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Sombor Index over the Tensor and Cartesian Products of Monogenic Semigroup Graphs

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Abstract: Consider a simple graph G with vertex set $V(G)$ and edge set $E(G)$. A graph invariant for G is a number related to the structure of G, which is invariant under the symmetry of G. The Sombor index of G is a new graph invariant defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{(d_u)^2 + (d_v)^2}$. In this work, we connected the theory of the Sombor index with abstract algebra. We computed this topological index over the tensor and Cartesian products of a monogenic semigroup graph by presenting two different algorithms; the obtained results are illustrated by examples.

Keywords: monogenic semigroups; graphs; tensor product; Cartesian product; indices

1. Introduction

Let $G(V,E)$ be a simple graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ as the vertex and the edge sets. The degree of a vertex $v \in V(G)$ will be indicated by d_v. The edge adjacent to the vertices u and v will be specified by uv. In graph G, the distance between any two vertices u and v is stipulated by $d(u,v)$ and is determined as the length of the shortest path linking u and v. For further information on the basics of graph theory, please see reference [1].

A finite multiplicative monogenic semigroup (with zero) is given below

$$S_M = \left\{0, x, x^2, x^3, \ldots, x^n\right\}, \tag{1}$$

in which the authors put into effect in [2]. Whilst the graph $\Gamma(S_M)$ is specified by modifying the adjacent rule of vertices and sticking to the original stance. The vertices of $\Gamma(S_M)$ include all elements in S_M, except zero. For any two different vertices, x^i and x^j in $\Gamma(S_M)$ that $(1 \leq i, j \leq n)$ are adjoined to one another, if and only if $i + j > n$. For detailed information about monogenic semigroup graphs, see [3–5].

Zero divisor graphs are the bases of monogenic semigroup graphs [2]. Zero divisor graphs were first conducted on commutative rings [6]; following this study, the researchers worked on commutative and non-commutative rings [7–9]. Following the studies of zero divisor graphs on rings, in references [10,11], the authors utilized the information in commutative and noncommutative semigroups.

In chemistry, studies on topological indices have been carried out for more than half a century [12]. Recently, topological indices have been thoroughly detailed in mathematics. Indices such as these are utilised in creating structural properties of molecules, equipping us with data for industrial science, applied physics, biochemistry, environmental science, and toxicology [13]. In ref. [14], Gutman introduced a graph-based topological index named the Sombor index. It was first used in chemistry [15–20]; soon after, it captured the attention of mathematicians [2,21–24]. Network science used the modeling dynamical effect of biology and social technological complex systems [25]. The Sombor index became popular for military use as well [26]. Since its inception (less than one year after being published), the vast interest of mathematicians researching the Sombor index has been

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astounding. We believe that the Sombor index needs to be investigated in-more depth. Our research produces results based on the Sombor index category of algebraic structures, such as monogenic semigroups.

Graph operations play a pivotal role in mathematical chemistry due to the substantial importance they provide in evaluating numerous graph operations of simple graphs, such as chemical interest graphs. In addition, their importance, i.e., in the construction of bigger graphs out of smaller ones, is beneficial in the identification and decomposition of large graphs. Research into the extension of graphs is quintessential in applied sciences (see [27,28]). It is with this concept that the tensor and Cartesian products have been determined. The researchers in [29,30] estimated the Wiener index of the Cartesian product of graphs and in [31] the authors calculated the Szeged index of the Cartesian product of graphs. In 2011, Yarahmadi calculated certain topological indices, such as the Zagreb indices and Harary and Schultz indices [32].

In this paper, we use G_1 and G_2 as two simple graphs. The vertex set of the tensor product and Cartesian product of G_1 and G_2 are indicated by $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, respectively. For the tensor product, necessary and sufficient conditions are applied for any $u = (u_1, u_2), v = (v_1, v_2)$ in $V(G_1) \times V(G_2)$ to be joined $(u_1, v_1) \in E(G_1)$ and $(u_2, v_2) \in E(G_2)$. For the Cartesian product, necessary and sufficient conditions for any $u = (u_1, u_2), v = (v_1, v_2)$ in $V(G_1) \times V(G_2)$ to be associated is $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $(u_1, v_1) \in E(G_1)$ and $u_2 = v_2$.

In this study, the S_M^1 monogenic semigroup and S_M^2 monogenic semigroup were used, as given below, respectively,

$$S_M^1 = \left\{x_1, x_1^2, x_1^3, \ldots, x_1^m\right\} \cup \{0\} \quad and \quad S_M^2 = \left\{x_2, x_2^2, x_2^3, \ldots, x_2^n\right\} \cup \{0\}. \tag{2}$$

The vertex set of the tensor product and the Cartesian product of S_M^1 and S_M^2 is given as:

$$\begin{aligned}\Big\{&(x_1, x_2), (x_1^2, x_2), \ldots, (x_1^m, x_2), (x_1, x_2^2), (x_1^2, x_2^2), \ldots, (x_1^m, x_2^2), \ldots, (x_1, x_2^{n-1}),\\ &(x_1^2, x_2^{n-1}), \ldots, (x_1^m, x_2^{n-1}), (x_1, x_2^n), (x_1^2, x_2^n), \ldots, (x_1^m, x_2^n)\Big\}.\end{aligned} \tag{3}$$

The Sombor index invented by Gutman [14] is a vertex degree-based topological index, which is narrowed down as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(d_u)^2 + (d_v)^2}. \tag{4}$$

In addition, for a real number r, we designate by $\lfloor r \rfloor$ the greatest integer $\leq r$ and by $\lceil r \rceil$, the least integer $\geq r$. It is quite apparent that $r - 1 < \lfloor r \rfloor \leq r$ and $r \leq \lceil r \rceil < r + 1$. In addition, for a natural number n, we have

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2} & if \quad n \quad is \quad even \\ \frac{n+1}{2} & if \quad n \quad is \quad odd. \end{cases} \tag{5}$$

Here, any two vertices (x_1^i, x_2^j) and (x_1^a, x_2^b) are connected to each other if and only if

$$x_1^i x_1^a \in E(\Gamma(S_M^1)) \Leftrightarrow x_1^i x_1^a = 0 \Leftrightarrow i + a \geq m + 1 \tag{6}$$

and

$$x_2^j x_2^b \in E(\Gamma(S_M^2)) \Leftrightarrow x_2^j x_2^b = 0 \Leftrightarrow j + b \geq n + 1. \tag{7}$$

Our research calculates results based on the Sombor index of Cartesian and tensor products of monogenic semigroup graphs. Algorithms in Sections 2 and 4 are given for the purpose of detecting vertex neighborhoods. With the help of these algorithms, the properties of the tensor and Cartesian products, respectively, will be used to calculate the Sombor indices on these products more easily. These algorithms are calculated independently of each other and the algorithm given in [23] using tensor and Cartesian product definitions.

2. An Algorithm for the Tensor Product of the Monogenic Semigroup Graph

To make some simplifications in our calculations, we provide our results in Section 3 below; we present this algorithm on the neighborhood of the vertices on $\Gamma(S_M^1)\otimes\Gamma(S_M^2)$ by considering the definition of the monogenic semigroup graph.

If m is even (n is even or odd):

$I_{m,n}$: the vertex (x_1^m, x_2^n) is linked to every vertex (x_1^i, x_2^j) $(1 \leq i \leq m-1, \quad 1 \leq j \leq n-1)$.

$I_{m,n-1}$: the vertex (x_1^m, x_2^{n-1}) is linked to every vertex (x_1^i, x_2^j) $(1 \leq i \leq m-1, \quad 2 \leq j \leq n, \quad j \neq n-1)$.

$I_{m,n-2}$: the vertex $(x_1^m x_2^{n-2})$ is linked to every vertex (x_1^i, x_2^j) $(1 \leq i \leq m-1, 3 \leq j \leq n$ $j \neq n-2)$.

$\vdots$

$I_{m,1}$: the vertex (x_1^m, x_2^1) is linked to every vertex (x_1^i, x_2^j) $(1 \leq i \leq m-1, j = n)$.

$I_{m-1,n}$: the vertex (x_1^{m-1}, x_2^n) is linked to every vertex (x_1^i, x_2^j) $(2 \leq i \leq m-2, 1 \leq j \leq n-1)$.

$\vdots$

$I_{\frac{m}{2}+1,1}$: the vertex $(x_1^{\frac{m}{2}+1}, x_2)$ is linked to $(x_1^{\frac{m}{2}}, x_2^n)$.

By keeping these steps in this algorithm, we have two possibilities, depending on whether n is even or odd:

If m is odd and n is even:

$I_{\frac{m-1}{2}+2,1}$: the vertex $(x_1^{\frac{m-1}{2}+2}, x_2^1)$ is linked to $(x_1^{\frac{m-1}{2}}, x_2^n)$ and $(x_1^{\frac{m-1}{2}+1}, x_2^n)$.

If m is odd and n is odd:

$I_{\frac{m-1}{2}+2,1}$: the vertex $(x_1^{\frac{m-1}{2}+2}, x_2^1)$ is linked to $(x_1^{\frac{m-1}{2}}, x_2^n)$ and $(x_1^{\frac{m-1}{2}+1}, x_2^n)$.

In the following lemma, the vertex degrees are given as:

$$\begin{aligned}&(x_1,x_2),(x_1^2,x_2),\ldots,(x_1^m,x_2),(x_1,x_2^2),(x_1^2,x_2^2),\ldots,(x_1^m,x_2^2),\ldots,(x_1,x_2^{n-1}),\\&(x_1^2,x_2^{n-1}),\ldots,(x_1^m,x_2^{n-1}),(x_1,x_2^n),(x_1^2,x_2^n),\ldots,(x_1^m,x_2^n)\in\Gamma(S_M^1)\otimes\Gamma(S_M^2).\end{aligned} \tag{8}$$

These vertex degrees are denoted by

$$(d_1,d_1'),(d_2,d_1'),\ldots(d_m,d_1'),(d_1,d_2'),(d_2,d_2'),\ldots(d_m,d_2'),\ldots(d_1,d_n'),(d_2,d_n'),\ldots(d_m,d_n'). \tag{9}$$

Several investigations exist on the degree series with respect to this series; we referred to [33,34].

Lemma 1.

$$\begin{aligned}(d_1,d_1') = 1, \quad (d_2,d_1') = 2, \quad \ldots \quad ,(d_{\lceil\frac{m}{2}\rceil},d_1') = \lceil\tfrac{m}{2}\rceil,(d_{\lceil\frac{m}{2}\rceil+1},d_1') = \lceil\tfrac{m}{2}\rceil, \quad \ldots,\\ (d_m,d_1') = m-1\ldots,(d_{\lceil\frac{m}{2}\rceil+1},d'_{\lceil\frac{n}{2}\rceil+1}) = (\lceil\tfrac{m}{2}\rceil)(\lceil\tfrac{n}{2}\rceil).\end{aligned} \tag{10}$$

Remark 1. *Considering special observations in Lemma 1, the recurrent phrases utilized that follows*

$$d_{\lceil\frac{n}{2}\rceil} = \left\lceil\frac{n}{2}\right\rceil = d_{\lceil\frac{n}{2}\rceil+1}. \tag{11}$$

Consequently, the degree of d_n is noted by $n-1$, in spite of the amount of vertices by n.

3. Computing the Sombor Index of the Tensor Products of Monogenic Semigroup Graphs

Our data acquired in this area will produce an accurate formula of the Sombor index over the tensor products of monogenic semigroup graphs by using the above algorithm.

Theorem 1. *For any monogenic semigroup, S_M^1 and S_M^2, the Sombor index over the tensor product of two monogenic semigroup graphs, $\Gamma(S_M^1) \otimes \Gamma(S_M^2)$, is*

$$SO(\Gamma(S_M^1) \otimes \Gamma(S_M^2)) = \begin{cases} \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=1}^{\frac{m}{2}} \sum_{i=1}^{\frac{n}{2}} \sqrt{((t-1)(r-1))^2 + (bi)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=1}^{\frac{m}{2}} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{((t-1)(r-1))^2 + (b(i-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=\frac{m}{2}+1}^{m} \sum_{i=1}^{\frac{n}{2}} \sqrt{((t-1)(r-1))^2 + ((b-1)i)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=\frac{m}{2}+1}^{m} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{((t-1)(r-1))^2 + ((b-1)(i-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=1}^{\frac{n}{2}} \sum_{b=1}^{\frac{m}{2}} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{((t-1)r)^2 + (b(i-1))^2} \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=1}^{\frac{n}{2}} \sum_{b=\frac{m}{2}+1}^{m} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{((t-1)r)^2 + ((b-1)(i-1))^2}. \end{cases} \tag{12}$$

In the formula given above, t, b, and r, m will be taken in accordance with the rules $t + b \geq m + 1$ and $r + i \geq n + 1$.

Proof. Our primary focus was to methodically formulize $SO(\Gamma(S_M^1) \otimes \Gamma(S_M^2))$ concerning the sum of the degrees. The calculation includes the tally of several pieces, thereafter determining each separately. The calculation given in Section 2 is utilized and will determine the structures of the degrees of vertices. Equations (5), (10) and Remark 1 will allow us to evaluate further.

If m and n is even:

$$\begin{aligned}
[SO](\Gamma(S_M^1) \otimes \Gamma(S_M^2)) &= \sqrt{((d_m-1)(d'_n-1))^2 + (d_1 d'_1)^2} + \sqrt{((d_m-1)(d'_n-1))^2 + (d_1 d'_2)^2} \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + (d_1 d'_{\frac{n}{2}})^2} + \sqrt{((d_m-1)(d'_n-1))^2 + (d_1(d'_{\frac{n}{2}+1}-1))^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + (d_1(d'_{n-1}-1))^2} + \sqrt{((d_m-1)(d'_n-1))^2 + (d_2 d'_1)^2} \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + (d_2 d'_2)^2} + \ldots + \sqrt{((d_m-1)(d'_n-1))^2 + (d_2 d'_{\frac{n}{2}})^2} \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + (d_2 d'_{\frac{n}{2}+1}-1)^2} + \ldots + \sqrt{((d_m-1)(d'_n-1))^2 + (d_2(d'_{n-1}-1))^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_1))^2} + \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{\frac{m}{2}+1}-1)d'_2)^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{\frac{n}{2}}))^2} + \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{\frac{n}{2}+1}-1))^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{n-1}-1))^2} + \ldots + \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{m-1}-1)d'_1)^2} \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{m-1}-1)d'_2)^2} + \ldots + \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{m-1}-1)d'_{\frac{n}{2}})^2} \\
&+ \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{m-1}-1)(d'_{\frac{n}{2}+1}-1))^2} + \ldots + \sqrt{((d_m-1)(d'_n-1))^2 + ((d_{m-1}-1)(d'_{n-1}-1))^2} \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_1 d'_2)^2} + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_1 d'_3)^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_1 d'_{\frac{n}{2}})^2} + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_1(d'_{\frac{n}{2}+1}-1))^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_1(d'_{n-2}-1))^2} + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_2 d'_2)^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_2 d'_3)^2} + \ldots + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_2 d'_{\frac{n}{2}})^2} \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + (d_2(d'_{\frac{n}{2}+1}-1))^2} + \ldots \sqrt{(d_m d'_{n-1}-1)^2 + (d_2(d'_{n-2}-1)^2)} + \ldots \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + ((d_{\frac{m}{2}+1}-1)d'_2)^2} + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + ((d_{\frac{m}{2}+1}-1)d'_3)^2} + \ldots \\
&+ \sqrt{((d_m-1)(d'_{n-1}-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{\frac{n}{2}}))^2} + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{\frac{n}{2}+1}-1))^2} \\
&+ \ldots + \sqrt{((d_m-1)(d'_{n-1}-1))^2 + ((d_{\frac{m}{2}+1}-1)(d'_{n-2}-1))^2} + \ldots + \sqrt{((d_{\frac{m}{2}+1}-1)d'_1)^2 + (d_{\frac{m}{2}} d'_n)^2}.
\end{aligned} \tag{13}$$

Consequently the Sombor index $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))$ is noted as the sum following

$$\begin{aligned}[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2)) &= [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n)} + [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n-1)} + \dots \\ &+ [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,1)} + [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,n)} \\ &+ [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,n-1)} + \dots + [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,1)} \\ &+ [SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(\frac{m}{2}+1,1)}.\end{aligned} \tag{14}$$

Whilst estimating the Sombor index value, the minutest amount is acquired after several calculations. Where n is odd, we utilize the equality $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ given in (5). Then we obtain

$$\begin{aligned}[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n)} &= \sqrt{((m-1)(n-1))^2+(1.1)^2} + \sqrt{((m-1)(n-1))^2+(1.2)^2} + \dots \\ &+ \sqrt{((m-1)(n-1))^2+(1.(n-1))^2} + \sqrt{((m-1)(n-1))^2+(2.1)^2} \\ &+ \sqrt{((m-1)(n-1))^2+(2.2)^2} + \dots + \sqrt{((m-1)(n-1))^2+(2.(n-1))^2} + \dots \\ &+ \sqrt{((m-1)(n-1))^2+((\frac{m}{2}+1-1).1)^2} + \sqrt{((m-1)(n-1))^2+((\frac{m}{2}+1-1).2)^2} + \dots \\ &+ \sqrt{((m-1)(n-1))^2+((\frac{m}{2}+1-1).\frac{n}{2})^2} + \sqrt{((m-1)(n-1))^2+((\frac{m}{2}+1-1).(\frac{n}{2}+1-1))^2} + \dots \\ &+ \sqrt{((m-1)(n-1))^2+(((\frac{m}{2}+1-1).(n-1-1))^2} + \dots + \sqrt{((m-1)(n-1))^2+((m-1-1).1)^2} \\ &+ \sqrt{((m-1)(n-1))^2+((m-1-1).2)^2} + \dots + \sqrt{((m-1)(n-1))^2+((m-1-1).\frac{n}{2})^2} \\ &+ \sqrt{((m-1)(n-1))^2+((m-1-1).(\frac{n}{2}+1-1))^2} + \dots \\ &+ \sqrt{((m-1)(n-1))^2+((m-1-1).(n-1-1))^2}.\end{aligned} \tag{15}$$

The above equation can be given briefly with the sum symbol as follows

$$\begin{aligned}[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n)} &= \sum_{b=1}^{\frac{m}{2}}\sum_{i=1}^{\frac{n}{2}} \sqrt{((m-1)(n-1))^2+(bi)^2} \\ &+ \sum_{b=1}^{\frac{m}{2}}\sum_{i=\frac{n}{2}+1}^{n-1} \sqrt{((m-1)(n-1))^2+(b(i-1))^2} \\ &+ \sum_{b=\frac{m}{2}+1}^{m-1}\sum_{i=1}^{\frac{n}{2}} \sqrt{((m-1)(n-1))^2+((b-1)i)^2} \\ &+ \sum_{b=\frac{m}{2}+1}^{m-1}\sum_{i=\frac{n}{2}+1}^{n-1} \sqrt{((m-1)(n-1))^2+((b-1)(i-1))^2}.\end{aligned} \tag{16}$$

If similar operations in $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n)}$ are applied to $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n-1)}$, we obtain

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,n-1)} &= \sum_{b=1}^{\frac{m}{2}}\sum_{i=2}^{\frac{n}{2}}\sqrt{((m-1)((n-1)-1))^2+(bi)^2} \\
&+\sum_{b=1}^{\frac{m}{2}}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((m-1)(n-1))^2+(b(i-1))^2} \qquad (i\neq n-1) \\
&+\sum_{b=\frac{m}{2}+1}^{m-1}\sum_{i=2}^{\frac{n}{2}}\sqrt{((m-1)(n-1))^2+((b-1)i)^2} \\
&+\sum_{b=\frac{m}{2}+1}^{m-1}\sum_{i=\frac{n}{2}+1}^{n-1}\sqrt{((m-1)(n-1))^2+((b-1)(i-1))^2}. \qquad (i\neq n-1)
\end{aligned} \tag{17}
$$

If it continues in this way, the following equalities will be obtained for $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,\frac{n}{2}+1)}$, $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,\frac{n}{2})}$, and $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,1)}$, respectively,

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,\frac{n}{2})} &= \sum_{b=1}^{\frac{m}{2}}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((m-1)(\frac{n}{2}))^2+(b(i-1))^2} \\
&+\sum_{b=\frac{m}{2}+1}^{m-1}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((m-1)(\frac{n}{2}))^2+((b-1)(i-1))^2}
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m,1)} &= \sum_{b=1}^{\frac{m}{2}}\sqrt{((m-1)1)^2+(b(n-1))^2} \\
&+\sum_{b=\frac{m}{2}+1}^{m-1}\sqrt{((m-1)1)^2+((b-1)(n-1))^2}.
\end{aligned} \tag{19}
$$

In this way, $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,n)}$, $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,n-1)},\ldots,$ $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(m-1,1)}$, $\ldots,$ $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(\frac{m}{2}+1,n)}$, $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(\frac{m}{2}+1,n-1)}$, $\ldots,$ $[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2))_{(\frac{m}{2}+1,1)}$ are calculated one-by-one to obtain a general sum formula as given below:

$$
SO(\Gamma(S_M^1)\otimes\Gamma(S_M^2)) = \begin{cases}
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=\frac{n}{2}+1}^{n}\sum_{b=1}^{\frac{m}{2}}\sum_{i=1}^{\frac{n}{2}}\sqrt{((t-1)(r-1))^2+(bi)^2}+ \\
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=\frac{n}{2}+1}^{n}\sum_{b=1}^{\frac{m}{2}}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((t-1)(r-1))^2+(b(i-1))^2}+ \\
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=\frac{n}{2}+1}^{n}\sum_{b=\frac{m}{2}+1}^{m}\sum_{i=1}^{\frac{n}{2}}\sqrt{((t-1)(r-1))^2+((b-1)i)^2}+ \\
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=\frac{n}{2}+1}^{n}\sum_{b=\frac{m}{2}+1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((t-1)(r-1))^2+((b-1)(i-1))^2}+ \\
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=1}^{\frac{n}{2}}\sum_{b=1}^{\frac{m}{2}}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((t-1)r)^2+(b(i-1))^2}+ \\
\sum_{t=\frac{m}{2}+1}^{m}\sum_{r=1}^{\frac{n}{2}}\sum_{b=\frac{m}{2}+1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\sqrt{((t-1)r)^2+((b-1)(i-1))^2}.
\end{cases} \tag{20}
$$

□

4. An Algorithm for the Cartesian Product of the Monogenic Semigroup Graph

We introduced this algorithm to the neighborhood of the vertices on $\Gamma(S_M^1)\times\Gamma(S_M^2)$, taking into consideration the details of the monogenic semigroup graphs. Our main results will be presented in Section 5 with our calculations.

If m is even (n is even or odd):

$I_{m,n}$: the vertex (x_1^m, x_2^n) is linked to (x_1^i, x_2^n) and (x_1^m, x_2^j) $(1\leq i\leq m-1, \quad 1\leq j\leq n-1)$.

$I_{m,n-1}$: the vertex (x_1^m, x_2^{n-1}) is linked to (x_1^i, x_2^{n-1}) and (x_1^m, x_2^j) $(1 \le i \le m-1,\quad 2 \le j \le n-2)$.

$I_{m,n-2}$: the vertex $(x_1^m x_2^{n-2})$ is linked to (x_1^i, x_2^{n-2}) and (x_1^m, x_2^j) $(1 \le i \le m-1,\quad 3 \le j \le n-3)$.

$\vdots$

$I_{m,1}$: the vertex (x_1^m, x_2^1) is linked to (x_1^i, x_2^1) $(1 \le i \le m-1)$.

$I_{m-1,n}$: the vertex (x_1^{m-1}, x_2^n) is linked to (x_1^i, x_2^n) and (x_1^{m-1}, x_2^j) $(2 \le i \le m-2, 1 \le j \le n-1)$.

$\vdots$

$I_{\frac{m}{2}+1,1}$: The vertex $(x_1^{\frac{m}{2}+1}, x_2^1)$ is linked to $(x_1^{\frac{m}{2}}, x_2^1)$.

By continuing these steps, if m is odd, the following situation will occur depending on whether n is even or odd.

If m is odd (n is even or odd):

$I_{\frac{m-1}{2}+2,1}$: the vertex $(x_1^{\frac{m-1}{2}+2}, x_2^1)$ is linked to (x_1^i, x_2^1) $(\frac{m-1}{2} \le i \le \frac{m-1}{2}+1)$.

5. Computing the Sombor Index of the Cartesian Products of Monogenic Semigroup Graphs

In this section, by using the above algorithm, the formula of the Sombor index of the Cartesian products of the monogenic semigroup graphs will be given.

Theorem 2. *For any monogenic semigroup, S_M^1 and S_M^2, the Sombor index over the Cartesian products of two monogenic semigroup graphs $\Gamma(S_M^1) \times \Gamma(S_M^2)$ is*

$$SO(\Gamma(S_M^1) \times \Gamma(S_M^2)) = \begin{cases} \sum_{t=\frac{m}{2}+1}^{m} \sum_{b=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sqrt{((t-1)+(r-1))^2+(b+(r-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{b=1}^{\frac{m}{2}} \sum_{r=1}^{\frac{n}{2}} \sqrt{((t-1)+r)^2+(b+r)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=\frac{m}{2}+1}^{m} \sqrt{((t-1)+(r-1))^2+((b-1)+(r-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=1}^{\frac{n}{2}} \sqrt{((t-1)+(r-1))^2+((t-1)+i)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{i=\frac{n}{2}+1}^{n} \sum_{r=\frac{n}{2}+1}^{n} \sqrt{((t-1)+(r-1))^2+((t-1)+(i-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=1}^{\frac{n}{2}} \sum_{b=\frac{m}{2}+1}^{m-1} \sqrt{((t-1)+r)^2+((b-1)+r)^2} + \\ \sum_{t=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=1}^{\frac{n}{2}} \sqrt{(t+(r-1))^2+(t+i)^2} + \\ \sum_{t=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{(t+(r-1))^2+(t+(i-1))^2}. \end{cases} \tag{21}$$

In the formula given above, t, b, and r, m will be taken in accordance with the rules $t + b \ge m+1$, $t \succ b$ and $r + i \ge n+1$, $r \succ i$.

Proof. Since our primary focus is to formulize $SO(\Gamma(S_M^1) \times \Gamma(S_M^2))$ concerning the total number of degrees, we need to treat the sum as the sum of the total amount of different blocks, thereafter determined individually. The calculation given in Section 2 is utilized and will determine the structures of the degrees of vertices with the addition of Equations (5), (10), and Remark 1.

If m and n are even:

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\times\Gamma(S_M^2)) = {} & \sqrt{((d_m-1)+(d'_n-1))^2+(d_1+d'_n-1)^2}+\sqrt{((d_m-1)+(d'_n-1))^2+(d_2+(d'_n-1))^2}+\dots \\
& +\sqrt{((d_m-1)+(d'_n-1))^2+(d_{\frac{m}{2}}+(d'_n-1))^2}+\sqrt{((d_m-1)+(d'_n-1))^2+((d_{\frac{m}{2}+1}-1)+(d'_n-1))^2}+\dots \\
& +\sqrt{((d_m-1)+(d'_n-1))^2+((d_{m-1}-1)+(d'_n-1))^2}+\sqrt{((d_m-1)+(d'_n-1))^2+((d_m-1)+d'_1)^2} \\
& +\sqrt{((d_m-1)+(d'_n-1))^2+((d_m-1)+d'_{\frac{n}{2}})^2}+\sqrt{((d_m-1)+(d'_n-1))^2+((d_m-1)+d'_{\frac{n}{2}+1}-1)^2}\dots \\
& +\sqrt{((d_m-1)(d'_n-1))^2+((d_m-1)+(d'_{n-1}-1))^2}+\sqrt{((d_m-1)+(d'_{n-1}-1))^2+(d_1+(d'_{n-1}-1)^2} \\
& +\sqrt{((d_m-1)+(d'_{n-1}-1))^2+(d_2+(d'_{n-1}-1)^2}+\dots+\sqrt{((d_m-1)+(d'_{n-1}-1))^2+(d_{\frac{m}{2}}+(d'_{n-1}-1))^2} \\
& +\sqrt{((d_m-1)(d'_{n-1}-1))^2+((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2}\dots \\
& +\sqrt{((d_m-1)+(d'_{n-1}-1))^2+((d_{m-1}-1)+(d'_{n-1}-1))^2}+\sqrt{((d_m-1)+(d'_{n-1}-1))^2+((d_m-1)+d'_2)^2} \\
& +\dots+\sqrt{((d_m-1)+(d'_{n-1}-1))^2+((d_m-1)+(d'_{n-2}-1))^2}+\dots \\
& +\sqrt{((d_m-1)+d'_{\frac{n}{2}})^2+(d_1+d'_{\frac{n}{2}})^2}+\sqrt{((d_m-1)+d'_{\frac{n}{2}})^2+(d_2+d'_{\frac{n}{2}})^2}+\dots \\
& +\sqrt{((d_m-1)+d'_{\frac{n}{2}})^2+((d_{m-1}-1)(d'_{n-1}-1))^2}+\sqrt{((d_m-1)+(d'_n-1))^2+((d_{m-1}-1)+d'_1)^2} \\
& +\sqrt{((d_m-1)+(d'_n-1))^2+((d_{m-1}-1)+d'_2)^2}+\dots \\
& +\sqrt{((d_{m-1}-1)+(d'_n-1))^2+((d_{m-1}-1)+(d'_{n-1}-1))^2} \\
& +\sqrt{((d_{m-1}-1)+(d'_{n-1}-1))^2+(d_2+(d'_{n-1}-1))^2}+\sqrt{((d_{m-1}-1)+(d'_{n-1}-1))^2+(d_3+(d'_{n-1}-1))^2}+\dots \\
& +\sqrt{((d_{m-1}-1)+(d'_{n-1}-1))^2+((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2}+\dots \\
& +\sqrt{((d_{m-1}-1)+(d'_{n-1}-1))^2+((d_{m-1}-1)+d'_2)^2}+\dots \\
& +\sqrt{((d_{m-1}-1)+(d'_{n-1}-1))^2+((d_{m-1}-1)+d'_{\frac{n}{2}})^2} \\
& +\dots+\sqrt{((d_{m-1}-1)+d'_{\frac{n}{2}})^2+(d_2+d'_{\frac{n}{2}})^2}+\sqrt{((d_{m-1}-1)+d'_{\frac{n}{2}})^2+(d_3+d'_{\frac{n}{2}})^2} \\
& +\dots+\sqrt{((d_{m-1}-1)+d'_{\frac{n}{2}})^2+((d_{m-2}-1)+d'_{\frac{n}{2}})^2}+\sqrt{((d_{m-1}-1)+d'_1)^2+(d_2+d'_1)^2} \\
& +\sqrt{((d_{m-1}-1)+d'_1)^2+(d_3+d'_1)^2}+\dots+\sqrt{((d_{m-1}-1)+d'_1)^2+((d_{m-2}-1)+d'_1)^2} \\
& +\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_n-1))^2+(d_{\frac{m}{2}}+(d'_n-1))^2}+\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_n-1))^2+((d_{\frac{m}{2}+1}-1)+d'_1)^2} \\
& +\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_n-1))^2+((d_{\frac{m}{2}+1}-1)+d'_2)^2}+\dots \\
& +\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_n-1))^2+((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2} \\
& +\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2+(d_{\frac{m}{2}}+(d'_{n-1}-1))^2}+\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2+((d_{\frac{m}{2}+1}-1)+d'_2)^2+} \\
& +\dots+\sqrt{((d_{\frac{m}{2}+1}-1)+(d'_{n-1}-1))^2+((d_{\frac{m}{2}+1}-1)+(d'_{n-2}-1))^2} \\
& +\sqrt{((d_{\frac{m}{2}+1}-1)+d'_{\frac{n}{2}})^2+(d_{\frac{m}{2}}+d'_{\frac{n}{2}})^2}+\dots+\sqrt{((d_{\frac{m}{2}+1}-1)+d'_1)^2+(d_{\frac{m}{2}}+d'_1)^2} \\
& +\sqrt{(d_{\frac{m}{2}}+(d'_n-1))^2+(d_{\frac{m}{2}}+d'_1)^2}+\dots+\sqrt{(d_{\frac{m}{2}}+(d'_n-1))^2+(d_{\frac{m}{2}}+d'_{\frac{n}{2}})^2}+\dots \\
& +\sqrt{(d_{\frac{m}{2}}+(d'_n-1))^2+(d_{\frac{m}{2}}+(d'_{n-1}-1))^2}+\sqrt{(d_{\frac{m}{2}}+(d'_{n-1}-1))^2+(d_{\frac{m}{2}}+d'_2)^2}+\dots \\
& +\sqrt{(d_{\frac{m}{2}}+d'_{n-1}-1)^2+(d_{\frac{m}{2}}+(d'_{n-2}-1))^2}+\sqrt{(d_2+(d'_n-1))^2+(d_2+d'_1)^2} \\
& +\sqrt{(d_2+(d'_n-1))^2+(d_2+d'_2)^2}+\dots+\sqrt{(d_2+(d'_n-1))^2+(d_2+(d'_{n-1}-1))^2} \\
& +\sqrt{(d_2+(d'_{n-1}-1))^2+(d_2+(d'_{n-2}-1))^2}+\dots+\sqrt{(d_2+(d'_{n-1}-1))^2+(d_2+d'_{\frac{n}{2}})^2} \\
& +\sqrt{(d_1+(d'_n-1))^2+(d_1+d'_1)^2}+\sqrt{(d_1+(d'_n-1))^2+(d_1+d'_2)^2}+\dots \\
& +\sqrt{(d_1+(d'_n-1))^2+(d_1+(d'_{n-1}-1))^2}+\sqrt{(d_1+(d'_{n-1}-1))^2+(d_1+d'_2)^2} \\
& +\dots+\sqrt{(d_1+(d'_{n-1}-1))^2+(d_1+(d'_{n-2}-1))^2}.
\end{aligned}
\tag{22}
$$

Consequently, the Sombor index of $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))$ is stated as the sum below

$$\begin{aligned}[SO](\Gamma(S^1_M) \times \Gamma(S^2_M)) = {} & [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n)} + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n-1)} + \ldots + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,1)} \\ & + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m-1,n)} + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m-1,n-1)} + \ldots + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m-1,1)} \\ & + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(1,n)} + \ldots + [SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(1,\frac{n}{2}+1)}.\end{aligned} \tag{23}$$

Whilst estimating the Sombor index value, the minutest amount is acquired after several calculations. Where n is odd, we utilize the equality $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ given in (5). Then we have

$$\begin{aligned}[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n)} = {} & \sqrt{((m-1)+(n-1))^2 + (1+(n-1))^2} + \sqrt{((m-1)+(n-1))^2 + ((2-2)+(n-1))^2} + \ldots \\ & + \sqrt{((m-1)+(n-1))^2 + (\frac{m}{2}+(n-1))^2} + \sqrt{((m-1)+(n-1))^2 + ((\frac{m}{2}+1-1)+(n-1))^2} + \ldots \\ & + \sqrt{((m-1)+(n-1))^2 + (((m-1)-1)+(n-1))^2} + \sqrt{((m-1)+(n-1))^2 + ((m-1)+1)^2} \\ & + \sqrt{((m-1)+(n-1))^2 + ((m-1)+\frac{n}{2})^2} + \sqrt{((m-1)+(n-1))^2 + ((m-1)+(\frac{n}{2}+1-1))^2}.\end{aligned} \tag{24}$$

The above equation can be given briefly with the sum as follows

$$\begin{aligned}[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n)} = {} & \sum_{b=1}^{\frac{m}{2}} \sqrt{((m-1)+(n-1))^2 + (b+(n-1))^2} \\ & + \sum_{b=\frac{m}{2}+1}^{m-1} \sqrt{((m-1)+(n-1))^2 + ((b-1)+(n-1))^2} \\ & + \sum_{i=1}^{\frac{n}{2}} \sqrt{((m-1)+(n-1))^2 + ((m-1)+i)^2} \\ & + \sum_{i=\frac{n}{2}+1}^{n-1} \sqrt{((m-1)+(n-1))^2 + ((m-1)+(i-1))^2}.\end{aligned} \tag{25}$$

If similar operations applied in $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n)}$ are applied to $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n-1)}$, we obtain

$$\begin{aligned}[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,n-1)} = {} & \sum_{b=1}^{\frac{m}{2}} \sqrt{((m-1)+((n-1)-1))^2 + (b+((n-1)-1))^2} \\ & + \sum_{b=\frac{m}{2}+1}^{m-1} \sqrt{((m-1)+((n-1)-2))^2 + ((b-1)+((n-1)-1))^2} \\ & + \sum_{b=\frac{m}{2}+1}^{m-1} \sum_{i=2}^{\frac{n}{2}} \sqrt{((m-1)+(n-1))^2 + ((b-1)+i)^2} \\ & + \sum_{i=2}^{\frac{n}{2}} \sqrt{((m-1)+((n-1)-1))^2 + ((m-1)+i)^2} \\ & + \sum_{i=\frac{n}{2}+1}^{n-2} \sqrt{((m-1)+((n-1)-1))^2 + ((m-1)+(i-1))^2}.\end{aligned} \tag{26}$$

If it is continued in this way, the following equalities are obtained for $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,\frac{n}{2}+1)}$, $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,\frac{n}{2})}$ and $[SO](\Gamma(S^1_M) \times \Gamma(S^2_M))_{(m,1)}$, respectively,

$$[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(m,\frac{n}{2})} = \sqrt{((m-1) + (\frac{n}{2} - \frac{n}{2}))^2 + ((b-b) + (\frac{n}{2} - \frac{n}{2}))^2} \quad (27)$$

and

$$\begin{aligned}[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(m,1)} &= \sum_{b=1}^{\frac{m}{2}} \sqrt{((m-1)+1)^2 + (b+1)^2} \\ &+ \sum_{b=\frac{m}{2}+1}^{m-1} \sqrt{((m-1)+1)^2 + ((b-1)+(n-1))^2}.\end{aligned} \quad (28)$$

In this way, $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(m-1,n)}$, $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(m-1,n-1)}, \ldots,$ $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(m-1,1)}$, $\ldots$, $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(\frac{m}{2}+1,n)}$, $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(\frac{m}{2}+1,n-1)}$, $\ldots$, $[SO](\Gamma(S_M^1) \times \Gamma(S_M^2))_{(\frac{m}{2}+1,1)}$ are calculated one-by-one to obtain a general sum formula as given below:

$$SO(\Gamma(S_M^1) \times \Gamma(S_M^2)) = \begin{cases} \sum_{t=\frac{m}{2}+1}^{m} \sum_{b=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sqrt{((t-1)+(r-1))^2 + (b+(r-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{b=1}^{\frac{m}{2}} \sum_{r=1}^{\frac{n}{2}} \sqrt{((t-1)+r)^2 + (b+r)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{b=\frac{m}{2}+1}^{m} \sqrt{((t-1)+(r-1))^2 + ((b-1)+(r-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=1}^{\frac{n}{2}} \sqrt{((t-1)+(r-1))^2 + ((t-1)+i)^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{i=\frac{n}{2}+1}^{n} \sum_{r=\frac{n}{2}+1}^{n} \sqrt{((t-1)+(r-1))^2 + ((t-1)+(i-1))^2} + \\ \sum_{t=\frac{m}{2}+1}^{m} \sum_{r=1}^{\frac{n}{2}} \sum_{b=\frac{m}{2}+1}^{m-1} \sqrt{((t-1)+r)^2 + ((b-1)+r)^2} \\ \sum_{t=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=1}^{\frac{n}{2}} \sqrt{(t+(r-1))^2 + (t+i)^2} \\ \sum_{t=1}^{\frac{m}{2}} \sum_{r=\frac{n}{2}+1}^{n} \sum_{i=\frac{n}{2}+1}^{n} \sqrt{(t+(r-1))^2 + (t+(i-1))^2}. \end{cases} \quad (29)$$

□

The examples given below show the calculation of the Sombor index of the tensor product and Cartesian product of $\Gamma(S_{M_6})$ and $\Gamma(S_{M_4})$, to support the main theorems.

Example 1. *Let S_{M_6} and S_{M_4} be a monogenic semigroup that is given in the following:*

$$S_{M_6} = \left\{x, x^2, x^3, x^4, x^5, x^6\right\} \cup \{0\}, \qquad S_{M_4} = \left\{x, x^2, x^3, x^4\right\} \cup \{0\}. \quad (30)$$

Now we will calculate the Sombor index of the $\Gamma(S_{M_6}) \otimes \Gamma(S_{M_4})$ graph by using the technique given in Theorem 1.

By using the formula, which is given in the following

$$SO(\Gamma(S_M^1) \otimes \Gamma(S_M^2)) = \begin{cases} \sum_{t=4}^{6} \sum_{r=3}^{4} \sum_{b=1}^{3} \sum_{i=1}^{2} \sqrt{((t-1)(r-1))^2 + (bi)^2} + \\ \sum_{t=4}^{6} \sum_{r=3}^{4} \sum_{b=1}^{3} \sum_{i=3}^{4} \sqrt{((t-1)(r-1))^2 + (b(i-1))^2} + \\ \sum_{t=4}^{6} \sum_{r=3}^{4} \sum_{b=4}^{6} \sum_{i=1}^{2} \sqrt{((t-1)(r-1))^2 + ((b-1)i)^2} + \\ \sum_{t=4}^{6} \sum_{r=3}^{4} \sum_{b=4}^{6} \sum_{i=3}^{4} \sqrt{((t-1)(r-1))^2 + ((b-1)(i-1))^2} + \\ \sum_{t=4}^{6} \sum_{r=1}^{2} \sum_{b=1}^{3} \sum_{i=3}^{4} \sqrt{((t-1)r)^2 + (b(i-1))^2} + \\ \sum_{t=4}^{6} \sum_{r=1}^{2} \sum_{b=4}^{6} \sum_{i=3}^{4} \sqrt{((t-1)r)^2 + ((b-1)(i-1))^2}, \end{cases} \quad (31)$$

we obtain

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\otimes\Gamma(S_M^2)) = {} & \sqrt{4^2+6^2}+2\sqrt{4^2+9^2}+\sqrt{5^2+3^2}+\sqrt{5^2+6^2}+2\sqrt{5^2+9^2}+2\sqrt{5^2+9^2} \\
& +2\sqrt{5^2+12^2}+2\sqrt{6^2+6^2}+2\sqrt{8^2+4^2}+6\sqrt{8^2+6^2}+5\sqrt{8^2+9^2}+2\sqrt{9^2+3^2} \\
& +6\sqrt{9^2+6^2}+2\sqrt{10^2+2^2}+2\sqrt{10^2+3^2}+2\sqrt{10^2+4^2}+6\sqrt{10^2+6^2}+2\sqrt{10^2+8^2} \\
& +4\sqrt{10^2+9^2}+\sqrt{10^2+12^2}+\sqrt{12^2+2^2}+2\sqrt{12^2+3^2}+2\sqrt{12^2+4^2}+4\sqrt{12^2+6^2} \\
& +\sqrt{15^2+1^2}+3\sqrt{15^2+2^2}+\sqrt{15^2+3^2}+3\sqrt{15^2+4^2}+3\sqrt{15^2+6^2}+2\sqrt{15^2+8^2}.
\end{aligned}
\tag{32}
$$

Example 2. *The Cartesian products of the graphs $\Gamma(S_M^1)$ and $\Gamma(S_M^2)$ are given below. By using the formula given in Theorem 2, we will calculate the Sombor index of $(\Gamma(S_M^1)\times\Gamma(S_M^2)$.*

By utilizing the formula given as

$$
SO(\Gamma(S_M^1)\times\Gamma(S_M^2)) = \begin{cases}
\sum_{t=4}^{6}\sum_{b=1}^{3}\sum_{r=3}^{4}\sqrt{((t-1)+(r-1))^2+(b+(r-1))^2}+ \\
\sum_{t=4}^{6}\sum_{b=1}^{3}\sum_{r=1}^{2}\sqrt{((t-1)+r)^2+(b+r)^2}+ \\
\sum_{t=4}^{6}\sum_{r=3}^{4}\sum_{b=4}^{6}\sqrt{((t-1)+(r-1))^2+((b-1)+(r-1))^2}+ \\
\sum_{t=4}^{6}\sum_{r=3}^{4}\sum_{i=1}^{2}\sqrt{((t-1)+(r-1))^2+((t-1)+i)^2}+ \\
\sum_{t=4}^{6}\sum_{i=3}^{4}\sum_{r=3}^{4}\sqrt{((t-1)+(r-1))^2+((t-1)+(i-1))^2}+ \\
\sum_{t=4}^{6}\sum_{r=1}^{2}\sum_{b=4}^{5}\sqrt{((t-1)+r)^2+((b-1)+r)^2} \\
\sum_{t=1}^{3}\sum_{r=3}^{4}\sum_{i=1}^{2}\sqrt{(t+(r-1))^2+(t+i)^2} \\
\sum_{t=1}^{3}\sum_{r=3}^{4}\sum_{i=3}^{4}\sqrt{(t+(r-1))^2+(t+(i-1))^2},
\end{cases}
\tag{33}
$$

we have

$$
\begin{aligned}
[SO](\Gamma(S_M^1)\times\Gamma(S_M^2)) = {} & \sqrt{3^2+3^2}+\sqrt{4^2+2^2}+2\sqrt{4^2+3^2}+2\sqrt{4^2+4^2}+2\sqrt{5^2+3^2}+4\sqrt{5^2+4^2}+4\sqrt{5^2+5^2} \\
& +\sqrt{6^2+2^2}+2\sqrt{6^2+3^2}+6\sqrt{6^2+4^2}+9\sqrt{6^2+5^2}+2\sqrt{7^2+3^2}+2\sqrt{7^2+4^2}+6\sqrt{7^2+5^2} \\
& +6\sqrt{7^2+6^2}+\sqrt{7^2+7^2}+\sqrt{8^2+4^2}+\sqrt{8^2+5^2}+3\sqrt{8^2+6^2}+3\sqrt{8^2+7^2}.
\end{aligned}
\tag{34}
$$

As we understand, the Sombor index of a graph of tensor and Cartesian products, of two monogenic semigroups, are easily found by considering the exact formula obtained in the main theorems (Figures 1–4).

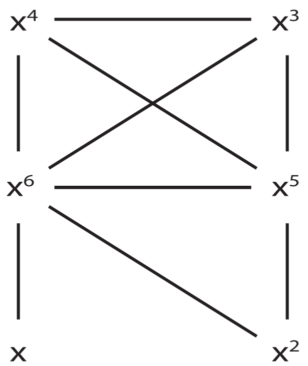

Figure 1. S_{M_6} monogenic semigroup graph.

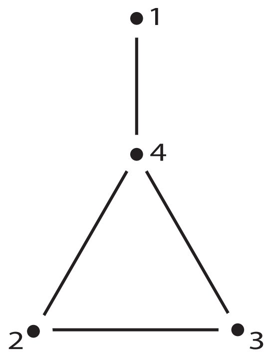

Figure 2. S_{M_4} monogenic semigroup graph.

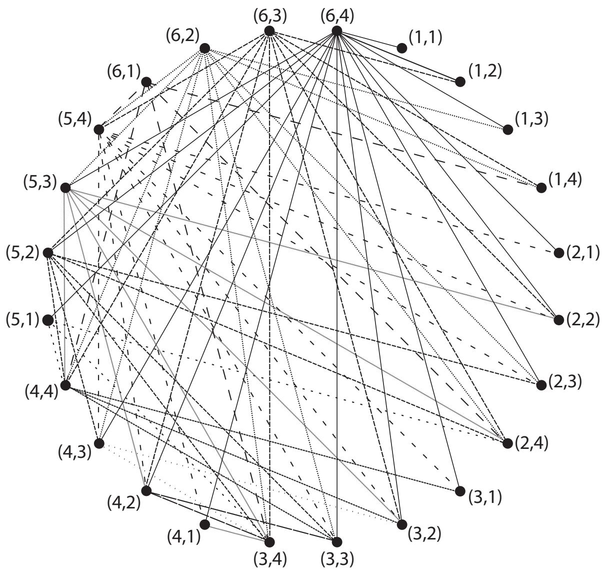

Figure 3. Tensor products of S_{M_6} and S_{M_4} monogenic semigroup graphs.

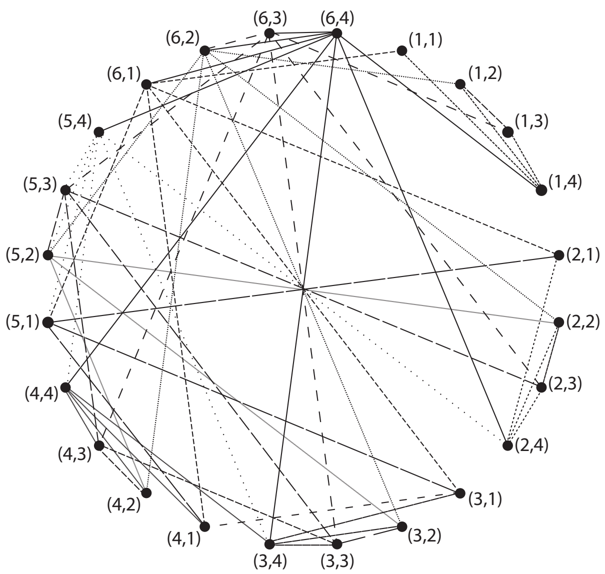

Figure 4. Cartesian products of S_{M_6} and S_{M_4}, monogenic semigroup graphs.

6. Conclusions

In the mathematical and chemical literature, there are so-called vertex-degree-based topological indices. One of the newest vertex-degree-based indices is the Sombor index; it soon became very popular, with around 100 published papers. The Sombor index differs from earlier vertex-degree-based topological indices because it has a peculiar geometric interpretation. Extending the theory of the Sombor index beyond classical combinatorics and linear algebra is of great importance, as it confirms that the Sombor index concept is not just one of the numerous vertex-degree-based graph invariants, but it has a deeper geometry-related meaning (as shown in [14]). Numerous research papers on the Sombor index and its variants have applied methods involving standard combinatorial optimization or linear algebra. In this work and in [23] we used the Sombor index in the semigroup theory as an exception. We connected the Sombor index with abstract algebra. In particular, we computed this index over the tensor and Cartesian products of a monogenic semigroup graph; the obtained results are supported by examples. To better comprehend the Sombor index, see [14]. Calculating the Sombor index (of disjunctive and corona products) over a graph of a monogenic semigroup remains an open problem.

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References

1. West, D.B. *An Introduction to Graph Theory*; Prentice-Hall: Upper Saddle River, NJ, USA, 1996.
2. Das, K.C.; Çevik, A.S.; Cangül, I.N.; Shang, Y. On Sombor index. *Symmetry* **2021** *13*, 140. [CrossRef]
3. Akgüneş, N.; Das, K.C.; Çevik, A.S. *Topological Indices on a Graph of Monogenic Semigroups*; Gutman, I., Ed.; Topics in Chemical Graph Theory, Mathematical Chemistry Monographs; University of Kragujevac and Faculty of Science Kragujevac: Kragujevac, Serbia, 2014.
4. Akgüneş, N.; Çağan, B. On the dot product of graphs over monogenic semigroups. *Appl. Math. Comput.* **2018**, *322*, 1–5. [CrossRef]
5. Akgüneş, N. A further note on the graph of monogenic semigroups. *Konuralp J. Math.* **2018**, *6*, 49–53.
6. Beck, I. Coloring of Commutating Rings. *J. Algebra* **1988**, *116*, 208–226. [CrossRef]
7. Anderson, D.D.; Naseer, M. Beck's coloring of a commutative ring. *J. Algebra* **1991**, *159*, 500–514. [CrossRef]
8. Anderson, D.F.; Livingston, P. The Zero-divisor Graph of Commutative Ring. *J. Algebra* **1999**, *217*, 434–447. [CrossRef]
9. Anderson, D.F.; Badawi, A. On the Zero-Divisor Graph of a Ring. *Commun. Algebra* **2008**, *36*, 3073–3092. [CrossRef]
10. DeMeyer, F.R.; DeMeyer, L. Zero-Divisor Graphs of Semigroups. *J. Algebra* **2005**, *283*, 190–198. [CrossRef]
11. DeMeyer, F.R.; McKenzie, T.; Schneider, K. The Zero-Divisor Graph of a Commutative Semigroup. *Semigroup Forum* **2002**, *65*, 206–214. [CrossRef]
12. Todeschini, R.; Consonni, V. *Molecular Descriptors for Chemoinformatics*; Wiley VCH: Weinheim, Germany, 2009.
13. Devillers, J.; Balaban, A.T. *Topological Indices and Related Descriptors in QSAR and QSPR*; Gordon and Breach: Amsterdam, NY, USA , 1999.
14. Gutman, I. Geometric approach to degree-based topological indices: Sombor indices. *MATCH Commun. Math. Comput. Chem.* **2021**, *86*, 11–16.
15. Alikhani, S.; Ghanbari, N. Sombor index of polymers. *MATCH Commun. Math. Comput. Chem.* **2021**, *86*, 715–728.
16. Amin, S.; Rehman Virk, A.U.; Rehman, M.A.; Shah, N.A. Analysis of dendrimer generation by Sombor indices. *J. Chem.* **2021**, *2021*, 9930645. [CrossRef]
17. Cruz, R.; Gutman, I.; Rada, J. Sombor index of chemical graphs. *Appl. Math. Comput.* **2021**, *399*, 126018. [CrossRef]
18. Liu, H.; You, L.; Huang, Y. Ordering chemical graphs by Sombor indices and its applications. *MATCH Commun. Math. Comput. Chem.* **2022**, *87*, 5–22. [CrossRef]
19. Liu, H.; Gutman, I.; You, L.; Huang, Y. Sombor index: Review of extremal results and bounds. *J. Math. Chem.* **2022**, *66*, 771–798. [CrossRef]
20. Redžepović, I. Chemical applicability of Sombor indices. *J. Serb. Chem. Soc.* **2021**, *86*, 445–457. [CrossRef]
21. Horoldagva, B.; Xu, C. On Sombor index of graphs. *MATCH Commun. Math. Comput. Chem.* **2021**, *86*, 713–793.
22. Milovanovic, I.; Milovanovic, E.; Ali, A.; Matejic, M. Some results on the Sombor indices of graphs. *Contrib. Math.* **2021**, *3*, 59–67.
23. Oğuz Ünal, S. An application of Sombor index over a special class of semigroup graph. *J. Math.* **2021**, *2021*, 3273117. [CrossRef]

24. Rada, J.; Rodríguez; J. M.; Sigarreta, J. M. General properties on Sombor indices. *Discrete Appl. Math.* **2021**, *299*, 87–97. [CrossRef]
25. Shang, Y. Sombor index and degree-related properties of simplicial networks. *Appl. Math. Comput.* **2022**, *419*, 126881. [CrossRef]
26. Gutman, I. Spectrum and energy of the Sombor matrix. *Mil. Tech. Cour.* **2021**, *69*, 551–561. [CrossRef]
27. Imrich, W. Factoring cardinal product graphs in polynomial time: Discrete metric spaces. *Discrete. Math.* **1998**, *192*, 119–144. [CrossRef]
28. Imrich, W.; Klavzar, S. *Product Graphs: Structure and Recognition, Wiley-Interscience Series in Discrete Mathematics and Optimization*; Wiley-Interscience: New York, NY, USA, 2000; p. xvi+358.
29. Graovac, A.; Pisanski, T. On the Wiener index of a graph. *J. Math. Chem.* **1991**, *8*, 53–62. [CrossRef]
30. Yeh, Y.N.; Gutman, I. On the sum of all distances in composite graphs. *Discrete Math.* **1994**, *135*, 359–365. [CrossRef]
31. Klavzar, S.; Rajapakse, A.; Gutman, I. The Szeged and the Wiener index of graphs. *Appl. Math. Lett.* **1996**, *9*, 45–49. [CrossRef]
32. Yarahmadi, Z. Computing some topological indices of tensor product of graphs. *Iran. J. Math. Chem.* **2011**, *2*, 109–118.
33. Akgüneş, N.; Çevik, A.S. A new bound of radius of irregularity index. *Appl. Math. Comput.* **2013**, *219*, 5750–5753. [CrossRef]
34. Das, K.C.; Akgüneş, N.; Çevik, A.S. On a graph of monogenic semigroup. *J. Ineq. Appl.* **2013**, *2013*, 44. [CrossRef]