy\}$. A DM perceives preferences in the big set only if also in the small set, rather than vice versa; e.g., the DM can perceive the preference $x\mathrm{R}y$ in the set of chocolates $\{wxy\}$, but not in the set of chocolates $\{wxyz\}$. Moreover, the DM's choices satisfy Condition 2 when the DM's preferences are transitive; e.g., the DM's perceived preferences in the set of chocolates $\{wxz\}$ are $w\mathrm{R}_{\{wxz\}}x$, $x\mathrm{R}_{\{wxz\}}z$, $w\mathrm{R}_{\{wxz\}}z$, and the chocolate w is chosen in the small set of chocolates $\{wz\}$ and hence also in the big set of chocolates $\{wxz\}$. Finally, the choices should attain the given utility threshold; e.g., the chosen chocolates in the set of chocolates $\{wxy\}$ are those with utility values no smaller than $\mu(\{wxy\}) = 0.4$.

The choices in Table 3 are consistent with Proposition 7. For example, the DM's choices in the set of chocolates $\{vwxy\}$ are v, w, and y. This implies that each chocolate of v, w, and y is acceptable. Her preferences are $v\mathrm{R}_{\{vwxy\}}x$, $w\mathrm{R}_{\{vwxy\}}x$, so v and w are incomparable in the set of chocolates $\{vwxy\}$ for her. This implies that if $v\mathrm{R}_{\{vwxy\}}x$, then $w\mathrm{R}_{\{vwxy\}}x$, and hence the DM's preferences are negatively transitive. Moreover, the preference $v\mathrm{R}_{\{vwxy\}}x$ is perceived in the set of chocolates $\{vwxy\}$ and hence also in the set of chocolates $\{vwxz\}$. The chocolates v and w are chosen in the set of chocolates $\{vwxy\}$ and hence also in the set of chocolates $\{vwxz\}$. This implies that the DM's choices satisfy Condition 3.

5. Conclusions

The framing effect and preference reversal phenomenon show that individual preferences are not stable and consistent. Therefore, it is difficult for us to explain and predict the behaviors of human beings by rational models. This paper proposes a boundedly rational choice model based on weakly consistent preference on the premise of incomplete preferences. Our main contribution is that we investigate the proposed contraction and expansion consistency conditions and characterize boundedly rational choice behaviors based on the weak consistency of preferences under different conditions. Moreover, rational choice behaviors combined with the defined interval ordinal numbers in the choices of chocolates are analyzed.

The choice overload phenomenon can be explained by the model in this paper. When faced with more choices, people show an attitude of indecisiveness because of the limitations of rationality. It can be interpreted as "more choices increase the difficulty of decision making, and therefore individual preferences may change, which may result in weak consistency of preferences." Therefore, the model in this paper provides theoretical guidance and an applicable value for satisficing decision-making. Moreover, the model in this paper can be extended to the following research directions for future studies: (1) The hesitant fuzzy preference relation is a useful tool for the DM to reveal her preference information over a set of alternatives. The DM's preferences in hesitant fuzzy preference relation are usually incomplete because of the limitations of professional knowledge, experience, and time pressure (such as Zhang et al. [33]). How to study incomplete hesitant fuzzy preference relations under bounded rationality is a research direction for the future. (2) It is also interesting to extend the boundedly rational decision-making model to consensus-reaching in linguistic group decision-making, random decision forests, and so on (such as Zhang and Li [34], Wang et al. [35]).

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