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Solving a System of Sylvester-like Quaternion Matrix Equations

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Abstract: Using the ranks and Moore-Penrose inverses of involved matrices, in this paper we establish some necessary and sufficient solvability conditions for a system of Sylvester-type quaternion matrix equations, and give an expression of the general solution to the system when it is solvable. As an application of the system, we consider a special symmetry solution, named the η-Hermitian solution, for a system of quaternion matrix equations. Moreover, we present an algorithm and a numerical example to verify the main results of this paper.

Keywords: Sylvester-type matrix equation; quaternion matrix; rank; Moore–Penrose inverse; η-Hermitian matrix

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1. Introduction

In 1952, Roth [1] studied the following one-sided generalized Sylvester matrix equation for the first time

$$A_1X + YB_1 = C_1, \tag{1}$$

which is widely used in system and control theory. Since then, many researches have paid attention to Sylvester-type matrix equations (e.g., [2–5]) because of their wide range of applications, such as in descriptor system control theory [6], neural networks [7], robust, feedback [8], graph theory [9] and other areas. For instance, Baksalary and Kala [10] established a necessary and sufficient condition for Equation (1) to have a solution and gave an expression of its general solution. In [11], Baksalary and Kala give a solvability condition for the equation

$$AXB + CYD = E. \tag{2}$$

Wang investigated Equation (2) over arbitrary regular rings with identity [12].

In 1843, the very famous mathematician Hamilton discovered the quaternion. It is well known that quaternion algebra, denoted by $\mathbb{H}$, is an associative and non-commutative division algebra over the real number field $\mathbb{R}$, where

$$\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Since the 1970s, quaternions and the quaternion matrix have been studied a lot (e.g., [13–16]). The widespread applications of quaternions and the quaternion matrix include theoretical mechanics, optics, computer graphics, flight mechanics and aerospace technology, quantum physics, signal processing and so on (e.g., [17–20]).

In the last decade, the study of Sylvester-type matrix equations was extended to $\mathbb{H}$ (e.g., [21–28]). In 2012, Wang and He [29] presented the necessary and sufficient conditions for the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{3}$$

to be consistent and derived the expression of its general solution, which can be easily generalized to $\mathbb{H}$. For the Sylvester-type matrix equations with multiple variables and multiple equations, Wang [4] gave a solvability condition and the general solution to the system of Sylvester-type matrix equations

$$\begin{aligned} &A_3W = B_3,\ ZC_3 = D_3, \\ &A_5W + ZB_5 = D_4. \end{aligned} \tag{4}$$

Zhang [30] investigated the necessary and sufficient conditions for the solvability of the following system of Sylvester-like matrix equations

$$\begin{gathered} A_1X = B_1,\ XC_1 = D_1, \\ A_2Y = B_2,\ YC_2 = D_2, \\ ZC_3 = D_3,\ A_4V = B_4, \\ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \end{gathered} \tag{5}$$

and presented a formula of its general solution. We note that Equations (1)–(5) are the special cases of the following Sylvester-type quaternion matrix equations

$$\begin{gathered} A_1X = B_1,\ XC_1 = D_1, \\ A_2Y = B_2,\ YC_2 = D_2, \\ A_3W = B_3,\ ZC_3 = D_3, \\ A_5W + ZB_5 = D_4,\ A_4V = B_4, \\ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \end{gathered} \tag{6}$$

where A_i, B_i, C_j, D_k $(i = \overline{1,8},\ j = \overline{1,3},\ k = \overline{1,5})$ are given matrices over $\mathbb{H}$; X, Y, Z, V, W are unknown.

Motivated by the work mentioned above, in this paper we aim to investigate the solvability conditions and the general solutions to a more general system of a Sylvester-type quaternion matrix equation, Equation (6). In 2011, Took et al. [31] defined a special class of symmetric matrices, named η-Hermitian. For $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a quaternion matrix A is called η-Hermitian if $A = A^{\eta^*}$, where $A^{\eta^*} = -\eta A^* \eta$, A^* is the conjugate and transpose matrix of A. It is well known that η-Hermitian matrices have some applications in linear modeling (e.g., [32–34]) and so on.

As an application of (6), we derive the solvability conditions and an expression of the η-Hermitian solution to the system of matrix equations

$$\begin{gathered} A_4V = B_4, \\ A_1X = B_1,\ X = X^{\eta^*}, \\ A_2Y = B_2,\ Y = Y^{\eta^*}, \\ A_6V + (A_6V)^{\eta^*} + A_7XA_7^{\eta^*} + A_8YA_8^{\eta^*} = D_5,\ D_5 = D_5^{\eta^*}, \end{gathered} \tag{7}$$

where $A_i(i = 1,2,4,\overline{6,8}), B_1, B_2, B_4, D_5$ are given matrices over $\mathbb{H}$; X and Y are η-Hermitian matrices over $\mathbb{H}$.

We organize the rest of this article as follows: In Section 2, we introduce the basic knowledge of quaternions and Moore–Penrose inverse of a quaternion matrix, and review some matrix equations. In Section 3, we establish the solvability conditions for the system of (6) in terms of the Moore–Penrose inverses and the ranks of the coefficients' quaternion matrices in (6). In Section 4, we give an expression of the general solution to the system of (6), and illustrate the main results using a numerical example. In Section 5, we give some solvability conditions and an expression of the η-Hermitian solution to the system (7). Finally, we present a brief conclusion in Section 6 to end this paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand for the real number field and the set of all $m \times n$ matrix spaces over the quaternion algebra, respectively. The symbols $r(A)$, A^*, I and 0 are denoted by the rank of a given quaternion matrix A, the conjugate transpose of A, an identity matrix, and a zero matrix with appropriate sizes, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{l\times k}$ is defined to be the unique matrix, denoted by $A^\dagger$, satisfying

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

Moreover, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ represent two projectors. Clearly, $(L_A)^{\eta^*} = R_{A^{\eta^*}}$ and $(R_A)^{\eta^*} = L_{A^{\eta^*}}$ of A.

The following lemma was given by Marsaglia and Stynan [35], which is also available over $\mathbb{H}$.

Lemma 1 ([35])**.** *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times k}, C \in \mathbb{H}^{l\times n}, D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$. Then,*

$$r\begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 2 ([36])**.** *Let A_1 and C_1 be known matrices with feasible dimensions over $\mathbb{H}$. Then, the matrix equation $A_1X = C_1$ has a solution if and only if $R_{A_1}C_1 = 0$. In this case, its general solution is expressed as*

$$X = A_1^\dagger C_1 + L_{A_1}T_1,$$

where T_1 is an arbitrary matrix of an appropriate size.

Lemma 3 ([36])**.** *Let B_1 and D_1 be known matrices with allowable dimensions over $\mathbb{H}$. Then, the matrix equation $YB_1 = D_1$ has a solution if and only if $D_1L_{B_1} = 0$. In this case, its general solution is*

$$Y = D_1B_1^\dagger + T_2R_{B_1},$$

where T_2 is an arbitrary matrix of an appropriate size.

Lemma 4 ([37])**.** *Let A_1, B_1, C_1 and C_2 be the given matrices. Then, the system of matrix equations*

$$A_1Y = C_1,\ YB_1 = C_2$$

is consistent if and only if

$$R_{A_1}C_1 = 0,\ C_2L_{B_1} = 0,\ A_1C_2 = C_1B_1.$$

In this case, its general solution is

$$Y = A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + L_{A_1}T_3R_{B_1},$$

where T_3 is an arbitrary matrix of an appropriate size.

Lemma 5 ([10])**.** *Let A, B and C be given over $\mathbb{H}$. Then, the Equation (1) is solvable if and only if $R_ACL_B = 0$. Under this condition, the general solution to Equation (1) can be expressed as*

$$X = A^{\dagger}C - U_1B + L_AU_2,$$
$$Y = R_ACB^{\dagger} + AU_1 + U_3R_B,$$

where U_1, U_2 and U_3 are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

Lemma 6 ([38])**.** *Consider the following matrix equation over $\mathbb{H}$*

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B, \tag{8}$$

where A_i, B_i $(i = \overline{1,4})$, B are given and the others are unknown. Let

$$\begin{aligned}
&R_{A_1}A_2 = A_{11},\ R_{A_1}A_3 = A_{22},\ R_{A_1}A_4 = A_{33},\ B_2L_{B_1} = B_{11},\ B_{22}L_{B_{11}} = N_1,\\
&B_3L_{B_1} = B_{22},\ B_4L_{B_1} = B_{33},\ R_{A_{11}}A_{22} = M_1,\ S_1 = A_{22}L_{M_1},\ R_{A_1}BL_{B_1} = T_1,\\
&C = R_{M_1}R_{A_{11}},\ C_1 = CA_{33},\ C_2 = R_{A_{11}}A_{33},\ C_3 = R_{A_{22}}A_{33},\ C_4 = A_{33},\\
&D = L_{B_{11}}L_{N_1},\ D_1 = B_{33},\ D_2 = B_{33}L_{B_{22}},\ D_3 = B_{33}L_{B_{11}},\ D_4 = B_{33}D,\\
&E_1 = CT_1,\ E_2 = R_{A_{11}}T_1L_{B_{22}},\ E_3 = R_{A_{22}}T_1L_{B_{11}},\ E_4 = T_1D,\\
&C_{11} = (L_{C_2},\ L_{C_4}),\ D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\ C_{22} = L_{C_1},\ D_{22} = R_{D_2},\ C_{33} = L_{C_3},\\
&D_{33} = R_{D_4},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}},\\
&M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_2 - F_1,\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M,\\
&F_{11} = C_2L_{C_1},\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2,\ F_{22} = C_4L_{C_3},\ G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4,\\
&F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger},\ F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}.
\end{aligned}$$

Then, the following statements are equivalent:

(1) *Equation (8) is consistent.*

(2)

$$R_{C_i}E_i = 0,\ E_iL_{D_i} = 0\ (i = \overline{1,4}),\ R_{E_{22}}EL_{E_{33}} = 0.$$

(3)

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r\begin{pmatrix} A_2 & A_3 & A_4 & A_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_4 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & A_4 & A_1 \end{pmatrix} + r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r\begin{pmatrix} A_4 & A_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_3 & A_1 \end{pmatrix} + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r\begin{pmatrix} A_3 & A_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1),$$

$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}.$$

In this case, the general solution to Equation (8) can be expressed as

$$X_1 = A_1^\dagger(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^\dagger U_1B_1 + L_{A_1}U_2,$$
$$X_2 = R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^\dagger + A_1A_1^\dagger U_1 + U_3R_{B_1},$$
$$Y_1 = A_{11}^\dagger TB_{11}^\dagger - A_{11}^\dagger A_{22}M_1^\dagger TB_{11}^\dagger - A_{11}^\dagger S_1A_{22}^\dagger TN_1^\dagger B_{22}B_{11}^\dagger - A_{11}^\dagger S_1U_4R_{N_1}B_{22}B_{11}^\dagger + L_{A_{11}}U_5 + U_6R_{B_{11}},$$
$$Y_2 = M_1^\dagger TB_{22}^\dagger + S_1^\dagger S_1A_{22}^\dagger TN_1^\dagger + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},$$
$$Y_3 = F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2}, \text{ or } Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},$$

where $T = T_1 - A_{33}Y_3B_{33}$; $U_i(i = \overline{1,8})$ *represents any matrix with appropriate dimensions over* $\mathbb{H}$,

$$V_1 = (I_m\ 0)\Big[C_{11}^\dagger(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^\dagger U_{11}D_{11} + L_{C_{11}}U_{12}\Big],$$
$$W_1 = (0\ I_m)\Big[C_{11}^\dagger(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^\dagger U_{11}D_{11} + L_{C_{11}}U_{12}\Big],$$
$$W_2 = \Big[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger + C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}}\Big]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$V_2 = \Big[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger + C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}}\Big]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$V_3 = E_{11}^\dagger FE_{33}^\dagger - E_{11}^\dagger E_{22}M^\dagger FE_{33}^\dagger - E_{11}^\dagger SE_{22}^\dagger FN^\dagger E_{44}E_{33}^\dagger - E_{11}^\dagger SU_{31}R_NE_{44}E_{33}^\dagger + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},$$
$$W_3 = M^\dagger FE_{44}^\dagger + S^\dagger SE_{22}^\dagger FN^\dagger + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},$$

where U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} *and* U_{42} *are any matrix with appropriate dimensions over* $\mathbb{H}$.

3. Solvability Conditions to the System (6)

The goal of this section is to give the necessary and sufficient conditions for the existence of a solution to system (6).

Theorem 1. *Let* $A_i \in \mathbb{H}^{m_i\times n_i}(i = \overline{1,4}), A_5 \in \mathbb{H}^{m\times n_3}, A_6 \in \mathbb{H}^{m\times n_4}, A_7 \in \mathbb{H}^{m\times n_1}, A_8 \in \mathbb{H}^{m\times n_2}, B_j \in \mathbb{H}^{m_j\times l_j}(j = \overline{1,2}), B_3 \in \mathbb{H}^{m_3\times q}, B_4 \in \mathbb{H}^{m_4\times l}, B_5 \in \mathbb{H}^{l_3\times q}, B_6 \in \mathbb{H}^{l_3\times l}, B_7 \in \mathbb{H}^{l_1\times l}, B_8 \in \mathbb{H}^{l_2\times l}, C_k \in \mathbb{H}^{l_k\times p_k}(k = \overline{1,3}), D_j \in \mathbb{H}^{n_j\times p_j}(j = \overline{1,2}), D_3 \in \mathbb{H}^{m\times l_2}, D_4 \in \mathbb{H}^{m\times q}$ *and* $D_5 \in \mathbb{H}^{m\times l}$. *Set*

$$A_{11} = A_5L_{A_3},\ B_{11} = R_{C_3}B_5,\ C_{11} = D_4 - A_5A_3^\dagger B_3 - D_3C_3^\dagger B_5,\ A_{22} = A_6L_{A_4}, \tag{9}$$
$$B_{22} = R_{B_{11}}R_{C_3}B_6,\ A_{33} = A_7L_{A_1},\ B_{33} = R_{C_1}B_7,\ A_{44} = A_8L_{A_2},\ B_{44} = R_{C_2}B_8, \tag{10}$$
$$A_{55} = A_{11},\ B_{55} = R_{C_3}B_6,\ M_1 = R_{A_{22}}A_{33},\ M_2 = R_{A_{22}}A_{44},\ M_3 = R_{A_{22}}A_{55}, \tag{11}$$
$$\begin{aligned} C_{22} &= D_5 - A_6A_4^\dagger B_4 - D_3C_3^\dagger B_6 - R_{A_{11}}C_{11}B_{11}^\dagger R_{C_3}B_6 \\ &- A_7(A_1^\dagger B_1 + L_{A_1}D_1C_1^\dagger)B_7 - A_8(A_2^\dagger B_2 + L_{A_2}D_2C_2^\dagger)B_8, \end{aligned} \tag{12}$$
$$N_1 = B_{33}L_{B_{22}},\ N_2 = B_{44}L_{B_{22}},\ N_3 = B_{55}L_{B_{22}},\ G_1 = N_2L_{N_1},\ H_1 = R_{M_1}M_2, \tag{13}$$
$$S_1 = M_2L_{H_1},\ T = R_{A_{22}}C_{22}L_{B_{22}},\ P = R_{H_1}R_{M_1},\ P_1 = PM_3,\ P_2 = R_{M_1}M_3, \tag{14}$$
$$P_3 = R_{M_2}M_3,\ P_4 = M_3,\ Q = L_{N_1}L_{G_1},\ Q_1 = N_3,\ Q_2 = N_3L_{N_2},\ Q_3 = N_3L_{N_1}, \tag{15}$$
$$Q_4 = N_3Q,\ E_1 = PT,\ E_2 = R_{M_1}TL_{N_2},\ E_3 = R_{M_2}TL_{N_1},\ E_4 = TQ, \tag{16}$$
$$E_{11} = (L_{P_2},\ L_{P_4}),\ F_{11} = \begin{pmatrix} R_{Q_1} \\ R_{Q_3} \end{pmatrix},\ E_{22} = L_{P_1},\ F_{22} = R_{Q_2},\ E_{33} = L_{P_3},\ F_{33} = R_{Q_4}, \tag{17}$$

$$M_{11} = R_{E_{11}}E_{22},\ M_{22} = R_{E_{11}}E_{33},\ M_{33} = F_{22}L_{F_{11}},\ M_{44} = F_{33}L_{F_{11}},\ M = R_{M_{11}}M_{22}, \tag{18}$$

$$N = M_{44}L_{M_{33}},\ F = F_2 - F_1,\ E = R_{E_{11}}FL_{F_{11}},\ S = M_{22}L_M,\ G_{11} = P_2L_{P_1}, \tag{19}$$

$$H_{11} = E_2 - P_2P_1^{\dagger}E_1Q_1^{\dagger}Q_2, \quad G_{22} = P_4L_{P_3}, \quad H_{22} = E_4 - P_4P_3^{\dagger}E_3Q_3^{\dagger}Q_4, \tag{20}$$

$$F_2 = P_1^{\dagger}E_1Q_1^{\dagger} + L_{P_1}P_2^{\dagger}E_2Q_2^{\dagger}, \quad F_1 = P_3^{\dagger}E_3Q_3^{\dagger} + L_{P_3}P_4^{\dagger}E_4Q_4^{\dagger}. \tag{21}$$

Then, the following statements are equivalent:

(1) *System (6) has a solution.*

(2)

$$A_1D_1 = B_1C_1,\ A_2D_2 = B_2C_2 \tag{22}$$

and

$$\begin{aligned} &R_{A_1}B_1 = 0,\ D_1L_{C_1} = 0,\ R_{A_2}B_2 = 0,\ D_2L_{C_2} = 0, \\ &R_{A_3}B_3 = 0,\ D_3L_{C_3} = 0,\ R_{A_4}B_4 = 0,\ R_{A_{11}}C_{11}L_{B_{11}} = 0, \\ &R_{P_i}E_i = 0,\ E_iL_{Q_i} = 0\ (i = \overline{1,4}),\ R_{M_{22}}EL_{M_{33}} = 0. \end{aligned} \tag{23}$$

(3) (22) *holds and*

$$r(A_1 \quad B_1) = r(A_1), r\begin{pmatrix} C_1 \\ D_1 \end{pmatrix} = r(C_1), r(A_2 \quad B_2) = r(A_2), r\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = r(C_2), \tag{24}$$

$$r(A_3 \quad B_3) = r(A_3), \quad r\begin{pmatrix} C_3 \\ D_3 \end{pmatrix} = r(C_3), \quad r(A_4 \quad B_4) = r(A_4), \tag{25}$$

$$r\begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_5 & C_3 \end{pmatrix}, \tag{26}$$

$$r\begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_4 & A_5 & D_3 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_8 & A_5 & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & 0 \end{pmatrix} + r\begin{pmatrix} B_6 & B_5 & C_3 \end{pmatrix}, \tag{27}$$

$$r\begin{pmatrix} D_5 & A_7 & A_6 & A_8D_2 & D_4 & A_5 & D_3 \\ B_8 & 0 & 0 & C_2 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_6 & A_5 \\ A_1 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_8 & C_2 & 0 & 0 \\ B_6 & 0 & B_5 & C_3 \end{pmatrix}, \tag{28}$$

$$r\begin{pmatrix} D_5 & A_8 & A_6 & A_7D_1 & D_4 & A_5 & D_3 \\ B_7 & 0 & 0 & C_1 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_8 & A_6 & A_5 \\ A_2 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_7 & C_1 & 0 & 0 \\ B_6 & 0 & B_5 & C_3 \end{pmatrix}, \tag{29}$$

$$
r\begin{pmatrix} D_5 & A_6 & A_7D_1 & A_8D_2 & D_4 & A_5 & D_3 \\ B_7 & 0 & C_1 & 0 & 0 & 0 & 0 \\ B_8 & 0 & 0 & C_2 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix}
= r\begin{pmatrix} B_7 & C_1 & 0 & 0 & 0 \\ B_8 & 0 & C_2 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & C_3 \end{pmatrix} + r\begin{pmatrix} A_6 & A_5 \\ A_4 & 0 \\ 0 & A_3 \end{pmatrix}, \tag{30}
$$

$$
r\begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_3 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_8 & A_6 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_6 & C_3 \end{pmatrix}, \tag{31}
$$

$$
r\begin{pmatrix} D_5 & A_7 & A_6 & A_8D_2 & D_3 \\ B_8 & 0 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_8 & C_2 & 0 \\ B_6 & 0 & C_3 \end{pmatrix} + r\begin{pmatrix} A_7 & A_6 \\ A_1 & 0 \\ 0 & A_4 \end{pmatrix}, \tag{32}
$$

$$
r\begin{pmatrix} D_5 & A_8 & A_6 & A_7D_1 & D_3 \\ B_7 & 0 & 0 & C_1 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_2B_8 & A_2 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_7 & C_1 & 0 \\ B_6 & 0 & C_3 \end{pmatrix} + r\begin{pmatrix} A_8 & A_6 \\ A_2 & 0 \\ 0 & A_4 \end{pmatrix}, \tag{33}
$$

$$
r\begin{pmatrix} D_5 & A_6 & A_7D_1 & A_8D_2 & D_3 \\ B_7 & 0 & C_1 & 0 & 0 \\ B_8 & 0 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_4 & A_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_7 & C_1 & 0 & 0 \\ B_8 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & C_3 \end{pmatrix} + r\begin{pmatrix} A_6 \\ A_4 \end{pmatrix}, \tag{34}
$$

$$
r\begin{pmatrix}
D_5 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & D_4 & 0 & 0 & 0 & A_5 & D_3 & 0 & 0 \\
B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
0 & 0 & 0 & D_5 & A_8 & A_6 & 0 & 0 & A_7D_1 & D_4 & 0 & 0 & 0 & A_5 & D_3 \\
0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\
B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\
B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0
\end{pmatrix}
$$

$$= r\begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} A_7 & A_6 & 0 & 0 & A_5 & 0 \\ 0 & 0 & A_8 & A_6 & 0 & A_5 \\ A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_3 \end{pmatrix}. \tag{35}$$

Proof. (1) $\Leftrightarrow$ (2)

The proof is divided into three parts:

- Firstly, we divide the system (6) into the following:

$$\begin{gathered} A_3W = B_3,\ ZC_3 = D_3,\ A_4V = B_4, \\ A_1X = B_1,\ XC_1 = D_1,\ A_2Y = B_2,\ YC_2 = D_2, \end{gathered} \tag{36}$$

$$A_5Z + WB_5 = D_4, \tag{37}$$

$$A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \tag{38}$$

and consider the solvability conditions and the general solution to the system of matrices of Equation (36). For more information, see *Step 1*.

- Secondly, substituting the W and Z obtained in the first step into Equation (37) yields

$$A_{11}T_3 + T_4B_{11} = C_{11}, \tag{39}$$

where A_{11}, B_{11} and C_{11} are defined by (9); T_3 and T_4 are unknowns. For more information, see *Step 2*.

- Finally, by substituting the X, Y, Z, and V obtained from the above two steps into Equation (38), we obtain a matrix equation with the following form

$$A_{22}T_5 + U_3B_{22} + A_{33}T_1B_{33} + A_{44}T_2B_{44} + A_{55}U_1B_{55} = C_{22}, \tag{40}$$

where A_{ii}, B_{ii} $(i = \overline{2,5})$ and C_{22} are given by (9)–(12); T_1, T_2, T_5, U_1 and U_3 are unknowns. For more information, see *Step 3*.

We can obtain the results from the following steps: First, we consider the solvability conditions and the expression of the general solutions to the system of the matrix Equation (36).

Step 1. It follows from Lemmas 2–4 that system (36) has a solution if and only if (22) holds and

$$\begin{aligned} &R_{A_1}B_1 = 0,\ D_1L_{C_1} = 0,\ R_{A_2}B_2 = 0,\ D_2L_{C_2} = 0, \\ &R_{A_3}B_3 = 0,\ D_3L_{C_3} = 0,\ R_{A_4}B_4 = 0. \end{aligned} \tag{41}$$

In this case, the general solution to system (36) can be written as

$$\begin{gathered} X = A_1^{\dagger}B_1 + L_{A_1}D_1C_1^{\dagger} + L_{A_1}T_1R_{C_1}, \\ Y = A_2^{\dagger}B_2 + L_{A_2}D_2C_2^{\dagger} + L_{A_2}T_2R_{C_2}, \\ W = A_3^{\dagger}B_3 + L_{A_3}T_3,\ Z = D_3C_3^{\dagger} + T_4R_{C_3},\ V = A_4^{\dagger}B_4 + L_{A_4}T_5, \end{gathered} \tag{42}$$

where T_i $(i = \overline{1,5})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Step 2. Substituting W, Z in (42) into (37) yields (39). According to Lemma 5, it follows that Equation (39) has a solution if and only if

$$R_{A_{11}} C_{11} L_{B_{11}} = 0. \tag{43}$$

In this case, the general solution to Equation (39) can be expressed as

$$T_3 = A_{11}^{\dagger} C_1 - U_1 B_{11} + L_{A_{11}} U_2, \tag{44}$$
$$T_4 = R_{A_{11}} C_{11} B_{11}^{\dagger} + A_{11} U_1 + U_3 R_{B_{11}}, \tag{45}$$

where U_1, U_2 and U_3 are any matrix with appropriate sizes over $\mathbb{H}$.

Substituting (45) into $Z = D_3 C_3^{\dagger} + T_4 R_{C_3}$ yields

$$Z = D_3 C_3^{\dagger} + R_{A_{11}} C_{11} B_{11}^{\dagger} R_{C_3} + A_{11} U_1 R_{C_3} + U_3 R_{B_{11}} R_{C_3}. \tag{46}$$

Step 3. By substituting X, Y, V in (42) and Z in (46) into (38), we obtain Equation (40). By using Lemma 6, Equation (40) is consistent if and only if

$$R_{P_i} E_i = 0,\ E_i L_{Q_i} = 0\ (i = \overline{1,4}),\ R_{M_{22}} E L_{M_{33}} = 0, \tag{47}$$

namely,

$$r\begin{pmatrix} C_{22} & A_{33} & A_{44} & A_{55} & A_{22} \\ B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_{22}) + r\begin{pmatrix} A_{33} & A_{44} & A_{55} & A_{22} \end{pmatrix}, \tag{48}$$

$$r\begin{pmatrix} C_{22} & A_{33} & A_{55} & A_{22} \\ B_{44} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{33} & A_{55} & A_{22} \end{pmatrix} + r\begin{pmatrix} B_{44} \\ B_{22} \end{pmatrix}, \tag{49}$$

$$r\begin{pmatrix} C_{22} & A_{44} & A_{55} & A_{22} \\ B_{33} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{44} & A_{55} & A_{22} \end{pmatrix} + r\begin{pmatrix} B_{33} \\ B_{22} \end{pmatrix}, \tag{50}$$

$$r\begin{pmatrix} C_{22} & A_{55} & A_{22} \\ B_{33} & 0 & 0 \\ B_{44} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{44} \\ B_{22} \end{pmatrix} + r\begin{pmatrix} A_{55} & A_{22} \end{pmatrix}, \tag{51}$$

$$r\begin{pmatrix} C_{22} & A_{33} & A_{44} & A_{22} \\ B_{55} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{33} & A_{44} & A_{22} \end{pmatrix} + r\begin{pmatrix} B_{55} \\ B_{22} \end{pmatrix}, \tag{52}$$

$$r\begin{pmatrix} C_{22} & A_{33} & A_{22} \\ B_{44} & 0 & 0 \\ B_{55} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{44} \\ B_{55} \\ B_{22} \end{pmatrix} + r\begin{pmatrix} A_{33} & A_{22} \end{pmatrix}, \tag{53}$$

$$r\begin{pmatrix} C_{22} & A_{44} & A_{22} \\ B_{33} & 0 & 0 \\ B_{55} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{55} \\ B_{22} \end{pmatrix} + r\begin{pmatrix} A_{44} & A_{22} \end{pmatrix}, \tag{54}$$

$$r\begin{pmatrix} C_{22} & A_{22} \\ B_{33} & 0 \\ B_{44} & 0 \\ B_{55} & 0 \\ B_{22} & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{44} \\ B_{55} \\ B_{22} \end{pmatrix} + r(A_{22}), \tag{55}$$

$$r\begin{pmatrix} C_{22} & A_{33} & A_{22} & 0 & 0 & 0 & A_{55} \\ B_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_{44} & A_{22} & A_{55} \\ 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\ B_{55} & 0 & 0 & B_{55} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{44} & 0 \\ B_{22} & 0 \\ 0 & B_{33} \\ 0 & B_{22} \\ B_{55} & B_{55} \end{pmatrix} + r\begin{pmatrix} A_{33} & A_{22} & 0 & 0 & A_{55} \\ 0 & 0 & A_{44} & A_{22} & A_{55} \end{pmatrix}. \tag{56}$$

In this case, the general solution to Equation (40) can be expressed as

$$\begin{aligned} T_5 &= A_{22}^{\dagger}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55}) + A_{22}^{\dagger}V_1B_{22} + L_{A_{22}}V_2, \\ U_3 &= R_{A_{22}}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55})B_{22}^{\dagger} + A_{22}A_{22}^{\dagger}V_1 + V_3R_{B_{22}}, \\ T_1 &= M_1^{\dagger}T_{11}N_1^{\dagger} - M_1^{\dagger}M_2H_1^{\dagger}T_{11}N_1^{\dagger} - M_1^{\dagger}S_1M_2^{\dagger}T_{11}G_1^{\dagger}N_2N_1^{\dagger} - M_1^{\dagger}S_1V_4R_{G_1}N_2N_1^{\dagger} \\ &\quad + L_{M_1}V_5 + V_6R_{N_1}, \\ T_2 &= H_1^{\dagger}T_{11}N_2^{\dagger} + S_1^{\dagger}S_1M_2^{\dagger}T_{11}G_1^{\dagger} + L_{H_1}L_{S_1}V_7 + V_8R_{N_2} + L_{H_1}V_4R_{G_1}, \\ U_1 &= F_1 + L_{P_2}W_1 + W_2R_{Q_1} + L_{P_1}W_3R_{Q_2}, \ or \quad U_1 = F_2 - L_{P_2}W_4 - W_5R_{Q_3} - L_{P_3}W_6R_{Q_4}, \end{aligned}$$

where $T_{11} = T - M_3U_1N_3,\ V_i(i = \overline{1,8})$ are any matrix with suitable dimensions over $\mathbb{H}$,

$$\begin{aligned} W_1 &= \begin{bmatrix} I_m & 0 \end{bmatrix}\left[E_{11}^{\dagger}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^{\dagger}U_{11}F_{11} + L_{E_{11}}U_{12}\right], \\ W_4 &= \begin{bmatrix} 0 & I_m \end{bmatrix}\left[E_{11}^{\dagger}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^{\dagger}U_{11}F_{11} + L_{E_{11}}U_{12}\right], \\ W_2 &= \left[R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^{\dagger} + E_{11}E_{11}^{\dagger}U_{11} + U_{21}R_{F_{11}}\right]\begin{bmatrix} I_n \\ 0 \end{bmatrix}, \\ W_5 &= \left[R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^{\dagger} + E_{11}E_{11}^{\dagger}U_{11} + U_{21}R_{F_{11}}\right]\begin{bmatrix} 0 \\ I_n \end{bmatrix}, \\ W_3 &= M_{11}^{\dagger}FM_{33}^{\dagger} - M_{11}^{\dagger}M_{22}M^{\dagger}FM_{33}^{\dagger} - M_{11}^{\dagger}SM_{22}^{\dagger}FN^{\dagger}M_{44}M_{33}^{\dagger} - M_{11}^{\dagger}SU_{31}R_NM_{44}M_{33}^{\dagger} \\ &\quad + L_{M_{11}}U_{32} + U_{33}R_{M_{33}}, \\ W_6 &= M^{\dagger}FM_{44}^{\dagger} + S^{\dagger}SM_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{M_{44}}, \end{aligned}$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ and U_{42} are any matrix with suitable dimensions over $\mathbb{H}$.

To sum up, the system of matrices of Equation (6) has a solution if and only if (41), (43) and (47) hold.

(2) $\Leftrightarrow$ (3) We divide it into three parts to prove its equivalence.

Part 1. In this part, we prove that (41) holds if and only if (24) and (25) hold. According to Lemma 1, it is easy to show that (41) holds if and only if (24) and (25) hold.

Part 2. In this part, we prove that (43) $\Longleftrightarrow$ (26). It follows from Lemma 1 and elementary operations that

$$
\begin{aligned}
(43) \Leftrightarrow\ & r\begin{pmatrix} C_{11} & A_{11} \\ B_{11} & 0 \end{pmatrix} = r(A_{11}) + r(B_{11}) \\
\Leftrightarrow\ & r\begin{pmatrix} C_{11} & A_5L_{A_3} \\ R_{C_3}B_5 & 0 \end{pmatrix} = r(A_5L_{A_3}) + r(R_{C_3}B_5) \\
\Leftrightarrow\ & r\begin{pmatrix} D_4 - A_5A_3^{\dagger}B_3 - D_3C_3^{\dagger}B_5 & A_5 & 0 \\ B_5 & 0 & C_3 \\ 0 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_5 & C_3 \end{pmatrix} \\
\Leftrightarrow\ & r\begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_5 & C_3 \end{pmatrix} \Leftrightarrow (26).
\end{aligned}
$$

Part 3. In this part, we show that (47) holds if and only if (27) to (35) hold. By using Lemma 6, (47) holds if and only if (48) to (56) hold. Hence, we only show that (48) to (56) hold if and only if (27) to (35) hold, respectively. We first prove that (48) $\Leftrightarrow$ (27).

Note that

$$X_0 = A_1^{\dagger}B_1 + L_{A_1}D_1C_1^{\dagger},\ Y_0 = A_2^{\dagger}B_2 + L_{A_2}D_2C_2^{\dagger},\ Z_0 = D_3C_3^{\dagger},\ V_0 = A_4^{\dagger}B_4,\ W_0 = A_3^{\dagger}B_3$$

are the special solution to the equations

$$
\begin{aligned}
& A_1X = B_1,\ XC_1 = D_1, \\
& A_2Y = B_2,\ YC_2 = D_2, \\
& A_3W = B_3,\ ZC_3 = D_3,\ A_4V = B_4,
\end{aligned}
$$

respectively. Then, we have that

$$C_{11} = D_4 - A_5W_0 - Z_0B_5, \tag{57}$$
$$C_{22} = D_5 - A_6V_0 - Z_0B_6 - R_{A_{11}}C_{11}B_{11}^{\dagger}R_{C_3}B_6 - A_7X_0B_7 - A_8Y_0B_8. \tag{58}$$

It follows from Lemma 1 and elementary operations to (47) that

$$
\begin{aligned}
(48) \Leftrightarrow\ & r\begin{pmatrix} C_{22} & A_7 & A_8 & A_{11} & A_6 & 0 \\ R_{C_3}B_6 & 0 & 0 & 0 & 0 & B_{11} \\ 0 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & 0 \end{pmatrix} = r\begin{pmatrix} R_{C_3}B_6 & B_{11} \end{pmatrix} + r\begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \\
\Leftrightarrow\ & r\begin{pmatrix} D_5 - Z_0B_6 & A_7 & A_8 & A_6 & C_{11} & A_{11} \\ R_{C_3}B_6 & 0 & 0 & 0 & B_{11} & 0 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} R_{C_3}B_6 & B_{11} \end{pmatrix}
\end{aligned}
$$

$$\Leftrightarrow r\begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_4 & A_5 & D_3 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_8 & A_5 & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & 0 \end{pmatrix} + r\begin{pmatrix} B_6 & B_5 & C_3 \end{pmatrix} \Leftrightarrow (27).$$

Similarly, we can prove that $R_{P_2}E_2 = 0 \Leftrightarrow$ (28), $R_{P_3}E_3 = 0 \Leftrightarrow$ (29), $R_{P_4}E_4 = 0 \Leftrightarrow$ (30) and $E_iL_{Q_i} = 0$ $(i = \overline{1,4})$ hold if and only if (31) to (34) hold, respectively. Next, we show that $R_{M_{22}}EL_{M_{33}} = 0 \Leftrightarrow$ (35). According to Lemma 1 and elementary operations, we have that

$$R_{M_{22}}EL_{M_{33}} = 0 \Leftrightarrow r\begin{pmatrix} E & D_{22} \\ D_{33} & 0 \end{pmatrix} = r(D_{22}) + r(D_{33})$$

$$\Leftrightarrow r\begin{pmatrix} C_{22} & A_{33} & A_{22} & 0 & 0 & 0 & A_{55} \\ B_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_{44} & A_{22} & A_{55} \\ 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\ B_{55} & 0 & 0 & B_{55} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{44} & 0 \\ B_{22} & 0 \\ 0 & B_{33} \\ 0 & B_{22} \\ B_{55} & B_{55} \end{pmatrix} + r\begin{pmatrix} A_{33} & A_{22} & 0 & 0 & A_{55} \\ 0 & 0 & A_{44} & A_{22} & A_{55} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} C_{22} & A_7 & A_6 & 0 & 0 & 0 & A_{11} & 0 & 0 & 0 & 0 & 0 \\ B_8 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 \\ R_{C_3}B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_8 & A_6 & A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & R_{C_3}B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 \\ B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 \\ R_{C_3}B_6 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 \\ 0 & R_{C_3}B_6 & 0 & 0 & 0 & B_{11} & 0 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 \end{pmatrix} + r\begin{pmatrix} A_7 & A_6 & 0 & 0 & A_{11} \\ 0 & 0 & A_8 & A_6 & A_{11} \\ A_1 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} A_7 & A_6 & 0 & 0 & A_{11} & 0 \\ 0 & 0 & A_8 & A_6 & 0 & A_{11} \\ A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \end{pmatrix}$$

$$
= r\begin{pmatrix}
D_5 - Z_0B_6 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & C_{11} & 0 & 0 & 0 & A_{11} & 0 & 0 & 0 \\
B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
0 & 0 & 0 & Z_0B_6 - D_5 & A_8 & A_6 & 0 & 0 & -A_7D_1 & -C_{11} & 0 & 0 & 0 & A_{11} & 0 \\
0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\
B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\
B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\Leftrightarrow r\begin{pmatrix}
B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\
0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\
0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\
B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0
\end{pmatrix}
+ r\begin{pmatrix}
A_7 & A_6 & 0 & 0 & A_5 & 0 \\
0 & 0 & A_8 & A_6 & 0 & A_5 \\
A_1 & 0 & 0 & 0 & 0 & 0 \\
0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & A_2 & 0 & 0 & 0 \\
0 & 0 & 0 & A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & A_3 & 0 \\
0 & 0 & 0 & 0 & 0 & A_3
\end{pmatrix}
$$

$$
= r\begin{pmatrix}
D_5 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & D_4 & 0 & 0 & 0 & A_5 & D_3 & 0 & 0 \\
B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
0 & 0 & 0 & D_5 & A_8 & A_6 & 0 & 0 & A_7D_1 & D_4 & 0 & 0 & 0 & A_5 & D_3 \\
0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\
B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\
B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0
\end{pmatrix}
\Leftrightarrow (35).
$$

□

4. The General Solution to the System (6)

In this section, we give an expression for the general solution of Equation (6) by using the Moore–Penrose inverse. According to the proof of Theorem 1, we obtain the following theorem:

Theorem 2. *The general solution to system* (6) *can be expressed as follows when the solvability conditions are met:*

$$
\begin{aligned}
&X = A_1^{\dagger}B_1 + L_{A_1}D_1C_1^{\dagger} + L_{A_1}T_1R_{C_1}, \quad Y = A_2^{\dagger}B_2 + L_{A_2}D_2C_2^{\dagger} + L_{A_2}T_2R_{C_2}, \\
&Z = D_3C_3^{\dagger} + R_{A_{11}}C_{11}B_{11}^{\dagger}R_{C_3} + A_{11}U_1R_{C_3} + U_3R_{B_{11}}R_{C_3}, \\
&W = A_3^{\dagger}B_3 + L_{A_3}A_{11}^{\dagger}C_1 - L_{A_3}A_{11}^{\dagger}U_1B_{11} + L_{A_3}L_{A_{11}}U_2,
\end{aligned}
$$

$$
V = A_4^{\dagger}B_4 + L_{A_4}A_{22}^{\dagger}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55}) + L_{A_4}A_{22}^{\dagger}V_1B_{22} + L_{A_4}L_{A_{22}}V_2,
$$

where $T_{11} = T - M_3U_1N_3$, $V_i(i = \overline{1,8})$ *are arbitrary matrices with appropriate sizes.*

$$T_1 = M_1^\dagger T_{11}N_1^\dagger - M_1^\dagger M_2H_1^\dagger T_{11}N_1^\dagger - M_1^\dagger S_1M_2^\dagger T_{11}G_1^\dagger N_2N_1^\dagger - M_1^\dagger S_1V_4R_{G_1}N_2N_1^\dagger$$
$$+ L_{M_1}V_5 + V_6R_{N_1},$$
$$T_2 = H_1^\dagger T_{11}N_2^\dagger + S_1^\dagger S_1M_2^\dagger T_{11}G_1^\dagger + L_{H_1}L_{S_1}V_7 + V_8R_{N_2} + L_{H_1}V_4R_{G_1},$$

$$U_3 = R_{A_{22}}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55})B_{22}^\dagger + A_{22}A_{22}^\dagger V_1 + V_3R_{B_{22}},$$
$$U_1 = F_1 + L_{P_2}W_1 + W_2R_{Q_1} + L_{P_1}W_3R_{Q_2},\ or\quad U_1 = F_2 - L_{P_2}W_4 - W_5R_{Q_3} - L_{P_3}W_6R_{Q_4},$$
$$W_1 = \begin{bmatrix} I_m & 0 \end{bmatrix}\Big[E_{11}^\dagger(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^\dagger U_{11}F_{11} + L_{E_{11}}U_{12}\Big],$$
$$W_4 = \begin{bmatrix} 0 & I_m \end{bmatrix}\Big[E_{11}^\dagger(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^\dagger U_{11}F_{11} + L_{E_{11}}U_{12}\Big],$$
$$W_2 = \Big[R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^\dagger + E_{11}E_{11}^\dagger U_{11} + U_{21}R_{F_{11}}\Big]\begin{bmatrix} I_n \\ 0 \end{bmatrix},$$
$$W_5 = \Big[R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^\dagger + E_{11}E_{11}^\dagger U_{11} + U_{21}R_{F_{11}}\Big]\begin{bmatrix} 0 \\ I_n \end{bmatrix},$$
$$W_3 = M_{11}^\dagger FM_{33}^\dagger - M_{11}^\dagger M_{22}M^\dagger FM_{33}^\dagger - M_{11}^\dagger SM_{22}^\dagger FN^\dagger M_{44}M_{33}^\dagger - M_{11}^\dagger SU_{31}R_NM_{44}M_{33}^\dagger$$
$$+ L_{M_{11}}U_{32} + U_{33}R_{M_{33}},$$
$$W_6 = M^\dagger FM_{44}^\dagger + S^\dagger SM_{22}^\dagger FN^\dagger + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{M_{44}},$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ *and* U_{42} *are arbitrary matrices over* $\mathbb{H}$ *of appropriate sizes.*

Next, we discuss the special cases of the system of matrices of Equation (6). Letting A_3, B_3, A_5, B_5 and D_4 vanish yields the following:

Corollary 1. *Suppose that* A_i, B_i, C_j, D_j $(i = \overline{1,4},\ j = \overline{1,5})$ *and* E_1 *are given, denote*

$$A_6 = A_4L_{A_1},\quad B_6 = R_{B_1}B_4,\quad C_6 = C_4L_{A_2},\quad D_6 = R_{B_2}D_4,\quad C_7 = C_5L_{A_3},\quad D_7 = R_{B_3}D_5,$$
$$E_6 = E_1 - A_4A_1^\dagger C_1 - D_1B_1^\dagger B_4 - C_4\Big(A_2^\dagger C_2 + L_{A_2}D_2B_2^\dagger\Big)D_4 - C_5\Big(A_3^\dagger C_3 + L_{A_3}D_3B_3^\dagger\Big)D_5,$$
$$A = R_{A_6}C_6,\quad B = D_6L_{B_6},\quad C = R_{A_6}C_7,\quad D = D_7L_{B_6},$$

$$E = R_{A_6}E_6L_{B_6},\quad M = R_AC,\quad N = DL_B,\quad S = CL_M.$$

Then, the following statements are equivalent:
(1)*System (5) is consistent.*
(2)

$$R_{A_i}C_i = 0,\quad D_iL_{B_i} = 0\ (i = 1,2,3),\quad A_2D_2 = C_2B_2,\quad A_3D_3 = C_3B_3,$$
$$R_AE = MM^\dagger E,\quad EL_B = EN^\dagger N,\quad R_AEL_D = 0,\quad R_CEL_B = 0.$$

(3)

$$r\begin{pmatrix} A_i & C_i \end{pmatrix} = r(A_i), \quad r\begin{pmatrix} B_i \\ D_i \end{pmatrix} = r(B_i)\ (i = 1,2,3), \quad A_2D_2 = C_2B_2, \quad A_3D_3 = C_3B_3,$$

$$r\begin{pmatrix} E_1 & A_4 & D_1 & C_4D_2 & C_5D_3 \\ B_4 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & B_2 & 0 \\ D_5 & 0 & 0 & 0 & B_3 \\ C_1 & A_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_4 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_2 & 0 \\ D_5 & 0 & 0 & B_2 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_4 & C_5 & D_1 \\ B_4 & 0 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 \\ C_3D_5 & 0 & 0 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 & C_4 & C_5 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_4 & D_1 & C_5D_3 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_5 & 0 & 0 & 0 & B_3 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 \\ D_5 & 0 & B_3 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_5 & D_1 & C_4D_2 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_3D_5 & 0 & A_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 & C_5 \\ A_1 & 0 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{pmatrix}.$$

In this case, the general solution to system (5) can be expressed as

$$\begin{aligned}
X_1 &= A_1^{\dagger}C_1 + L_{A_1}U_1, \quad X_2 = D_1B_1^{\dagger} + U_2R_{B_1}, \\
X_3 &= A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger} + L_{A_2}U_3R_{B_2}, \\
X_4 &= A_3^{\dagger}C_3 + L_{A_3}D_3B_3^{\dagger} + L_{A_3}U_4R_{B_3}, \\
U_1 &= A_6^{\dagger}(E_6 - C_6U_3D_6 - C_7U_4D_7) - A_6^{\dagger}W_2B_6 + L_{A_6}W_1, \\
U_2 &= R_{A_6}(E_6 - C_6U_3D_6 - C_7U_4D_7)B_6^{\dagger} + A_6A_6^{\dagger}W_2 + W_3R_{B_6}, \\
U_3 &= A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}SV_4R_NDB^{\dagger} + L_AV_1 + V_2R_B, \\
U_4 &= M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_SV_3 + L_MV_4R_N + V_5R_D,
\end{aligned}$$

where $V_i, W_j (i = \overline{1,5},\ j = \overline{1,3})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Remark 1. *The above corollary is from the important findings of [30].*

Letting A_i, B_i, C_j, D_j $(i = 1,2,4,6,7,8, j = 1,2)$ and D_5 vanish, we have the following:

Corollary 2. *Given $A_3, B_3, C_3, D_3, A_5, B_5$ and D_4 of feasible dimensions over $\mathbb{H}$. Set $A_{11} = A_5L_{A_3}$, $B_{11} = R_{C_3}C$ and $E_{11} = D_4 - A_5A_3^{\dagger}B_3 - D_3C_3^{\dagger}B_5$. Then, the following statements are equivalent:*

(1) *System (4) is consistent.*

$$(2)\ r\begin{pmatrix} A_3 & B_3 \end{pmatrix} = r(A_3), r\begin{pmatrix} C_3 \\ D_3 \end{pmatrix} = r(C_3), r\begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r\begin{pmatrix} B_5 & C_3 \end{pmatrix}.$$

In this case, the general solution to system (4) *can be expressed as*

$$\begin{aligned} W &= A_3^{\dagger}B_3 + L_{A_3}\big(A_{11}^{\dagger}E_{11} - A_3^{\dagger}W_2B_{11} + L_{A_{11}}W_1\big), \\ Z &= D_3C_3^{\dagger} + \big(R_{A_{11}}E_{11}B_{11}^{\dagger} + A_{11}A_{11}^{\dagger}W_2 + W_3R_{B_{11}}\big)R_{C_3}, \end{aligned}$$

where W_1, W_2 *and* W_3 *are arbitrary matrices over* $\mathbb{H}$ *of appropriate sizes.*

Remark 2. *The above corollary is from the vital investigation of [4].*

Finally, we give Algorithm 1 and an example to illustrate the main results of this paper.

Algorithm 1: Algorithm for solving Equation (6)
(1) Feed the values of A_i, B_i, C_j, D_k $(i = \overline{1,8},\ j = \overline{1,3},\ k = \overline{1,5})$ with conformable shapes over $\mathbb{H}$. (2) Compute the symbols in (9) to (21). (3) Check (22), (23) or rank equalities in (24) to (35) hold or not. If no, then return "inconsistent". (4) Otherwise, compute X, Y, Z, V, W.

Example 1. *Consider the matrix of Equation* (6)*. Assume*

$$A_1 = \begin{pmatrix} -1-j & i-j \\ i & -1 \\ 1-i & -j \end{pmatrix},\ A_2 = \begin{pmatrix} i+j & 1+k \\ i+k & k \\ 1+i & 1+i+k \end{pmatrix},\ A_3 = \begin{pmatrix} j+k & -1+j \\ -i+k & i \\ i+j & -j+k \end{pmatrix},\ A_4 = \begin{pmatrix} 1+j & 2+k \\ 1-i+k & 2j+2k \\ 1+i+j+k & 1+i+j \end{pmatrix},$$

$$A_5 = \begin{pmatrix} -j & -j+k \\ -1-j+k & k \\ 1-j & i+k \end{pmatrix},\ A_6 = \begin{pmatrix} 1+i+j+k & 1 \\ 0 & 1+i \\ 1+j & 1+i \end{pmatrix},\ A_7 = \begin{pmatrix} i+j & 1+i \\ 1+i+k & 1+j \\ 1+i+j & 1+j \end{pmatrix},\ A_8 = \begin{pmatrix} -1+i+j-k & -i \\ -1+j-k & i-j \\ -1+j+k & -1+i \end{pmatrix},$$

$$B_1 = \begin{pmatrix} i-5j & -1+i \\ -2+i & -1-2i+j \\ 1-2i-j-k & -2+i-j+k \end{pmatrix},\ B_2 = \begin{pmatrix} 1+i+j-k & -1+3i+j+k \\ -1+2i+j+k & -1+i+2j \\ 1+3i-j & -3+3i \end{pmatrix},\ B_3 = \begin{pmatrix} -2-2i+j-k & -3+2j-3k & -3-2j+k \\ -1-k & 2-i+4j & 1-i+k \\ -1+3k & -3-j-2k & 1-2i-j \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 2+i-j+3k & 2+i+5j-2k & 2j \\ 1+3i+2j+3k & 2-3i+2j-k & 1-2i+j \\ 3+3i+k & 3i+5j & 2j+2k \end{pmatrix},\ B_5 = \begin{pmatrix} i & 1+j+k & 1+i+k \\ 0 & i & 1+i+j \end{pmatrix}, B_6 = \begin{pmatrix} 2+2i+j+k & 2+2j+k & 1+i+j \\ 1+2i+j+2k & 1+2i+k & 2+2i+2j+k \end{pmatrix},$$

$$B_7 = \begin{pmatrix} k & i+j & i+k \\ i+k & j & 1 \end{pmatrix}, B_8 = \begin{pmatrix} j & j & 1+i+j+k \\ 1 & 1+j+k & 1+j \end{pmatrix},\ C_1 = \begin{pmatrix} 1+i+k & 1+i+k & 1+i+k \\ 1+i & k & 0 \end{pmatrix},\ C_2 = \begin{pmatrix} i & -1 & 0 \\ 1-i+j-k & -i-j & i+k \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & k & 0 \\ 1+j+k & i+j+k & 1+i+k \end{pmatrix},\ D_1 = \begin{pmatrix} i+j & 1+2i+j-k & 1+2i+j \\ 1+4i+4k & 1+i+4k & 1+2i+j+3k \end{pmatrix},\ D_2 = \begin{pmatrix} -i-j+k & -1-i+j & 0 \\ 4+2i+3k & i-j-3k & -3-i-j+k \end{pmatrix},$$

$$D_3 = \begin{pmatrix} -i+j+2k & -1+j+2k & i+j+2k \\ -1-j+2k & -3+2k & -2+i+k \\ 0 & -1-j & 0 \end{pmatrix},\ D_4 = \begin{pmatrix} i+2k & 3+j+5k & 4+k \\ 1+j & -2-6i+5j+4k & -i+j+3k \\ -1+i+3j+k & i+7k & -i+3k \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 7i-2j+9k & 2+10i+13j+10k & 4i+6j \\ -7+8i-j+7k & -12+2i+10j-k & -5+5i+j-k \\ -7+4i-4j+6k & -8+2i+6j-4k & -11+9i+j-8k \end{pmatrix}.$$

Computing directly yields

$$r\begin{pmatrix} A_i & B_i \end{pmatrix} = r(A_i) = 2, \quad r\begin{pmatrix} C_j \\ D_j \end{pmatrix} = r(C_j) = 2\ (i = \overline{1,4},\ j = \overline{1,3}),$$

$$(26) = 4,\ (27) = 10,\ (28) = 10,\ (29) = 10,\ (30) = 10,$$

$$(31) = 8,\ (32) = 8,\ (33) = 8,\ (34) = 8,\ (35) = 24.$$

All rank equations hold. Thus, according to Theorem 1, the system of matrix equations has a solution, and the general solution to the system can be expressed as

$$X=\begin{pmatrix}1+j & -1\\ 2+k & 1+i-j\end{pmatrix},\quad Y=\begin{pmatrix}1+i-j & 0\\ 1 & 2i-j+k\end{pmatrix},\quad Z=\begin{pmatrix}1+i & 1+k\\ 1+j+k & i+k\\ i+k & 0\end{pmatrix},$$

$$V=\begin{pmatrix}1 & 2+i-k & 1+j\\ 1-j+k & i+j & 0\end{pmatrix},\quad W=\begin{pmatrix}j & 3i+j+k & j\\ 1+i & i+j & -i+2j\end{pmatrix},$$

5. The General Solution to the System (7) with η-Hermicity

As an application of the results of system (6), we study the necessary and sufficient conditions for system (7) to have a solution involving η-Hermicity and derive a formula of its general solution, where X, Y are η-Hermitian matrices.

Theorem 3. *Given A_i, B_j $(i=1,2,7,8,\ j=1,2,\overline{5,8}), C_3, D_3, D_4$ of appropriate dimensions over $\mathbb{H}$. Set*

$$A_{22}=A_6L_{A_4},\ A_{33}=A_7L_{A_1},\ A_{44}=A_8L_{A_2},$$

$$C_{22}=D_5-A_6^{\dagger}A_4B_4-A_7A_1^{\dagger}(A_1^{\dagger})^{\eta^*}+L_{A_1}C_1^{\eta^*}(C_1^{\dagger})^{\eta^*}A_7^{\eta^*}-A_8A_2^{\dagger}(A_2^{\dagger})^{\eta^*}+L_{A_2}C_2^{\eta^*}(C_2^{\dagger})^{\eta^*})A_8^{\eta^*},$$

$$M_1=R_{A_{22}}A_{33},\ M_2=R_{A_{22}}A_{44},\ T=R_{A_{22}}C_{22}R_{A_{22}}^{\eta^*},\ M=R_{M_1}M_2,\ S=M_2L_M.$$

Then, the following statements are equivalent:
(1) *System (7) has a solution.*
(2)

$$R_{A_1}B_1=0,\ R_{A_2}B_2=0,\ R_{A_4}B_4=0,$$
$$R_{M_1}R_MT=0,\ R_{A_{22}}T(R_{A_{44}})^{\eta^*}=0.$$

(3)

$$r\begin{pmatrix}A_1 & B_1\end{pmatrix}=r(A_1),\quad r\begin{pmatrix}A_2 & B_2\end{pmatrix}=r(A_2),\ r\begin{pmatrix}A_4 & B_4\end{pmatrix}=r(A_4),$$

$$r\begin{pmatrix}D_5 & A_6 & B_4^{\eta^*} & A_7B_1^{\eta^*} & A_8B_2^{\eta^*}\\ A_6^{\eta^*} & 0 & A_4^{\eta^*} & 0 & 0\\ A_7^{\eta^*} & 0 & 0 & A_1^{\eta^*} & 0\\ A_8^{\eta^*} & 0 & 0 & 0 & A_2^{\eta^*}\\ B_4 & A_4 & 0 & 0 & 0\end{pmatrix}=r\begin{pmatrix}A_6^{\eta^*} & A_4^{\eta^*} & 0 & 0\\ A_7^{\eta^*} & 0 & A_1^{\eta^*} & 0\\ A_8^{\eta^*} & 0 & 0 & A_1^{\eta^*}\end{pmatrix}+r\begin{pmatrix}A_4\\ A_6\end{pmatrix},$$

$$r\begin{pmatrix}D_5 & A_6 & A_7 & B_4^{\eta^*} & A_8B_2^{\eta^*}\\ A_6^{\eta^*} & 0 & 0 & A_4^{\eta^*} & 0\\ A_8^{\eta^*} & 0 & 0 & 0 & A_2^{\eta^*}\\ B_4 & A_4 & 0 & 0 & 0\\ B_1A_7^{\eta^*} & 0 & A_1 & 0 & 0\end{pmatrix}=r\begin{pmatrix}A_6^{\eta^*} & A_4^{\eta^*} & 0\\ A_8^{\eta^*} & 0 & A_2^{\eta^*}\end{pmatrix}+r\begin{pmatrix}A_6 & A_7\\ A_4 & 0\\ 0 & A_1\end{pmatrix}.$$

Under these conditions, the general solution with η-Hermicity to the system (7) can be stated as

$$\begin{aligned}
&V = A_4^{\dagger}B_4 + L_{A_4}U_1,\\
&X = A_1^{\dagger}B_1 + L_{A_1}B_1^{\eta^*}(A_1^{\dagger})^{\eta^*} + L_{A_1}U_2L_{A_1}^{\eta^*},\\
&Y = A_2^{\dagger}B_2 + L_{A_2}B_2^{\eta^*}(A_2^{\dagger})^{\eta^*} + L_{A_2}U_3L_{A_2}^{\eta^*},\\
&U_1 = A_{22}^{\dagger}(C_{22} - A_{33}U_2A_{33}^{\eta^*} - A_{44}U_3A_{44}^{\eta^*}) - A_{22}^{\dagger}W_2A_{22}^{\eta^*} + L_{A_{22}}W_1,\\
&U_2 = M_1^{\dagger}TM_1^{\eta^*} - M_1^{\dagger}M_2M^{\dagger}TM_1^{\eta^*} - M_1^{\dagger}SM_2^{\dagger}T(M^{\eta^*})^{\dagger}M_2^{\eta^*}M_1^{\eta^*} - M_1^{\dagger}SV_4(L_M)^{\eta^*}M_2^{\eta^*}M_1^{\eta^*}\\
&+ L_{M_1}V_1 + V_2(L_M)^{\eta^*},\\
&U_3 = M^{\dagger}TM_2^{\eta^*} + S^{\dagger}SM_2^{\dagger}TM^{\eta^*} + L_ML_SV_3 + L_MV_4(L_M)^{\eta^*} + V_5(L_{M_2})^{\eta^*},
\end{aligned}$$

where V_i $(i = \overline{1,5})$ and W_j $(j = \overline{1,3})$ are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

Proof. Since the solvability of the system (7) is equivalent to the system

$$\begin{aligned}
A_4V_1 = B_4,\ V_2(A_4)^{\eta^*} = (B_4)^{\eta^*},\ V_2 = (V_1)^{\eta^*},\\
A_1X_1 = B_1,\ X_1A_1^{\eta^*} = B_1^{\eta^*},\ X_1 = X_1^{\eta^*},\\
A_2Y_1 = B_2,\ Y_1A_2^{\eta^*} = B_2^{\eta^*},\ Y_1 = Y_1^{\eta^*},\\
A_6V_1 + V_2A_6^{\eta^*} + A_7X_1A_7^{\eta^*} + A_8Y_1A_8^{\eta^*} = D_5,\ D_5 = D_5^{\eta^*}.
\end{aligned} \tag{59}$$

If system (7) has a solution, say, (V, X, Y), then system (59) has a solution, $(V_1, V_2, X_1, Y_1) = (V, V^{\eta^*}, X, Y)$. Conversely, if system (59) has a solution (V_1, V_2, X_1, Y_1), then

$$(V, X, Y) = (\frac{V_1 + V_2^{\eta^*}}{2}, \frac{X_1 + X_1^{\eta^*}}{2}, \frac{X_2 + X_2^{\eta^*}}{2})$$

is the solution of (7). It follows from Corollary 1 that this proof can be completed. □

6. Conclusions

We established the solvability conditions for system (6) by using the Moore–Penrose inverses and ranks of the coefficient quaternion matrices in (6), and derived a formula of its general solution when it is solvable. In terms of applications, we derived the necessary and sufficient conditions for system (7) to have an η-Hermitian solution as well as the expression of the general solution. In addition, we used an algorithm and a numerical example to verify the main results of this paper. It is worth noting that the main results of (6) are available not only for $\mathbb{R}$ and $\mathbb{C}$, but also any division ring. Moreover, inspired by [39], we can investigate the system (6) tensor equations over the quaternion algebra.

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