

Solving a System of Sylvester-like Quaternion Matrix Equations

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Abstract: Using the ranks and Moore–Penrose inverses of involved matrices, in this paper we establish some necessary and sufficient solvability conditions for a system of Sylvester-type quaternion matrix equations, and give an expression of the general solution to the system when it is solvable. As an application of the system, we consider a special symmetry solution, named the η -Hermitian solution, for a system of quaternion matrix equations. Moreover, we present an algorithm and a numerical example to verify the main results of this paper.

Keywords: Sylvester-type matrix equation; quaternion matrix; rank; Moore–Penrose inverse; η -Hermitian matrix

1. Introduction

In 1952, Roth [1] studied the following one-sided generalized Sylvester matrix equation for the first time

$$A_1X + YB_1 = C_1, \quad (1)$$

which is widely used in system and control theory. Since then, many researches have paid attention to Sylvester-type matrix equations (e.g., [2–5]) because of their wide range of applications, such as in descriptor system control theory [6], neural networks [7], robust feedback [8], graph theory [9] and other areas. For instance, Baksalary and Kala [10] established a necessary and sufficient condition for Equation (1) to have a solution and gave an expression of its general solution. In [11], Baksalary and Kala give a solvability condition for the equation

$$AXB + CYD = E. \quad (2)$$

Wang investigated Equation (2) over arbitrary regular rings with identity [12].

In 1843, the very famous mathematician Hamilton discovered the quaternion. It is well known that quaternion algebra, denoted by \mathbb{H} , is an associative and non-commutative division algebra over the real number field \mathbb{R} , where

$$\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Since the 1970s, quaternions and the quaternion matrix have been studied a lot (e.g., [13–16]). The widespread applications of quaternions and the quaternion matrix include theoretical mechanics, optics, computer graphics, flight mechanics and aerospace technology, quantum physics, signal processing and so on (e.g., [17–20]).



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In the last decade, the study of Sylvester-type matrix equations was extended to \mathbb{H} (e.g., [21–28]). In 2012, Wang and He [29] presented the necessary and sufficient conditions for the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{3}$$

to be consistent and derived the expression of its general solution, which can be easily generalized to \mathbb{H} . For the Sylvester-type matrix equations with multiple variables and multiple equations, Wang [4] gave a solvability condition and the general solution to the system of Sylvester-type matrix equations

$$\begin{aligned} A_3W &= B_3, \quad ZC_3 = D_3, \\ A_5W + ZB_5 &= D_4. \end{aligned} \tag{4}$$

Zhang [30] investigated the necessary and sufficient conditions for the solvability of the following system of Sylvester-like matrix equations

$$\begin{aligned} A_1X &= B_1, \quad XC_1 = D_1, \\ A_2Y &= B_2, \quad YC_2 = D_2, \\ ZC_3 &= D_3, \quad A_4V = B_4, \\ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 &= D_5, \end{aligned} \tag{5}$$

and presented a formula of its general solution. We note that Equations (1)–(5) are the special cases of the following Sylvester-type quaternion matrix equations

$$\begin{aligned} A_1X &= B_1, \quad XC_1 = D_1, \\ A_2Y &= B_2, \quad YC_2 = D_2, \\ A_3W &= B_3, \quad ZC_3 = D_3, \\ A_5W + ZB_5 &= D_4, \quad A_4V = B_4, \\ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 &= D_5, \end{aligned} \tag{6}$$

where A_i, B_i, C_j, D_k ($i = \overline{1,8}, j = \overline{1,3}, k = \overline{1,5}$) are given matrices over \mathbb{H} ; X, Y, Z, V, W are unknown.

Motivated by the work mentioned above, in this paper we aim to investigate the solvability conditions and the general solutions to a more general system of a Sylvester-type quaternion matrix equation, Equation (6). In 2011, Took et al. [31] defined a special class of symmetric matrices, named η -Hermitian. For $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a quaternion matrix A is called η -Hermitian if $A = A^{\eta*}$, where $A^{\eta*} = -\eta A^* \eta$, A^* is the conjugate and transpose matrix of A . It is well known that η -Hermitian matrices have some applications in linear modeling (e.g., [32–34]) and so on.

As an application of (6), we derive the solvability conditions and an expression of the η -Hermitian solution to the system of matrix equations

$$\begin{aligned} A_4V &= B_4, \\ A_1X &= B_1, \quad X = X^{\eta*}, \\ A_2Y &= B_2, \quad Y = Y^{\eta*}, \\ A_6V + (A_6V)^{\eta*} + A_7XA_7^{\eta*} + A_8YA_8^{\eta*} &= D_5, \quad D_5 = D_5^{\eta*}, \end{aligned} \tag{7}$$

where $A_i (i = 1, 2, 4, \overline{6, 8}), B_1, B_2, B_4, D_5$ are given matrices over \mathbb{H} ; X and Y are η -Hermitian matrices over \mathbb{H} .

We organize the rest of this article as follows: In Section 2, we introduce the basic knowledge of quaternions and Moore–Penrose inverse of a quaternion matrix, and review some matrix equations. In Section 3, we establish the solvability conditions for the system of (6) in terms of the Moore–Penrose inverses and the ranks of the coefficients’ quaternion matrices in (6). In Section 4, we give an expression of the general solution to the system of (6), and illustrate the main results using a numerical example. In Section 5, we give some solvability conditions and an expression of the η -Hermitian solution to the system (7). Finally, we present a brief conclusion in Section 6 to end this paper.

2. Preliminaries

Let \mathbb{R} and $\mathbb{H}^{m \times n}$ stand for the real number field and the set of all $m \times n$ matrix spaces over the quaternion algebra, respectively. The symbols $r(A), A^*, I$ and 0 are denoted by the rank of a given quaternion matrix A , the conjugate transpose of A , an identity matrix, and a zero matrix with appropriate sizes, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{l \times k}$ is defined to be the unique matrix, denoted by A^\dagger , satisfying

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A.$$

Moreover, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ represent two projectors. Clearly, $(L_A)^\eta = R_{A^\eta}$ and $(R_A)^\eta = L_{A^\eta}$ of A .

The following lemma was given by Marsaglia and Stynan [35], which is also available over \mathbb{H} .

Lemma 1 ([35]). Let $A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, C \in \mathbb{H}^{l \times n}, D \in \mathbb{H}^{j \times k}$ and $E \in \mathbb{H}^{l \times i}$. Then,

$$r \begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r \begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 2 ([36]). Let A_1 and C_1 be known matrices with feasible dimensions over \mathbb{H} . Then, the matrix equation $A_1 X = C_1$ has a solution if and only if $R_{A_1} C_1 = 0$. In this case, its general solution is expressed as

$$X = A_1^\dagger C_1 + L_{A_1} T_1,$$

where T_1 is an arbitrary matrix of an appropriate size.

Lemma 3 ([36]). Let B_1 and D_1 be known matrices with allowable dimensions over \mathbb{H} . Then, the matrix equation $Y B_1 = D_1$ has a solution if and only if $D_1 L_{B_1} = 0$. In this case, its general solution is

$$Y = D_1 B_1^\dagger + T_2 R_{B_1},$$

where T_2 is an arbitrary matrix of an appropriate size.

Lemma 4 ([37]). Let A_1, B_1, C_1 and C_2 be the given matrices. Then, the system of matrix equations

$$A_1 Y = C_1, Y B_1 = C_2$$

is consistent if and only if

$$R_{A_1}C_1 = 0, C_2L_{B_1} = 0, A_1C_2 = C_1B_1.$$

In this case, its general solution is

$$Y = A_1^\dagger C_1 + L_{A_1} C_2 B_1^\dagger + L_{A_1} T_3 R_{B_1},$$

where T_3 is an arbitrary matrix of an appropriate size.

Lemma 5 ([10]). Let A, B and C be given over \mathbb{H} . Then, the Equation (1) is solvable if and only if $R_A C L_B = 0$. Under this condition, the general solution to Equation (1) can be expressed as

$$\begin{aligned} X &= A^\dagger C - U_1 B + L_A U_2, \\ Y &= R_A C B^\dagger + A U_1 + U_3 R_B, \end{aligned}$$

where U_1, U_2 and U_3 are arbitrary matrices with appropriate sizes over \mathbb{H} .

Lemma 6 ([38]). Consider the following matrix equation over \mathbb{H}

$$A_1 X_1 + X_2 B_1 + A_2 Y_1 B_2 + A_3 Y_2 B_3 + A_4 Y_3 B_4 = B, \tag{8}$$

where $A_i, B_i (i = \overline{1,4}), B$ are given and the others are unknown. Let

$$\begin{aligned} R_{A_1} A_2 &= A_{11}, R_{A_1} A_3 = A_{22}, R_{A_1} A_4 = A_{33}, B_2 L_{B_1} = B_{11}, B_{22} L_{B_{11}} = N_1, \\ B_3 L_{B_1} &= B_{22}, B_4 L_{B_1} = B_{33}, R_{A_{11}} A_{22} = M_1, S_1 = A_{22} L_{M_1}, R_{A_1} B L_{B_1} = T_1, \\ C &= R_{M_1} R_{A_{11}}, C_1 = C A_{33}, C_2 = R_{A_{11}} A_{33}, C_3 = R_{A_{22}} A_{33}, C_4 = A_{33}, \\ D &= L_{B_{11}} L_{N_1}, D_1 = B_{33}, D_2 = B_{33} L_{B_{22}}, D_3 = B_{33} L_{B_{11}}, D_4 = B_{33} D, \\ E_1 &= C T_1, E_2 = R_{A_{11}} T_1 L_{B_{22}}, E_3 = R_{A_{22}} T_1 L_{B_{11}}, E_4 = T_1 D, \\ C_{11} &= (L_{C_2}, L_{C_4}), D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, C_{22} = L_{C_1}, D_{22} = R_{D_2}, C_{33} = L_{C_3}, \\ D_{33} &= R_{D_4}, E_{11} = R_{C_{11}} C_{22}, E_{22} = R_{C_{11}} C_{33}, E_{33} = D_{22} L_{D_{11}}, E_{44} = D_{33} L_{D_{11}}, \\ M &= R_{E_{11}} E_{22}, N = E_{44} L_{E_{33}}, F = F_2 - F_1, E = R_{C_{11}} F L_{D_{11}}, S = E_{22} L_M, \\ F_{11} &= C_2 L_{C_1}, G_1 = E_2 - C_2 C_1^\dagger E_1 D_1^\dagger D_2, F_{22} = C_4 L_{C_3}, G_2 = E_4 - C_4 C_3^\dagger E_3 D_3^\dagger D_4, \\ F_1 &= C_1^\dagger E_1 D_1^\dagger + L_{C_1} C_2^\dagger E_2 D_2^\dagger, F_2 = C_3^\dagger E_3 D_3^\dagger + L_{C_3} C_4^\dagger E_4 D_4^\dagger. \end{aligned}$$

Then, the following statements are equivalent:

- (1) Equation (8) is consistent.
- (2)

$$R_{C_i} E_i = 0, E_i L_{D_i} = 0 (i = \overline{1,4}), R_{E_{22}} E L_{E_{33}} = 0.$$

- (3)

$$\begin{aligned}
r \begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} &= r(B_1) + r \begin{pmatrix} A_2 & A_3 & A_4 & A_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} A_2 & A_4 & A_1 \end{pmatrix} + r \begin{pmatrix} B_3 \\ B_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} A_3 & A_4 & A_1 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r \begin{pmatrix} A_4 & A_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} A_2 & A_3 & A_1 \end{pmatrix} + r \begin{pmatrix} B_4 \\ B_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r \begin{pmatrix} A_2 & A_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r \begin{pmatrix} A_3 & A_1 \end{pmatrix}, \\
r \begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} &= r \begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1), \\
r \begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r \begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}.
\end{aligned}$$

In this case, the general solution to Equation (8) can be expressed as

$$\begin{aligned}
 X_1 &= A_1^\dagger(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^\dagger U_1B_1 + L_{A_1}U_2, \\
 X_2 &= R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^\dagger + A_1A_1^\dagger U_1 + U_3R_{B_1}, \\
 Y_1 &= A_{11}^\dagger TB_{11}^\dagger - A_{11}^\dagger A_{22}M_1^\dagger TB_{11}^\dagger - A_{11}^\dagger S_1A_{22}^\dagger TN_1^\dagger B_{22}B_{11}^\dagger - A_{11}^\dagger S_1U_4R_{N_1}B_{22}B_{11}^\dagger + L_{A_{11}}U_5 + U_6R_{B_{11}}, \\
 Y_2 &= M_1^\dagger TB_{22}^\dagger + S_1^\dagger S_1A_{22}^\dagger TN_1^\dagger + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1}, \\
 Y_3 &= F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2}, \text{ or } Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},
 \end{aligned}$$

where $T = T_1 - A_{33}Y_3B_{33}$; $U_i (i = \overline{1,8})$ represents any matrix with appropriate dimensions over \mathbb{H} ,

$$\begin{aligned}
 V_1 &= (I_m \ 0) \left[C_{11}^\dagger (F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^\dagger U_{11}D_{11} + L_{C_{11}}U_{12} \right], \\
 W_1 &= (0 \ I_m) \left[C_{11}^\dagger (F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^\dagger U_{11}D_{11} + L_{C_{11}}U_{12} \right], \\
 W_2 &= \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger + C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}} \right] \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\
 V_2 &= \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^\dagger + C_{11}C_{11}^\dagger U_{11} + U_{21}R_{D_{11}} \right] \begin{pmatrix} I_n \\ 0 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 V_3 &= E_{11}^\dagger FE_{33}^\dagger - E_{11}^\dagger E_{22}M^\dagger FE_{33}^\dagger - E_{11}^\dagger SE_{22}^\dagger FN^\dagger E_{44}E_{33}^\dagger - E_{11}^\dagger SU_{31}R_N E_{44}E_{33}^\dagger + L_{E_{11}}U_{32} + U_{33}R_{E_{33}}, \\
 W_3 &= M^\dagger FE_{44}^\dagger + S^\dagger SE_{22}^\dagger FN^\dagger + L_M L_S U_{41} + L_M U_{31}R_N - U_{42}R_{E_{44}},
 \end{aligned}$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ and U_{42} are any matrix with appropriate dimensions over \mathbb{H} .

3. Solvability Conditions to the System (6)

The goal of this section is to give the necessary and sufficient conditions for the existence of a solution to system (6).

Theorem 1. Let $A_i \in \mathbb{H}^{m_i \times n_i} (i = \overline{1,4}), A_5 \in \mathbb{H}^{m \times n_3}, A_6 \in \mathbb{H}^{m \times n_4}, A_7 \in \mathbb{H}^{m \times n_1}, A_8 \in \mathbb{H}^{m \times n_2}, B_j \in \mathbb{H}^{m_j \times l_j} (j = \overline{1,2}), B_3 \in \mathbb{H}^{m_3 \times q}, B_4 \in \mathbb{H}^{m_4 \times l}, B_5 \in \mathbb{H}^{l_3 \times q}, B_6 \in \mathbb{H}^{l_3 \times l}, B_7 \in \mathbb{H}^{l_1 \times l}, B_8 \in \mathbb{H}^{l_2 \times l}, C_k \in \mathbb{H}^{l_k \times p_k} (k = \overline{1,3}), D_j \in \mathbb{H}^{n_j \times p_j} (j = \overline{1,2}), D_3 \in \mathbb{H}^{m \times l_2}, D_4 \in \mathbb{H}^{m \times q}$ and $D_5 \in \mathbb{H}^{m \times l}$. Set

$$A_{11} = A_5L_{A_3}, B_{11} = R_{C_3}B_5, C_{11} = D_4 - A_5A_3^\dagger B_3 - D_3C_3^\dagger B_5, A_{22} = A_6L_{A_4}, \tag{9}$$

$$B_{22} = R_{B_{11}}R_{C_3}B_6, A_{33} = A_7L_{A_1}, B_{33} = R_{C_1}B_7, A_{44} = A_8L_{A_2}, B_{44} = R_{C_2}B_8, \tag{10}$$

$$A_{55} = A_{11}, B_{55} = R_{C_3}B_6, M_1 = R_{A_{22}}A_{33}, M_2 = R_{A_{22}}A_{44}, M_3 = R_{A_{22}}A_{55}, \tag{11}$$

$$\begin{aligned}
 C_{22} &= D_5 - A_6A_4^\dagger B_4 - D_3C_3^\dagger B_6 - R_{A_{11}}C_{11}B_{11}^\dagger R_{C_3}B_6 \\
 &\quad - A_7(A_1^\dagger B_1 + L_{A_1}D_1C_1^\dagger)B_7 - A_8(A_2^\dagger B_2 + L_{A_2}D_2C_2^\dagger)B_8,
 \end{aligned} \tag{12}$$

$$N_1 = B_{33}L_{B_{22}}, N_2 = B_{44}L_{B_{22}}, N_3 = B_{55}L_{B_{22}}, G_1 = N_2L_{N_1}, H_1 = R_{M_1}M_2, \tag{13}$$

$$S_1 = M_2L_{H_1}, T = R_{A_{22}}C_{22}L_{B_{22}}, P = R_{H_1}R_{M_1}, P_1 = PM_3, P_2 = R_{M_1}M_3, \tag{14}$$

$$P_3 = R_{M_2}M_3, P_4 = M_3, Q = L_{N_1}L_{G_1}, Q_1 = N_3, Q_2 = N_3L_{N_2}, Q_3 = N_3L_{N_1}, \tag{15}$$

$$Q_4 = N_3Q, E_1 = PT, E_2 = R_{M_1}TL_{N_2}, E_3 = R_{M_2}TL_{N_1}, E_4 = TQ, \tag{16}$$

$$E_{11} = (L_{P_2}, L_{P_4}), F_{11} = \begin{pmatrix} R_{Q_1} \\ R_{Q_3} \end{pmatrix}, E_{22} = L_{P_1}, F_{22} = R_{Q_2}, E_{33} = L_{P_3}, F_{33} = R_{Q_4}, \tag{17}$$

$$M_{11} = R_{E_{11}}E_{22}, M_{22} = R_{E_{11}}E_{33}, M_{33} = F_{22}L_{F_{11}}, M_{44} = F_{33}L_{F_{11}}, M = R_{M_{11}}M_{22}, \tag{18}$$

$$N = M_{44}L_{M_{33}}, F = F_2 - F_1, E = R_{E_{11}}FL_{F_{11}}, S = M_{22}L_M, G_{11} = P_2L_{P_1}, \tag{19}$$

$$H_{11} = E_2 - P_2P_1^\dagger E_1Q_1^\dagger Q_2, G_{22} = P_4L_{P_3}, H_{22} = E_4 - P_4P_3^\dagger E_3Q_3^\dagger Q_4, \tag{20}$$

$$F_2 = P_1^\dagger E_1Q_1^\dagger + L_{P_1}P_2^\dagger E_2Q_2^\dagger, F_1 = P_3^\dagger E_3Q_3^\dagger + L_{P_3}P_4^\dagger E_4Q_4^\dagger. \tag{21}$$

Then, the following statements are equivalent:

(1) System (6) has a solution.

(2)

$$A_1D_1 = B_1C_1, A_2D_2 = B_2C_2 \tag{22}$$

and

$$\begin{aligned} R_{A_1}B_1 &= 0, D_1L_{C_1} = 0, R_{A_2}B_2 = 0, D_2L_{C_2} = 0, \\ R_{A_3}B_3 &= 0, D_3L_{C_3} = 0, R_{A_4}B_4 = 0, R_{A_{11}}C_{11}L_{B_{11}} = 0, \\ R_{P_i}E_i &= 0, E_iL_{Q_i} = 0 (i = \overline{1,4}), R_{M_{22}}EL_{M_{33}} = 0. \end{aligned} \tag{23}$$

(3) (22) holds and

$$r(A_1 \ B_1) = r(A_1), r\begin{pmatrix} C_1 \\ D_1 \end{pmatrix} = r(C_1), r(A_2 \ B_2) = r(A_2), r\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = r(C_2), \tag{24}$$

$$r(A_3 \ B_3) = r(A_3), r\begin{pmatrix} C_3 \\ D_3 \end{pmatrix} = r(C_3), r(A_4 \ B_4) = r(A_4), \tag{25}$$

$$r\begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r(B_5 \ C_3), \tag{26}$$

$$r\begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_4 & A_5 & D_3 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_8 & A_5 & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & 0 \end{pmatrix} + r(B_6 \ B_5 \ C_3), \tag{27}$$

$$r\begin{pmatrix} D_5 & A_7 & A_6 & A_8D_2 & D_4 & A_5 & D_3 \\ B_8 & 0 & 0 & C_2 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_6 & A_5 \\ A_1 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_8 & C_2 & 0 & 0 \\ B_6 & 0 & B_5 & C_3 \end{pmatrix}, \tag{28}$$

$$r\begin{pmatrix} D_5 & A_8 & A_6 & A_7D_1 & D_4 & A_5 & D_3 \\ B_7 & 0 & 0 & C_1 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_8 & A_6 & A_5 \\ A_2 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_7 & C_1 & 0 & 0 \\ B_6 & 0 & B_5 & C_3 \end{pmatrix}, \tag{29}$$

$$r \begin{pmatrix} D_5 & A_6 & A_7D_1 & A_8D_2 & D_4 & A_5 & D_3 \\ B_7 & 0 & C_1 & 0 & 0 & 0 & 0 \\ B_8 & 0 & 0 & C_2 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} \tag{30}$$

$$= r \begin{pmatrix} B_7 & C_1 & 0 & 0 & 0 \\ B_8 & 0 & C_2 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & C_3 \end{pmatrix} + r \begin{pmatrix} A_6 & A_5 \\ A_4 & 0 \\ 0 & A_3 \end{pmatrix},$$

$$r \begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_3 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 \\ B_2B_8 & 0 & A_2 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_7 & A_8 & A_6 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_4 \end{pmatrix} + r \begin{pmatrix} B_6 & C_3 \end{pmatrix}, \tag{31}$$

$$r \begin{pmatrix} D_5 & A_7 & A_6 & A_8D_2 & D_3 \\ B_8 & 0 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_1B_7 & A_1 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_8 & C_2 & 0 \\ B_6 & 0 & C_3 \end{pmatrix} + r \begin{pmatrix} A_7 & A_6 \\ A_1 & 0 \\ 0 & A_4 \end{pmatrix}, \tag{32}$$

$$r \begin{pmatrix} D_5 & A_8 & A_6 & A_7D_1 & D_3 \\ B_7 & 0 & 0 & C_1 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_2B_8 & A_2 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_7 & C_1 & 0 \\ B_6 & 0 & C_3 \end{pmatrix} + r \begin{pmatrix} A_8 & A_6 \\ A_2 & 0 \\ 0 & A_4 \end{pmatrix}, \tag{33}$$

$$r \begin{pmatrix} D_5 & A_6 & A_7D_1 & A_8D_2 & D_3 \\ B_7 & 0 & C_1 & 0 & 0 \\ B_8 & 0 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & 0 & C_3 \\ B_4 & A_4 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_7 & C_1 & 0 & 0 \\ B_8 & 0 & C_2 & 0 \\ B_6 & 0 & 0 & C_3 \end{pmatrix} + r \begin{pmatrix} A_6 \\ A_4 \end{pmatrix}, \tag{34}$$

$$r \begin{pmatrix} D_5 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & D_4 & 0 & 0 & 0 & A_5 & D_3 & 0 & 0 \\ B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & D_5 & A_8 & A_6 & 0 & 0 & A_7D_1 & D_4 & 0 & 0 & 0 & A_5 & D_3 \\ 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\ B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 \end{pmatrix}$$

$$= r \begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} A_7 & A_6 & 0 & 0 & A_5 & 0 \\ 0 & 0 & A_8 & A_6 & 0 & A_5 \\ A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_3 \end{pmatrix}. \tag{35}$$

Proof. (1) \Leftrightarrow (2)

The proof is divided into three parts:

- Firstly, we divide the system (6) into the following:

$$\begin{aligned} A_3W &= B_3, \quad ZC_3 = D_3, \quad A_4V = B_4, \\ A_1X &= B_1, \quad XC_1 = D_1, \quad A_2Y = B_2, \quad YC_2 = D_2, \end{aligned} \tag{36}$$

$$A_5Z + WB_5 = D_4, \tag{37}$$

$$A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \tag{38}$$

and consider the solvability conditions and the general solution to the system of matrices of Equation (36). For more information, see Step 1.

- Secondly, substituting the W and Z obtained in the first step into Equation (37) yields

$$A_{11}T_3 + T_4B_{11} = C_{11}, \tag{39}$$

where A_{11}, B_{11} and C_{11} are defined by (9); T_3 and T_4 are unknowns. For more information, see Step 2.

- Finally, by substituting the X, Y, Z, and V obtained from the above two steps into Equation (38), we obtain a matrix equation with the following form

$$A_{22}T_5 + U_3B_{22} + A_{33}T_1B_{33} + A_{44}T_2B_{44} + A_{55}U_1B_{55} = C_{22}, \tag{40}$$

where A_{ii}, B_{ii} ($i = \overline{2,5}$) and C_{22} are given by (9)–(12); T_1, T_2, T_5, U_1 and U_3 are unknowns. For more information, see Step 3.

We can obtain the results from the following steps: First, we consider the solvability conditions and the expression of the general solutions to the system of the matrix Equation (36).

Step 1. It follows from Lemmas 2–4 that system (36) has a solution if and only if (22) holds and

$$\begin{aligned} R_{A_1}B_1 &= 0, \quad D_1L_{C_1} = 0, \quad R_{A_2}B_2 = 0, \quad D_2L_{C_2} = 0, \\ R_{A_3}B_3 &= 0, \quad D_3L_{C_3} = 0, \quad R_{A_4}B_4 = 0. \end{aligned} \tag{41}$$

In this case, the general solution to system (36) can be written as

$$\begin{aligned} X &= A_1^\dagger B_1 + L_{A_1} D_1 C_1^\dagger + L_{A_1} T_1 R_{C_1}, \\ Y &= A_2^\dagger B_2 + L_{A_2} D_2 C_2^\dagger + L_{A_2} T_2 R_{C_2}, \\ W &= A_3^\dagger B_3 + L_{A_3} T_3, \quad Z = D_3 C_3^\dagger + T_4 R_{C_3}, \quad V = A_4^\dagger B_4 + L_{A_4} T_5, \end{aligned} \tag{42}$$

where T_i ($i = \overline{1,5}$) are arbitrary matrices over \mathbb{H} with appropriate sizes.

Step 2. Substituting W, Z in (42) into (37) yields (39). According to Lemma 5, it follows that Equation (39) has a solution if and only if

$$R_{A_{11}}C_{11}L_{B_{11}} = 0. \tag{43}$$

In this case, the general solution to Equation (39) can be expressed as

$$T_3 = A_{11}^\dagger C_1 - U_1 B_{11} + L_{A_{11}} U_2, \tag{44}$$

$$T_4 = R_{A_{11}} C_{11} B_{11}^\dagger + A_{11} U_1 + U_3 R_{B_{11}}, \tag{45}$$

where U_1, U_2 and U_3 are any matrix with appropriate sizes over \mathbb{H} .

Substituting (45) into $Z = D_3 C_3^\dagger + T_4 R_{C_3}$ yields

$$Z = D_3 C_3^\dagger + R_{A_{11}} C_{11} B_{11}^\dagger R_{C_3} + A_{11} U_1 R_{C_3} + U_3 R_{B_{11}} R_{C_3}. \tag{46}$$

Step 3. By substituting X, Y, V in (42) and Z in (46) into (38), we obtain Equation (40). By using Lemma 6, Equation (40) is consistent if and only if

$$R_{P_i} E_i = 0, E_i L_{Q_i} = 0 (i = \overline{1,4}), R_{M_{22}} E L_{M_{33}} = 0, \tag{47}$$

namely,

$$r \begin{pmatrix} C_{22} & A_{33} & A_{44} & A_{55} & A_{22} \\ B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_{22}) + r \begin{pmatrix} A_{33} & A_{44} & A_{55} & A_{22} \end{pmatrix}, \tag{48}$$

$$r \begin{pmatrix} C_{22} & A_{33} & A_{55} & A_{22} \\ B_{44} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_{33} & A_{55} & A_{22} \end{pmatrix} + r \begin{pmatrix} B_{44} \\ B_{22} \end{pmatrix}, \tag{49}$$

$$r \begin{pmatrix} C_{22} & A_{44} & A_{55} & A_{22} \\ B_{33} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_{44} & A_{55} & A_{22} \end{pmatrix} + r \begin{pmatrix} B_{33} \\ B_{22} \end{pmatrix}, \tag{50}$$

$$r \begin{pmatrix} C_{22} & A_{55} & A_{22} \\ B_{33} & 0 & 0 \\ B_{44} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{33} \\ B_{44} \\ B_{22} \end{pmatrix} + r \begin{pmatrix} A_{55} & A_{22} \end{pmatrix}, \tag{51}$$

$$r \begin{pmatrix} C_{22} & A_{33} & A_{44} & A_{22} \\ B_{55} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_{33} & A_{44} & A_{22} \end{pmatrix} + r \begin{pmatrix} B_{55} \\ B_{22} \end{pmatrix}, \tag{52}$$

$$r \begin{pmatrix} C_{22} & A_{33} & A_{22} \\ B_{44} & 0 & 0 \\ B_{55} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{44} \\ B_{55} \\ B_{22} \end{pmatrix} + r \begin{pmatrix} A_{33} & A_{22} \end{pmatrix}, \tag{53}$$

$$r \begin{pmatrix} C_{22} & A_{44} & A_{22} \\ B_{33} & 0 & 0 \\ B_{55} & 0 & 0 \\ B_{22} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{33} \\ B_{55} \\ B_{22} \end{pmatrix} + r \begin{pmatrix} A_{44} & A_{22} \end{pmatrix}, \tag{54}$$

$$r \begin{pmatrix} C_{22} & A_{22} \\ B_{33} & 0 \\ B_{44} & 0 \\ B_{55} & 0 \\ B_{22} & 0 \end{pmatrix} = r \begin{pmatrix} B_{33} \\ B_{44} \\ B_{55} \\ B_{22} \end{pmatrix} + r(A_{22}), \tag{55}$$

$$r \begin{pmatrix} C_{22} & A_{33} & A_{22} & 0 & 0 & 0 & A_{55} \\ B_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_{44} & A_{22} & A_{55} \\ 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\ B_{55} & 0 & 0 & B_{55} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{44} & 0 \\ B_{22} & 0 \\ 0 & B_{33} \\ 0 & B_{22} \\ B_{55} & B_{55} \end{pmatrix} + r \begin{pmatrix} A_{33} & A_{22} & 0 & 0 & A_{55} \\ 0 & 0 & A_{44} & A_{22} & A_{55} \end{pmatrix}. \tag{56}$$

In this case, the general solution to Equation (40) can be expressed as

$$\begin{aligned} T_5 &= A_{22}^\dagger(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55}) + A_{22}^\dagger V_1B_{22} + L_{A_{22}}V_2, \\ U_3 &= R_{A_{22}}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55})B_{22}^\dagger + A_{22}A_{22}^\dagger V_1 + V_3R_{B_{22}}, \\ T_1 &= M_1^\dagger T_{11}N_1^\dagger - M_1^\dagger M_2H_1^\dagger T_{11}N_1^\dagger - M_1^\dagger S_1M_2^\dagger T_{11}G_1^\dagger N_2N_1^\dagger - M_1^\dagger S_1V_4R_{G_1}N_2N_1^\dagger \\ &\quad + L_{M_1}V_5 + V_6R_{N_1}, \\ T_2 &= H_1^\dagger T_{11}N_2^\dagger + S_1^\dagger S_1M_2^\dagger T_{11}G_1^\dagger + L_{H_1}L_{S_1}V_7 + V_8R_{N_2} + L_{H_1}V_4R_{G_1}, \\ U_1 &= F_1 + L_{P_2}W_1 + W_2R_{Q_1} + L_{P_1}W_3R_{Q_2}, \text{ or } U_1 = F_2 - L_{P_2}W_4 - W_5R_{Q_3} - L_{P_3}W_6R_{Q_4}, \end{aligned}$$

where $T_{11} = T - M_3U_1N_3$, $V_i(i = \overline{1,8})$ are any matrix with suitable dimensions over \mathbb{H} ,

$$\begin{aligned} W_1 &= \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} E_{11}^\dagger(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^\dagger U_{11}F_{11} + L_{E_{11}}U_{12} \end{bmatrix}, \\ W_4 &= \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} E_{11}^\dagger(F - E_{22}W_3F_{22} - E_{33}W_6F_{33}) - E_{11}^\dagger U_{11}F_{11} + L_{E_{11}}U_{12} \end{bmatrix}, \\ W_2 &= \begin{bmatrix} R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^\dagger + E_{11}E_{11}^\dagger U_{11} + U_{21}R_{F_{11}} \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \\ W_5 &= \begin{bmatrix} R_{E_{11}}(F - E_{22}W_3F_{22} - E_{33}W_6F_{33})F_{11}^\dagger + E_{11}E_{11}^\dagger U_{11} + U_{21}R_{F_{11}} \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \\ W_3 &= M_{11}^\dagger FM_{33}^\dagger - M_{11}^\dagger M_{22}M^\dagger FM_{33}^\dagger - M_{11}^\dagger SM_{22}^\dagger FN^\dagger M_{44}M_{33}^\dagger - M_{11}^\dagger SU_{31}R_N M_{44}M_{33}^\dagger \\ &\quad + L_{M_{11}}U_{32} + U_{33}R_{M_{33}}, \\ W_6 &= M^\dagger FM_{44}^\dagger + S^\dagger SM_{22}^\dagger FN^\dagger + L_M L_S U_{41} + L_M U_{31}R_N - U_{42}R_{M_{44}}, \end{aligned}$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ and U_{42} are any matrix with suitable dimensions over \mathbb{H} .

To sum up, the system of matrices of Equation (6) has a solution if and only if (41), (43) and (47) hold.

(2) \Leftrightarrow (3) We divide it into three parts to prove its equivalence.

Part 1. In this part, we prove that (41) holds if and only if (24) and (25) hold. According to Lemma 1, it is easy to show that (41) holds if and only if (24) and (25) hold.

Part 2. In this part, we prove that (43) \iff (26). It follows from Lemma 1 and elementary operations that

$$\begin{aligned}
 (43) &\iff r \begin{pmatrix} C_{11} & A_{11} \\ B_{11} & 0 \end{pmatrix} = r(A_{11}) + r(B_{11}) \\
 &\iff r \begin{pmatrix} C_{11} & A_5 L_{A_3} \\ R_{C_3} B_5 & 0 \end{pmatrix} = r(A_5 L_{A_3}) + r(R_{C_3} B_5) \\
 &\iff r \begin{pmatrix} D_4 - A_5 A_3^\dagger B_3 - D_3 C_3^\dagger B_5 & A_5 & 0 \\ & B_5 & 0 & C_3 \\ & 0 & A_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_5 & C_3 \end{pmatrix} \\
 &\iff r \begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_5 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_5 & C_3 \end{pmatrix} \iff (26).
 \end{aligned}$$

Part 3. In this part, we show that (47) holds if and only if (27) to (35) hold. By using Lemma 6, (47) holds if and only if (48) to (56) hold. Hence, we only show that (48) to (56) hold if and only if (27) to (35) hold, respectively. We first prove that (48) \iff (27).

Note that

$$X_0 = A_1^\dagger B_1 + L_{A_1} D_1 C_1^\dagger, Y_0 = A_2^\dagger B_2 + L_{A_2} D_2 C_2^\dagger, Z_0 = D_3 C_3^\dagger, V_0 = A_4^\dagger B_4, W_0 = A_3^\dagger B_3$$

are the special solution to the equations

$$\begin{aligned}
 A_1 X &= B_1, X C_1 = D_1, \\
 A_2 Y &= B_2, Y C_2 = D_2, \\
 A_3 W &= B_3, Z C_3 = D_3, A_4 V = B_4,
 \end{aligned}$$

respectively. Then, we have that

$$C_{11} = D_4 - A_5 W_0 - Z_0 B_5, \tag{57}$$

$$C_{22} = D_5 - A_6 V_0 - Z_0 B_6 - R_{A_{11}} C_{11} B_{11}^\dagger R_{C_3} B_6 - A_7 X_0 B_7 - A_8 Y_0 B_8. \tag{58}$$

It follows from Lemma 1 and elementary operations to (47) that

$$\begin{aligned}
 (48) &\iff r \begin{pmatrix} C_{22} & A_7 & A_8 & A_{11} & A_6 & 0 \\ R_{C_3} B_6 & 0 & 0 & 0 & 0 & B_{11} \\ 0 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & 0 \end{pmatrix} = r \begin{pmatrix} R_{C_3} B_6 & B_{11} \end{pmatrix} + r \begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \\
 &\iff r \begin{pmatrix} D_5 - Z_0 B_6 & A_7 & A_8 & A_6 & C_{11} & A_{11} \\ R_{C_3} B_6 & 0 & 0 & 0 & B_{11} & 0 \\ B_1 B_7 & A_1 & 0 & 0 & 0 & 0 \\ B_2 B_8 & 0 & A_2 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r \begin{pmatrix} R_{C_3} B_6 & B_{11} \end{pmatrix}
 \end{aligned}$$

$$\Leftrightarrow r \begin{pmatrix} D_5 & A_7 & A_8 & A_6 & D_4 & A_5 & D_3 \\ B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\ B_1 B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\ B_2 B_8 & 0 & A_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & A_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_7 & A_8 & A_5 & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & 0 \end{pmatrix} + r (B_6 \ B_5 \ C_3) \Leftrightarrow (27).$$

Similarly, we can prove that $R_{P_2}E_2 = 0 \Leftrightarrow (28)$, $R_{P_3}E_3 = 0 \Leftrightarrow (29)$, $R_{P_4}E_4 = 0 \Leftrightarrow (30)$ and $E_i L_{Q_i} = 0$ ($i = \overline{1,4}$) hold if and only if (31) to (34) hold, respectively. Next, we show that $R_{M_{22}}EL_{M_{33}} = 0 \Leftrightarrow (35)$. According to Lemma 1 and elementary operations, we have that

$$\begin{aligned} R_{M_{22}}EL_{M_{33}} = 0 &\Leftrightarrow r \begin{pmatrix} E & D_{22} \\ D_{33} & 0 \end{pmatrix} = r(D_{22}) + r(D_{33}) \\ &\Leftrightarrow r \begin{pmatrix} C_{22} & A_{33} & A_{22} & 0 & 0 & 0 & A_{55} \\ B_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_{44} & A_{22} & A_{55} \\ 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\ B_{55} & 0 & 0 & B_{55} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{44} & 0 \\ B_{22} & 0 \\ 0 & B_{33} \\ 0 & B_{22} \\ B_{55} & B_{55} \end{pmatrix} + r \begin{pmatrix} A_{33} & A_{22} & 0 & 0 & A_{55} \\ 0 & 0 & A_{44} & A_{22} & A_{55} \end{pmatrix} \\ &\Leftrightarrow r \begin{pmatrix} C_{22} & A_7 & A_6 & 0 & 0 & 0 & A_{11} & 0 & 0 & 0 & 0 \\ B_8 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 \\ R_{C_3} B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 \\ 0 & 0 & 0 & -C_{22} & A_8 & A_6 & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & R_{C_3} B_6 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 \\ B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= r \begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 \\ R_{C_3} B_6 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 \\ 0 & R_{C_3} B_6 & 0 & 0 & 0 & B_{11} & 0 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 \end{pmatrix} + r \begin{pmatrix} A_7 & A_6 & 0 & 0 & A_{11} \\ 0 & 0 & A_8 & A_6 & A_{11} \\ A_1 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 \end{pmatrix} \\ &\Leftrightarrow r \begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} A_7 & A_6 & 0 & 0 & A_{11} & 0 \\ 0 & 0 & A_8 & A_6 & 0 & A_{11} \\ A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$= r \begin{pmatrix} D_5 - Z_0B_6 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & C_{11} & 0 & 0 & 0 & A_{11} & 0 & 0 & 0 \\ B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & Z_0B_6 - D_5 & A_8 & A_6 & 0 & 0 & -A_7D_1 & -C_{11} & 0 & 0 & 0 & A_{11} & 0 \\ 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\ B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow r \begin{pmatrix} B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & B_5 & 0 & 0 & 0 & C_3 & 0 \\ 0 & B_7 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 & C_3 \\ B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} A_7 & A_6 & 0 & 0 & A_5 & 0 \\ 0 & 0 & A_8 & A_6 & 0 & A_5 \\ A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_3 \end{pmatrix}$$

$$= r \begin{pmatrix} D_5 & A_7 & A_6 & 0 & 0 & 0 & A_8D_2 & D_4 & 0 & 0 & 0 & A_5 & D_3 & 0 & 0 \\ B_8 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & D_5 & A_8 & A_6 & 0 & 0 & A_7D_1 & D_4 & 0 & 0 & 0 & A_5 & D_3 \\ 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & C_3 \\ B_6 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2B_8 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 & A_3 & 0 \end{pmatrix} \Leftrightarrow (35).$$

□

4. The General Solution to the System (6)

In this section, we give an expression for the general solution of Equation (6) by using the Moore–Penrose inverse. According to the proof of Theorem 1, we obtain the following theorem:

Theorem 2. *The general solution to system (6) can be expressed as follows when the solvability conditions are met:*

$$\begin{aligned} X &= A_1^\dagger B_1 + L_{A_1} D_1 C_1^\dagger + L_{A_1} T_1 R_{C_1}, \quad Y = A_2^\dagger B_2 + L_{A_2} D_2 C_2^\dagger + L_{A_2} T_2 R_{C_2}, \\ Z &= D_3 C_3^\dagger + R_{A_{11}} C_{11} B_{11}^\dagger R_{C_3} + A_{11} U_1 R_{C_3} + U_3 R_{B_{11}} R_{C_3}, \\ W &= A_3^\dagger B_3 + L_{A_3} A_{11}^\dagger C_1 - L_{A_3} A_{11}^\dagger U_1 B_{11} + L_{A_3} L_{A_{11}} U_2, \\ V &= A_4^\dagger B_4 + L_{A_4} A_{22}^\dagger (C_{22} - A_{33} T_1 B_{33} - A_{44} T_2 B_{44} - A_{55} U_1 B_{55}) + L_{A_4} A_{22}^\dagger V_1 B_{22} + L_{A_4} L_{A_{22}} V_2, \end{aligned}$$

where $T_{11} = T - M_3U_1N_3$, $V_i (i = \overline{1,8})$ are arbitrary matrices with appropriate sizes.

$$\begin{aligned}
 T_1 &= M_1^\dagger T_{11} N_1^\dagger - M_1^\dagger M_2 H_1^\dagger T_{11} N_1^\dagger - M_1^\dagger S_1 M_2^\dagger T_{11} G_1^\dagger N_2 N_1^\dagger - M_1^\dagger S_1 V_4 R_{G_1} N_2 N_1^\dagger \\
 &\quad + L_{M_1} V_5 + V_6 R_{N_1}, \\
 T_2 &= H_1^\dagger T_{11} N_2^\dagger + S_1^\dagger S_1 M_2^\dagger T_{11} G_1^\dagger + L_{H_1} L_{S_1} V_7 + V_8 R_{N_2} + L_{H_1} V_4 R_{G_1}, \\
 U_3 &= R_{A_{22}} (C_{22} - A_{33} T_1 B_{33} - A_{44} T_2 B_{44} - A_{55} U_1 B_{55}) B_{22}^\dagger + A_{22} A_{22}^\dagger V_1 + V_3 R_{B_{22}}, \\
 U_1 &= F_1 + L_{P_2} W_1 + W_2 R_{Q_1} + L_{P_1} W_3 R_{Q_2}, \text{ or } U_1 = F_2 - L_{P_2} W_4 - W_5 R_{Q_3} - L_{P_3} W_6 R_{Q_4}, \\
 W_1 &= \begin{bmatrix} I_m & 0 \end{bmatrix} \left[E_{11}^\dagger (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) - E_{11}^\dagger U_{11} F_{11} + L_{E_{11}} U_{12} \right], \\
 W_4 &= \begin{bmatrix} 0 & I_m \end{bmatrix} \left[E_{11}^\dagger (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) - E_{11}^\dagger U_{11} F_{11} + L_{E_{11}} U_{12} \right], \\
 W_2 &= \left[R_{E_{11}} (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) F_{11}^\dagger + E_{11} E_{11}^\dagger U_{11} + U_{21} R_{F_{11}} \right] \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \\
 W_5 &= \left[R_{E_{11}} (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) F_{11}^\dagger + E_{11} E_{11}^\dagger U_{11} + U_{21} R_{F_{11}} \right] \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \\
 W_3 &= M_{11}^\dagger F M_{33}^\dagger - M_{11}^\dagger M_{22} M^\dagger F M_{33}^\dagger - M_{11}^\dagger S M_{22}^\dagger F N^\dagger M_{44} M_{33}^\dagger - M_{11}^\dagger S U_{31} R_N M_{44} M_{33}^\dagger \\
 &\quad + L_{M_{11}} U_{32} + U_{33} R_{M_{33}}, \\
 W_6 &= M^\dagger F M_{44}^\dagger + S^\dagger S M_{22}^\dagger F N^\dagger + L_M L_S U_{41} + L_M U_{31} R_N - U_{42} R_{M_{44}},
 \end{aligned}$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ and U_{42} are arbitrary matrices over \mathbb{H} of appropriate sizes.

Next, we discuss the special cases of the system of matrices of Equation (6). Letting A_3, B_3, A_5, B_5 and D_4 vanish yields the following:

Corollary 1. Suppose that $A_i, B_i, C_j, D_j (i = \overline{1,4}, j = \overline{1,5})$ and E_1 are given, denote

$$\begin{aligned}
 A_6 &= A_4 L_{A_1}, \quad B_6 = R_{B_1} B_4, \quad C_6 = C_4 L_{A_2}, \quad D_6 = R_{B_2} D_4, \quad C_7 = C_5 L_{A_3}, \quad D_7 = R_{B_3} D_5, \\
 E_6 &= E_1 - A_4 A_1^\dagger C_1 - D_1 B_1^\dagger B_4 - C_4 \left(A_2^\dagger C_2 + L_{A_2} D_2 B_2^\dagger \right) D_4 - C_5 \left(A_3^\dagger C_3 + L_{A_3} D_3 B_3^\dagger \right) D_5, \\
 A &= R_{A_6} C_6, \quad B = D_6 L_{B_6}, \quad C = R_{A_6} C_7, \quad D = D_7 L_{B_6}, \\
 E &= R_{A_6} E_6 L_{B_6}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M.
 \end{aligned}$$

Then, the following statements are equivalent:

- (1) System (5) is consistent.
- (2)

$$\begin{aligned}
 R_{A_i} C_i &= 0, \quad D_i L_{B_i} = 0 \quad (i = 1, 2, 3), \quad A_2 D_2 = C_2 B_2, \quad A_3 D_3 = C_3 B_3, \\
 R_A E &= M M^\dagger E, \quad E L_B = E N^\dagger N, \quad R_A E L_D = 0, \quad R_C E L_B = 0.
 \end{aligned}$$

(3)

$$r(A_i \ C_i) = r(A_i), \quad r\begin{pmatrix} B_i \\ D_i \end{pmatrix} = r(B_i) \ (i = 1, 2, 3), \quad A_2D_2 = C_2B_2, \quad A_3D_3 = C_3B_3,$$

$$r\begin{pmatrix} E_1 & A_4 & D_1 & C_4D_2 & C_5D_3 \\ B_4 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & B_2 & 0 \\ D_5 & 0 & 0 & 0 & B_3 \\ C_1 & A_1 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_4 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_2 & 0 \\ D_5 & 0 & 0 & B_2 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_4 & C_5 & D_1 \\ B_4 & 0 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 \\ C_3D_5 & 0 & 0 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 & C_4 & C_5 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_4 & D_1 & C_5D_3 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_5 & 0 & 0 & 0 & B_3 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_3 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 \\ D_5 & 0 & B_3 \end{pmatrix},$$

$$r\begin{pmatrix} E_1 & A_4 & C_5 & D_1 & C_4D_2 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_3D_5 & 0 & A_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_4 & C_5 \\ A_1 & 0 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{pmatrix}.$$

In this case, the general solution to system (5) can be expressed as

$$\begin{aligned} X_1 &= A_1^\dagger C_1 + L_{A_1} U_1, \quad X_2 = D_1 B_1^\dagger + U_2 R_{B_1}, \\ X_3 &= A_2^\dagger C_2 + L_{A_2} D_2 B_2^\dagger + L_{A_2} U_3 R_{B_2}, \\ X_4 &= A_3^\dagger C_3 + L_{A_3} D_3 B_3^\dagger + L_{A_3} U_4 R_{B_3}, \\ U_1 &= A_6^\dagger (E_6 - C_6 U_3 D_6 - C_7 U_4 D_7) - A_6^\dagger W_2 B_6 + L_{A_6} W_1, \\ U_2 &= R_{A_6} (E_6 - C_6 U_3 D_6 - C_7 U_4 D_7) B_6^\dagger + A_6 A_6^\dagger W_2 + W_3 R_{B_6}, \\ U_3 &= A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S V_4 R_N D B^\dagger + L_A V_1 + V_2 R_B, \\ U_4 &= M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_5 V_3 + L_M V_4 R_N + V_5 R_D, \end{aligned}$$

where $V_i, W_j (i = \overline{1,5}, j = \overline{1,3})$ are arbitrary matrices over \mathbb{H} with appropriate sizes.

Remark 1. The above corollary is from the important findings of [30].

Letting $A_i, B_i, C_j, D_j (i = 1, 2, 4, 6, 7, 8, j = 1, 2)$ and D_5 vanish, we have the following:

Corollary 2. Given $A_3, B_3, C_3, D_3, A_5, B_5$ and D_4 of feasible dimensions over \mathbb{H} . Set $A_{11} = A_5 L_{A_3}$, $B_{11} = R_{C_3} C$ and $E_{11} = D_4 - A_5 A_3^\dagger B_3 - D_3 C_3^\dagger B_5$. Then, the following statements are equivalent:

- (1) System (4) is consistent.

$$(2) r(A_3 \ B_3) = r(A_3), r\begin{pmatrix} C_3 \\ D_3 \end{pmatrix} = r(C_3), r\begin{pmatrix} D_4 & A_5 & D_3 \\ B_5 & 0 & C_3 \\ B_3 & A_3 & 0 \end{pmatrix} = r\begin{pmatrix} A_5 \\ B_5 \ C_3 \end{pmatrix} + r(B_5 \ C_3).$$

In this case, the general solution to system (4) can be expressed as

$$W = A_3^\dagger B_3 + L_{A_3} (A_{11}^\dagger E_{11} - A_3^\dagger W_2 B_{11} + L_{A_{11}} W_1),$$

$$Z = D_3 C_3^\dagger + (R_{A_{11}} E_{11} B_{11}^\dagger + A_{11} A_{11}^\dagger W_2 + W_3 R_{B_{11}}) R_{C_3},$$

where W_1, W_2 and W_3 are arbitrary matrices over \mathbb{H} of appropriate sizes.

Remark 2. The above corollary is from the vital investigation of [4].

Finally, we give Algorithm 1 and an example to illustrate the main results of this paper.

Algorithm 1: Algorithm for solving Equation (6)

- (1) Feed the values of A_i, B_i, C_j, D_k ($i = \overline{1,8}, j = \overline{1,3}, k = \overline{1,5}$) with conformable shapes over \mathbb{H} .
 - (2) Compute the symbols in (9) to (21).
 - (3) Check (22), (23) or rank equalities in (24) to (35) hold or not. If no, then return “inconsistent”.
 - (4) Otherwise, compute X, Y, Z, V, W .
-

Example 1. Consider the matrix of Equation (6). Assume

$$A_1 = \begin{pmatrix} -1-j & i-j \\ i & -1 \\ 1-i & -j \end{pmatrix}, A_2 = \begin{pmatrix} i+j & 1+k \\ i+k & k \\ 1+i & 1+i+k \end{pmatrix}, A_3 = \begin{pmatrix} j+k & -1+j \\ -i+k & i \\ i+j & -j+k \end{pmatrix}, A_4 = \begin{pmatrix} 1+j & 2+k \\ 1-i+k & 2j+2k \\ 1+i+j+k & 1+i+j \end{pmatrix},$$

$$A_5 = \begin{pmatrix} -j & -j+k \\ -1-j+k & k \\ 1-j & i+k \end{pmatrix}, A_6 = \begin{pmatrix} 1+i+j+k & 1 \\ 0 & 1+i \\ 1+j & 1+i \end{pmatrix}, A_7 = \begin{pmatrix} i+j & 1+i \\ 1+i+k & 1+j \\ 1+i+j & 1+j \end{pmatrix}, A_8 = \begin{pmatrix} -1+i+j-k & -i \\ -1+j-k & i-j \\ -1+j+k & -1+i \end{pmatrix},$$

$$B_1 = \begin{pmatrix} i-5j & -1+i \\ -2+i & -1-2i+j \\ 1-2i-j-k & -2+i-j+k \end{pmatrix}, B_2 = \begin{pmatrix} 1+i+j-k & -1+3i+j+k \\ -1+2i+j+k & -1+i+2j \\ 1+3i-j & -3+3i \end{pmatrix}, B_3 = \begin{pmatrix} -2-2i+j-k & -3+2j-3k & -3-2j+k \\ -1-k & 2-i+4j & 1-i+k \\ -1+3k & -3-j-2k & 1-2i-j \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 2+i-j+3k & 2+i+5j-2k & 2j \\ 1+3i+2j+3k & 2-3i+2j-k & 1-2i+j \\ 3+3i+k & 3i+5j & 2j+2k \end{pmatrix}, B_5 = \begin{pmatrix} i & 1+j+k & 1+i+k \\ 0 & i & 1+i+j \end{pmatrix}, B_6 = \begin{pmatrix} 2+2i+j+k & 2+2j+k & 1+i+j \\ 1+2i+j+2k & 1+2i+k & 2+2i+2j+k \end{pmatrix},$$

$$B_7 = \begin{pmatrix} k & i+j & i+k \\ i+k & j & 1 \end{pmatrix}, B_8 = \begin{pmatrix} j & j & 1+i+j+k \\ 1 & 1+j+k & 1+j \end{pmatrix}, C_1 = \begin{pmatrix} 1+i+k & 1+i+k & 1+i+k \\ 1+i & k & 0 \end{pmatrix}, C_2 = \begin{pmatrix} i & -1 & 0 \\ 1-i+j-k & -i-j & i+k \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & k & 0 \\ 1+j+k & i+j+k & 1+i+k \end{pmatrix}, D_1 = \begin{pmatrix} i+j & 1+2i+j-k & 1+2i+j \\ 1+4i+4k & 1+i+4k & 1+2i+j+3k \end{pmatrix}, D_2 = \begin{pmatrix} -i-j+k & -1-i+j & 0 \\ 4+2i+3k & i-j-3k & -3-i-j+k \end{pmatrix},$$

$$D_3 = \begin{pmatrix} -i+j+2k & -1+j+2k & i+j+2k \\ -1-j+2k & -3+2k & -2+i+k \\ 0 & -1-j & 0 \end{pmatrix}, D_4 = \begin{pmatrix} i+2k & 3+j+5k & 4+k \\ 1+j & -2-6i+5j+4k & -i+j+3k \\ -1+i+3j+k & i+7k & -i+3k \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 7i-2j+9k & 2+10i+13j+10k & 4i+6j \\ -7+8i-j+7k & -12+2i+10j-k & -5+5i+j-k \\ -7+4i-4j+6k & -8+2i+6j-4k & -11+9i+j-8k \end{pmatrix}.$$

Computing directly yields

$$r(A_i \ B_i) = r(A_i) = 2, \quad r\begin{pmatrix} C_j \\ D_j \end{pmatrix} = r(C_j) = 2 \quad (i = \overline{1,4}, j = \overline{1,3}),$$

$$(26) = 4, (27) = 10, (28) = 10, (29) = 10, (30) = 10,$$

$$(31) = 8, (32) = 8, (33) = 8, (34) = 8, (35) = 24.$$

All rank equations hold. Thus, according to Theorem 1, the system of matrix equations has a solution, and the general solution to the system can be expressed as

$$X = \begin{pmatrix} 1+j & -1 \\ 2+k & 1+i-j \end{pmatrix}, \quad Y = \begin{pmatrix} 1+i-j & 0 \\ 1 & 2i-j+k \end{pmatrix}, \quad Z = \begin{pmatrix} 1+i & 1+k \\ 1+j+k & i+k \\ i+k & 0 \end{pmatrix}, \\
 V = \begin{pmatrix} 1 & 2+i-k & 1+j \\ 1-j+k & i+j & 0 \end{pmatrix}, \quad W = \begin{pmatrix} j & 3i+j+k & j \\ 1+i & i+j & -i+2j \end{pmatrix},$$

5. The General Solution to the System (7) with η -Hermiticity

As an application of the results of system (6), we study the necessary and sufficient conditions for system (7) to have a solution involving η -Hermiticity and derive a formula of its general solution, where X, Y are η -Hermitian matrices.

Theorem 3. Given A_i, B_j ($i = 1, 2, 7, 8, j = 1, 2, \overline{5, 8}$), C_3, D_3, D_4 of appropriate dimensions over \mathbb{H} . Set

$$A_{22} = A_6L_{A_4}, \quad A_{33} = A_7L_{A_1}, \quad A_{44} = A_8L_{A_2}, \\
 C_{22} = D_5 - A_6^\dagger A_4 B_4 - A_7 A_1^\dagger (A_1^\dagger)^\eta + L_{A_1} C_1^{\eta*} (C_1^\dagger)^\eta A_7^{\eta*} - A_8 A_2^\dagger (A_2^\dagger)^\eta + L_{A_2} C_2^{\eta*} (C_2^\dagger)^\eta A_8^{\eta*}, \\
 M_1 = R_{A_{22}} A_{33}, \quad M_2 = R_{A_{22}} A_{44}, \quad T = R_{A_{22}} C_{22} R_{A_{22}}^{\eta*}, \quad M = R_{M_1} M_2, \quad S = M_2 L_M.$$

Then, the following statements are equivalent:

- (1) System (7) has a solution.
- (2)

$$R_{A_1} B_1 = 0, \quad R_{A_2} B_2 = 0, \quad R_{A_4} B_4 = 0, \\
 R_{M_1} R_M T = 0, \quad R_{A_{22}} T (R_{A_{44}})^\eta = 0.$$

(3)

$$r(A_1 \ B_1) = r(A_1), \quad r(A_2 \ B_2) = r(A_2), \quad r(A_4 \ B_4) = r(A_4), \\
 r \begin{pmatrix} D_5 & A_6 & B_4^{\eta*} & A_7 B_1^{\eta*} & A_8 B_2^{\eta*} \\ A_6^{\eta*} & 0 & A_4^{\eta*} & 0 & 0 \\ A_7^{\eta*} & 0 & 0 & A_1^{\eta*} & 0 \\ A_8^{\eta*} & 0 & 0 & 0 & A_2^{\eta*} \\ B_4 & A_4 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_6^{\eta*} & A_4^{\eta*} & 0 & 0 \\ A_7^{\eta*} & 0 & A_1^{\eta*} & 0 \\ A_8^{\eta*} & 0 & 0 & A_1^{\eta*} \end{pmatrix} + r \begin{pmatrix} A_4 \\ A_6 \end{pmatrix}, \\
 r \begin{pmatrix} D_5 & A_6 & A_7 & B_4^{\eta*} & A_8 B_2^{\eta*} \\ A_6^{\eta*} & 0 & 0 & A_4^{\eta*} & 0 \\ A_8^{\eta*} & 0 & 0 & 0 & A_2^{\eta*} \\ B_4 & A_4 & 0 & 0 & 0 \\ B_1 A_7^{\eta*} & 0 & A_1 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_6^{\eta*} & A_4^{\eta*} & 0 \\ A_8^{\eta*} & 0 & A_2^{\eta*} \end{pmatrix} + r \begin{pmatrix} A_6 & A_7 \\ A_4 & 0 \\ 0 & A_1 \end{pmatrix}.$$

Under these conditions, the general solution with η -Hermiticity to the system (7) can be stated as

$$\begin{aligned}
 V &= A_4^\dagger B_4 + L_{A_4} U_1, \\
 X &= A_1^\dagger B_1 + L_{A_1} B_1^{\eta^*} (A_1^\dagger)^{\eta^*} + L_{A_1} U_2 L_{A_1}^{\eta^*}, \\
 Y &= A_2^\dagger B_2 + L_{A_2} B_2^{\eta^*} (A_2^\dagger)^{\eta^*} + L_{A_2} U_3 L_{A_2}^{\eta^*}, \\
 U_1 &= A_{22}^\dagger (C_{22} - A_{33} U_2 A_{33}^{\eta^*} - A_{44} U_3 A_{44}^{\eta^*}) - A_{22}^\dagger W_2 A_{22}^{\eta^*} + L_{A_{22}} W_1, \\
 U_2 &= M_1^\dagger T M_1^{\eta^*} - M_1^\dagger M_2 M^\dagger T M_1^{\eta^*} - M_1^\dagger S M_2^\dagger T (M^{\eta^*})^\dagger M_2^{\eta^*} M_1^{\eta^*} - M_1^\dagger S V_4 (L_M)^{\eta^*} M_2^{\eta^*} M_1^{\eta^*} \\
 &\quad + L_{M_1} V_1 + V_2 (L_M)^{\eta^*}, \\
 U_3 &= M^\dagger T M_2^{\eta^*} + S^\dagger S M_2^\dagger T M^{\eta^*} + L_M L_S V_3 + L_M V_4 (L_M)^{\eta^*} + V_5 (L_{M_2})^{\eta^*},
 \end{aligned}$$

where V_i ($i = \overline{1,5}$) and W_j ($j = \overline{1,3}$) are arbitrary matrices with appropriate sizes over \mathbb{H} .

Proof. Since the solvability of the system (7) is equivalent to the system

$$\begin{aligned}
 A_4 V_1 &= B_4, \quad V_2 (A_4)^{\eta^*} = (B_4)^{\eta^*}, \quad V_2 = (V_1)^{\eta^*}, \\
 A_1 X_1 &= B_1, \quad X_1 A_1^{\eta^*} = B_1^{\eta^*}, \quad X_1 = X_1^{\eta^*}, \\
 A_2 Y_1 &= B_2, \quad Y_1 A_2^{\eta^*} = B_2^{\eta^*}, \quad Y_1 = Y_1^{\eta^*}, \\
 A_6 V_1 + V_2 A_6^{\eta^*} + A_7 X_1 A_7^{\eta^*} + A_8 Y_1 A_8^{\eta^*} &= D_5, \quad D_5 = D_5^{\eta^*}.
 \end{aligned} \tag{59}$$

If system (7) has a solution, say, (V, X, Y) , then system (59) has a solution, $(V_1, V_2, X_1, Y_1) = (V, V^{\eta^*}, X, Y)$. Conversely, if system (59) has a solution (V_1, V_2, X_1, Y_1) , then

$$(V, X, Y) = \left(\frac{V_1 + V_2^{\eta^*}}{2}, \frac{X_1 + X_1^{\eta^*}}{2}, \frac{Y_1 + Y_1^{\eta^*}}{2} \right)$$

is the solution of (7). It follows from Corollary 1 that this proof can be completed. \square

6. Conclusions

We established the solvability conditions for system (6) by using the Moore–Penrose inverses and ranks of the coefficient quaternion matrices in (6), and derived a formula of its general solution when it is solvable. In terms of applications, we derived the necessary and sufficient conditions for system (7) to have an η -Hermitian solution as well as the expression of the general solution. In addition, we used an algorithm and a numerical example to verify the main results of this paper. It is worth noting that the main results of (6) are available not only for \mathbb{R} and \mathbb{C} , but also any division ring. Moreover, inspired by [39], we can investigate the system (6) tensor equations over the quaternion algebra.

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