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Kantorovich Type Generalization of Bernstein Type Rational Functions Based on (*p*,*q*)-Integers

Hayatem Hamal [1,*] **and Pembe Sabancigil** [2]

[1] Department of Mathematics, Tripoli University, Tripoli 22131, Libya
[2] Department of Mathematics, Eastern Mediterranean University, Gazimagusa 99628, T. R. North Cyprus; pembe.sabancigil@emu.edu.tr
* Correspondence: hafraj@yahoo.com; Tel.: +218-9053-3821-5279

Abstract: In this paper, we define a new Kantorovich-type (p,q)-generalization of the Balázs–Szabados operators. We derive a recurrence formula, and with the help of this formula, we give explicit formulas for the first and second-order moments, which follow a symmetric pattern. We estimate the second and fourth-order central moments. We examine the local approximation properties in terms of modulus of continuity, we give a Voronovskaja type theorem, and we give the weighted approximation properties of the operators.

Keywords: (*p*,*q*)-calculus; moments; Bernstein operators; Balázs–Szabados operators; Kantorovich-type operators; (*p*,*q*)-Balázs–Szabados operators

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1. Introduction

Bernstein operators have a long-standing history, and many studies have been written on them. Among all types of positive linear operators, they occupy a unique position because of their elegance and notable approximation properties (see [1]).

Bernstein type rational functions defined by Katalin Balázs in [2] are as follows:

$$R_n(f;x) = \frac{1}{(1+a_n x)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n x)^k \quad (n = 1,\, 2,\, \ldots)$$

where f is a real and single valued function which is defined on the unbounded interval $[0,\infty)$, a_n and b_n are real numbers which are selected suitably and do not depend on x. Later in 1982, Balázs and Szabados studied together and improved the estimation given in [2] by selecting a suitable a_n and b_n under some restrictions for $f(x)$ (see [3]).

Several q-generalizations of these operators have recently been studied by Hamal and Sabancigil ([4]), Doğru ([5]), and Özkan ([6]). On the other hand, the approximation properties of the q-Balázs–Szabados complex operators are studied by Mahmudov in [7] and by Ispir and Özkan in [8]. The Kantorovich-type q-analogue of Balázs–Szabados operators defined by Hamal and Sabancigil in [4] is as follows:

$$R^*_{n,q}(f,x) = \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q + q^k t}{b_n}\right)\, d_q t \tag{1}$$

where

$$r_{n,k}(q,x) = \tfrac{1}{(1+a_n x)^n}\left[\begin{array}{c} n \\ k \end{array}\right]_q (a_n x)^k \prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_n x\right),\ q\in(0,1),$$
$$a_n = [n]_q^{\beta-1},\ b_n = [n]_q^{\beta},\ 0<\beta\le \tfrac{2}{3},\ n\in\mathbb{N},\ x\ge 0$$

and f is a real-valued continuous function defined on $[0,\infty)$. The operators $R^{*}_{n,q}(f,x)$ are positive and linear operators. Compared to the previous q-analogues of Balázs–Szabados operators (see [7,8]), these operators have some advantages. The operators introduced by Mahmudov are summation type operators, which cannot be used to approximate integrable functions and the operators introduced by Özkan are positive if f is a non-decreasing function. New Kantorovich-type q-analogue of the Balázs –Szabados operators introduced in [4] also approximates the integrable functions, and they are positive even if f is not a non-decreasing function.

Additionally, the fast rise of (p,q)-calculus has encouraged many mathematicians in this subject to discover different generalizations. In the last decade, Mursaleen et al. defined and studied the (p,q)-analogue of many operators (see [9–15]). The (p,q)-generalization of Szász–Mirakjan operators was studied by Acar (see [16]), Kantorovich modification of (p,q)-Bernstein operators was studied by Acar and Aral (see [17]). A generalization of q-Balázs–Szabados operators based on (p,q)-integers which was studied by Özkan and İspir in [18] is as follows:

$$\Re_{n,p,q}(f,x)=\frac{1}{(1+a_nx)_{p,q}}\sum_{k=0}^{n}f\left(\frac{[k]_{p,q}}{q^{k-1}b_n}\right)p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}(a_nx)^k \tag{2}$$

where $f:[0,\infty)\to\mathbb{R}$ is a continuous function, $0<q<p\leq 1$, $n\in\mathbb{N}$, $x\in[0,\infty)$ and $a_n=[n]^{\beta-1}_{p,q}$, $b_n=[n]^{\beta}_{p,q}$, $0<\beta\leq\frac{2}{3}$.

On the other hand, another the (p,q)-generalization of Balázs–Szabados operators defined by Hamal and Sabancigil in [19] is as follows:

$$\begin{aligned}R_{n,p,q}(f,x)=\frac{1}{p^{n(n-1)/2}}\sum_{k=0}^{n}&\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}p^{k(k-1)/2}f\left(\frac{p^{n-k}[k]_{p,q}}{b_n}\right)\left(\frac{a_nx}{1+a_nx}\right)^k\\&\times\prod_{j=0}^{n-k-1}\left(p^j-q^j\frac{a_nx}{1+a_nx}\right)\end{aligned} \tag{3}$$

where $0<q<p\leq 1$, $a_n=[n]^{\beta-1}_{p,q}$, $b_n=[n]^{\beta}_{p,q}$, $0<\beta\leq\frac{2}{3}$, $n\in\mathbb{N}$, $x\geq 0$, f is a real-valued function defined on $[0,\infty)$.

These two operators defined by (2) and (3) are summation type operators and they are not capable of approximating integrable functions.

In this paper, we introduce a Kantorovich-type (p,q)-analogue of Balázs–Szabados operators by generalizing the new Kantorovich-type q-analogue of Balázs–Szabados operators, $R^{*}_{n,q}$, given by (1). We derive a recurrence formula, and by using this formula, we give explicit formulas for the first and second-order moments, which follow a symmetric pattern. We study some of the approximation properties of the new Kantorovich-type (p,q)-analogue of Balázs–Szabados operators in terms of the modulus of continuity, we prove a Voronovskaja-type theorem and we examine the weighted approximation properties of these new operators. Compared to the previous (p,q)-analogues of Balázs–Szabados operators defined in [18] and in [19], these new operators have an advantage of also approximating the integrable functions.

Before stating the main results for these operators, we will give some important notations and definitions of (p,q)-calculus. For any $p>0$, $q>0$ and a non-negative integer n, the (p,q)-integer of the number n is defined as:

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1}=\begin{cases}\frac{p^n-q^n}{p-q} & if\ p\neq q\neq 1\\ np^{n-1} & if\ p=q\neq 1\\ [n]_q & if\ p=1\\ n & if\ p=q=1\end{cases}.$$

One can easily see that, $[n]_{p,q} = p^{n(n-1)/2}[n]_{\frac{q}{p}}$.

(p,q)-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q} \text{ , } n \geq 1 \text{ and } [0]_{p,q}! = 1.$$

(p,q)-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} \text{ , } 0 \leq k \leq n,$$

and the formula of (p,q)-binomial expansion is

$$\begin{aligned}(ax+by)^n_{p,q} &= \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k \\ &= (ax+by)(pax+qby)\dots\left(p^{n-1}ax+q^{n-1}by\right)\end{aligned}$$

and

$$(x-y)^n_{p,q} = (x-y)(px-qy)\left(p^2x-q^2y\right)\dots\left(p^{n-1}x-q^{n-1}y\right).$$

From (p,q)-binomial expansion, we can see that

$$\sum_{k=0}^{n} p^{k(k-1)/2} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k(1-x)^{n-k}_{p,q} = p^{n(n-1)/2} \text{ , } x \in [0,1].$$

Let $f : C[0,a] \to \mathbb{R}$, the (p,q)-integral of f is defined by:

$$\int_0^a f(t) d_{p,q}t = (p-q)a \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right) \frac{q^k}{p^{k+1}} \text{ if } \left|\frac{p}{q}\right| > 1$$

The paper is organized as follows. In Section 2, we give the construction of the operators, we derive a recurrence formula, and we give explicit formulas for the first and second-order moments. In Section 3, we give an estimation of the central moments. In Section 4, we prove a local approximation theorem and a Voronovskaja-type theorem. In Section 5, we give weighted approximation properties of the operators.

2. Construction of the Operators and Their Moments

Definition 1. *Let $0 < q < p \leq 1$, we introduce a new Kantorovich-type (p,q)-analogue of the Balázs–Szabados operators by*

$$R^*_{n,p,q}(f,x) = \sum_{k=0}^{n} r^*_{n,k}(p,q,x) \int_0^1 f\left(\frac{p^{n-k}\left([k]_{p,q} + q^k t\right)}{b_n}\right) d_{p,q}\, t,$$

where

$$r^*_{n,k}(p,q,x) = \frac{1}{p^{n(n-1)/2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{k(k-1)/2} \left(\frac{a_n x}{1+a_n x}\right)^k \prod_{j=0}^{n-k-1} \left(p^j - q^j \frac{a_n x}{1+a_n x}\right)$$

and $a_n = [n]^{\beta-1}_{p,q}$, $b_n = [n]^{\beta}_{p,q}$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, $x \geq 0$, f is a real-valued function defined on $[0,\infty)$.

If $p=1$, these polynomials reduce to the new Kantorovich-type q-analogue of the Balázs–Szabados operators, which are defined by Hamal and Sabancigil in [4]. Moreover, we considered the following two special cases:

- If $0<p<q\leq 1$ or $1\leq p<q<\infty$ then the positivity property of the operators $R^*_{n,p,q}(f,x)$ fails.
- If $1\leq q<p<\infty$ then approximation by the new operators $R^*_{n,p,q}(f,x)$ becomes difficult because if p is large enough then the sequence $\left\{R^*_{n,p,q}\right\}_{n\in\mathbb{N}}$ may diverge.

Thus, in this paper, we study the approximation properties of the operators for $0<q<p\leq 1$.

In the following lemma, we give a recurrence formula for $R^*_{n,p,q}(t^m,x)$.

Lemma 1. *For all $n\in\mathbb{N},\ x\in[0,\infty),\ m\in\mathbb{Z}\cup\{0\}$, and $0<q<p\leq 1$, we have*

$$R^*_{n,p,q}(t^m,x)=\sum_{j=0}^{m}\binom{m}{j}\frac{1}{[m-j+1]_{p,q}}\left(\frac{p^n}{b_n}\right)^{m-j}$$
$$\times\sum_{i=0}^{m-j}\binom{m-j}{i}\left(\frac{a_n}{p^n}\right)^i(q^n-p^n)^iR_{n,p,q}(t^{i+j},x)$$

where $R_{n,p,q}(f,x)$ is the (p,q)-Balázs–Szabados operator defined by (3).

Proof. By direct calculations, the recurrence formula is obtained as follows:

$$R^*_{n,p,q}(t^m,x)=\sum_{k=0}^{n}r^*_{n,k}(p,q,x)\int_0^1\left(\frac{p^{n-k}\left([k]_{p,q}+q^kt\right)}{b_n}\right)^m d_{p,q}\,t,$$

by using the binomial expansion of $\left([k]_{p,q}+q^kt\right)^m$ and evaluating the (p,q)-integral we get

$$\begin{aligned}R^*_{n,p,q}(t^m,x)&=\sum_{k=0}^{n}r^*_{n,k}(p,q,x)\sum_{j=0}^{m}\binom{m}{j}\frac{1}{[m-j+1]_{p,q}}\frac{p^{(n-k)m}}{b_n^m}[k]^j_{p,q}q^{k(m-j)}\\
&=\sum_{j=0}^{m}\binom{m}{j}\frac{p^{nm}}{[m-j+1]_{p,q}b_n^m}\sum_{k=0}^{n}\sum_{i=0}^{m-j}\binom{m-j}{i}p^{-k(i+j)}\left(q^k-p^k\right)^i[k]^j_{p,q}r^*_{n,k}(p,q,x)\\
&=\sum_{j=0}^{m}\binom{m}{j}\frac{p^{nm}}{[m-j+1]_{p,q}b_n^m}\sum_{i=0}^{m-j}\binom{m-j}{i}(q-p)^i\sum_{k=0}^{n}p^{-k(i+j)}[k]^{i+j}_{p,q}r^*_{n,k}(p,q,x)\\
&=\sum_{j=0}^{m}\binom{m}{j}\frac{p^{n(m-j)}}{[m-j+1]_{p,q}b_n^{m-j}}\sum_{i=0}^{m-j}\binom{m-j}{i}\left(\frac{a_n}{p^n}\right)^i(q^n-p^n)^i\\
&\times\sum_{k=0}^{n}p^{(n-k)(i+j)}\frac{[k]^{i+j}_{p,q}}{b_n^{i+j}}r^*_{n,k}(p,q,x).\end{aligned}$$

Now, in the last equality, by using the definition of the operators $R_{n,p,q}(f,x)$ given by (3), we may write

$$R^*_{n,p,q}(t^m,x)=\sum_{j=0}^{m}\binom{m}{j}\frac{1}{[m-j+1]_{p,q}}\left(\frac{p^n}{b_n}\right)^{m-j}$$
$$\times\sum_{i=0}^{m-j}\binom{m-j}{i}\left(\frac{a_n}{p^n}\right)^i(q^n-p^n)^iR_{n,p,q}(t^{i+j},x).$$

Moments and central moments possess a great deal of importance in the approximation theory. In the following lemma, with the help of the recurrence formula we calculate the first, second, and the third-order moments of the operators $R^*_{n,p,q}(f,x)$. □

Lemma 2. *For all $n \in \mathbb{N},\ x \in [0,\infty)$ and $0 < q < p \leq 1$, we have the following equalities:*

$$R^*_{n,p,q}(1,x) = 1. \tag{4}$$

$$R^*_{n,p,q}(t,x) = \frac{p^n}{[2]_{p,q}b_n} + \frac{2q}{[2]_{p,q}}\left(\frac{x}{1+a_nx}\right). \tag{5}$$

$$\begin{aligned} R^*_{n,p,q}\left(t^2,x\right) &= \tfrac{p^{2n}}{[3]_{p,q}b_n^2} + \tfrac{\left(4q^3+5q^2p+3qp^2\right)p^{n-1}}{[2]_{p,q}[3]_{p,q}b_n}\left(\tfrac{x}{1+a_nx}\right) \\ &+ \tfrac{q[n-1]_{p,q}}{[n]_{p,q}}\tfrac{4q^3+q^2p+qp^2}{[2]_{p,q}[3]_{p,q}}\left(\tfrac{x}{1+a_nx}\right)^2. \end{aligned} \tag{6}$$

Proof.

$$\begin{aligned} R^*_{n,p,q}(1,x) &= \tfrac{1}{p^{n(n-1)/2}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\tfrac{a_nx}{1+a_nx}\right)^k \prod_{j=0}^{n-k-1}\left(p^j - q^j\tfrac{a_nx}{1+a_nx}\right) = 1, \\ R^*_{n,p,q}(t,x) &= \sum_{j=0}^{1}\begin{pmatrix} 1 \\ j \end{pmatrix}\tfrac{1}{[2-j]_{p,q}}\left(\tfrac{p^n}{b_n}\right)^{1-j}\sum_{i=0}^{1-j}\begin{pmatrix} 1-j \\ i \end{pmatrix}\left(\tfrac{a_n}{p^n}\right)^i (q^n-p^n)^i R_{n,p,q}\left(t^{i+j},x\right) \\ &= \tfrac{p^n}{b_n[2]_{p,q}}\left\{1+\tfrac{a_n}{p^n}(q^n-p^n)R_{n,p,q}(t,x)\right\} + R_{n,p,q}(t,x). \end{aligned}$$

Now, by using the formula for $R_{n,p,q}(t,x)$ which is given in [19], we get

$$R^*_{n,p,q}(t,x) = \frac{p^n}{[2]_{p,q}b_n} + \frac{2q}{[2]_{p,q}}\left(\frac{x}{1+a_nx}\right).$$

In a similar way,

$$\begin{aligned} R^*_{n,p,q}\left(t^2,x\right) &= \sum_{j=0}^{2}\begin{pmatrix} 2 \\ j \end{pmatrix}\tfrac{1}{[3-j]_{p,q}}\left(\tfrac{p^n}{b_n}\right)^{2-j}\sum_{i=0}^{2-j}\begin{pmatrix} 2-j \\ i \end{pmatrix}\left(\tfrac{a_n}{p^n}\right)^i (q^n-p^n)^i \\ &\times R_{n,p,q}\left(t^{i+j},x\right) \\ &= \tfrac{p^{2n}}{[3]_{p,q}b_n^2}\left\{1+\tfrac{2a_n}{p^n}(q^n-p^n)R_{n,p,q}(t,x) + \left(\tfrac{a_n}{p^n}\right)^2(q^n-p^n)^2R_{n,p,q}\left(t^2,x\right)\right\} \\ &+ \tfrac{2p^n}{[2]_{p,q}b_n}\left\{R_{n,p,q}(t,x)+\tfrac{a_n}{p^n}(q^n-p^n)R_{n,p,q}\left(t^2,x\right)\right\} + R_{n,p,q}\left(t^2,x\right), \end{aligned}$$

by simple calculations in the last equality, we get

$$\begin{aligned} R^*_{n,p,q}\left(t^2,x\right) &= \tfrac{p^{2n}}{[3]_{p,q}b_n^2} + \tfrac{\left(4q^3+5q^2p+3qp^2\right)p^{n-1}}{[2]_{p,q}[3]_{p,q}b_n}\left(\tfrac{x}{1+a_nx}\right) \\ &+ \tfrac{q[n-1]_{p,q}}{[n]_{p,q}}\tfrac{4q^3+q^2p+qp^2}{[2]_{p,q}[3]_{p,q}}\left(\tfrac{x}{1+a_nx}\right)^2. \end{aligned}$$

□

Remark 1. From Lemma 2, it can be easily seen that for $p = 1$, we obtain the moments of the new Kantorovich-type q-analogue of the Balázs–Szabados operators, $R^*_{n,q}(t^m,x)$ for m = 0, 1, 2, (see [4]).

3. Estimation of the Central Moments

In the next lemma, we present the estimations of the second and fourth-order central moments of the operators $R^*_{n,p,q}$.

Lemma 3. *For all $n \in \mathbb{N}$ and $0 < q < p \leq 1$, we have the following estimations:*

$$\left(R^*_{n,p,q}((t-x),x)\right)^2 \leq \frac{1}{b_n}\left\{\frac{1}{b_n} + \frac{(p^n-q^n)^2}{b_n}\left(\frac{1}{p+q} + \frac{1}{p-q}(a_n x)\right)^2\right\},\ x \in [0,\infty), \quad (7)$$

$$R^*_{n,p,q}\left((t-x)^2,x\right) \leq \frac{A_1}{b_n}\phi_n(p,q)(1+x)^2, \quad x \in [0,\infty), \quad (8)$$

$$R^*_{n,p,q}\left((t-x)^4,x\right) \ \leq \ \frac{A_2}{b_n^2}(1+x)^2,\ x \in [0,\infty), \quad (9)$$

where $A_1 > 0,\ A_2 > 0$ and $\phi_n(p,q) = \max\left\{p^{n-1},\ b_n - a_n p^{n-1},\ \frac{1}{[3]_{p,q} b_n}\right\}$.

Proof. First, we estimate $\left(R^*_{n,p,q}((t-x),x)\right)^2$. For $x \in [0,\infty)$,

$$\begin{aligned}
&\left(R^*_{n,p,q}((t-x),x)\right)^2 = \left(R^*_{n,p,q}(t,x) - xR^*_{n,p,q}(1,x)\right)^2\\
&\leq \left(\frac{p^n}{[2]_{p,q}b_n} - \frac{(p-q)}{[2]_{p,q}}\frac{x}{1+a_n x} - \frac{a_n x^2}{1+a_n x}\right)^2\\
&\leq \frac{2p^{2n}}{\left([2]_{p,q}b_n\right)^2} + 2\left(\frac{(p-q)}{[2]_{p,q}}\frac{x}{1+a_n x} + \frac{a_n x^2}{1+a_n x}\right)^2\\
&\leq \frac{1}{b_n}\left\{\frac{1}{b_n} + \frac{(p^n-q^n)^2}{b_n}\left(\frac{1}{p+q} + \frac{1}{p-q}(a_n x)\right)^2\right\}.
\end{aligned}$$

For the estimation of $R^*_{n,p,q}\left((t-x)^2,x\right)$, we use the formula of $R_{n,p,q}\left((t-x)^2,x\right)$, which is calculated in [19].

$$\begin{aligned}
R^*_{n,p,q}\left((t-x)^2,x\right) &= \sum_{k=0}^{n} r^*_{n,k}(p,q,x)\int_0^1\left(\frac{p^{n-k}\left([k]_{p,q}+q^k t\right)}{b_n} - x\right)^2 d_{p,q}\,t\\
&\leq 2\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\int_0^1\left(\frac{p^{n-k}q^k}{b_n}\right)^2 t^2 d_{p,q}\,t.\\
&+2\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\left(\frac{p^{n-k}[k]_{p,q}}{b_n} - x\right)^2\\
&\leq 2\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\frac{p^{2(n-k)}q^{2k}}{[3]_{p,q}b_n^2} + 2R_{n,p,q}\left((t-x)^2,x\right)\\
&\leq \frac{2}{[3]_{p,q}b_n^2} + 2\left\{\frac{p^{n-1}}{b_n}x + \left(1 - \frac{p^{n-1}}{[n]_{p,q}}\right)x^2\right\}\\
&\leq \frac{2}{b_n}\left\{\frac{1}{[3]_{p,q}b_n} + p^{n-1}x + \left(b_n - a_n p^{n-1}\right)x^2\right\}
\end{aligned}$$

and we may simplify the last expression as follows:

$$\frac{2}{b_n}\left\{\frac{1}{[3]_{p,q}b_n} + p^{n-1}x + \left(b_n - a_n p^{n-1}\right)x^2\right\} \leq \frac{A_1}{b_n}\phi_n(p,q)(1+x)^2$$

where $A_1 > 0$ and $\phi_n(p,q) = \max\left\{p^{n-1},\ b_n - a_n p^{n-1},\ \frac{1}{[3]_{p,q}b_n}\right\}$.

Now for $x \in [0,\infty)$, we use similar calculations for the estimation of $R^*_{n,p,q}\Big((t-x)^4, x\Big)$.

$$R^*_{n,p,q}\Big((t-x)^4, x\Big) = \sum_{k=0}^{n} r^*_{n,k}(p,q,x)\int_0^1 \left(\frac{p^{n-k}\Big([k]_{p,q}+q^k t\Big)}{b_n} - x\right)^4 d_{p,q}\, t$$
$$\leq 8\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\int_0^1 \left(\frac{q^k p^{(n-k)}}{b_n}\right)^4 t^4\, d_{p,q}\, t$$
$$+8\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\left(\frac{p^{n-k}[k]_{p,q}}{b_n} - x\right)^4$$

by evaluating the (p,q)-integral and using the formula of $R_{n,p,q}\Big((t-x)^4, x\Big)$ which is given in [19], we get

$$R^*_{n,p,q}\Big((t-x)^4, x\Big) \leq 8\sum_{k=0}^{n} r^*_{n,k}(p,q,x)\frac{q^{4k}p^{4(n-k)}}{b_n^4[5]_{p,q}} + 8R_{n,p,q}\Big((t-x)^4, x\Big)$$
$$\leq \frac{8}{b_n^4[5]_{p,q}} + \frac{8}{b_n^2}C_2\,\varphi(p,q)(1+x)^2$$

where $C_2 > 0$ and $\varphi(p,q) > 0$. Now we can write

$$R^*_{n,p,q}\Big((t-x)^4, x\Big) \leq \frac{8}{b_n^2}\left\{\frac{1}{[5]_{p,q}} + C_2\,\varphi(p,q)(1+x)^2\right\}$$
$$\leq \frac{A_2}{b_n^2}(1+x)^2,$$

where $A_2 > 0$. □

Remark 2. *To investigate the convergence results of the operators $R^*_{n,p,q}$, let $q = q_n$, $p = p_n$ be the sequences such that $0 < q_n < p_n \leq 1$. If $q_n \to 1$ as $n \to \infty$ then by the Squeeze Theorem, $p_n \to 1$ which implies $\lim\limits_{n\to\infty}[n]_{p_n,q_n} = \infty$.*

In the following lemma we give two limits that later will be used to prove the Voronovskaja-type theorem for the operators $R^*_{n,p,q}(f,x)$.

Lemma 4. *Assume that $0 < q_n < p_n < 1$, $q_n \to 1$, $q_n{}^n \to 1$ as $n \to \infty$ and $0 < \beta < \frac{1}{2}$. Then we have the following limits*

$$(i)\ \lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}((t-x),x) = \frac{1}{2}$$
$$(ii)\ \lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}\Big((t-x)^2,x\Big) = x,$$

where $a_{n,p_n,q_n} = [n]^{\beta-1}_{p_n,q_n}$ and $b_{n,p_n,q_n} = [n]^{\beta}_{p_n,q_n}$.

Proof. For the proof of this lemma, we use the formulas of $R^*_{n,p_n,q_n}(t,x)$ and $R^*_{n,p_n,q_n}(t^2,x)$, which are given in Lemma 2. The first statement is clear,

$$\lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}((t-x),x) = \lim_{n\to\infty} b_{n,p_n,q_n}\Big(R^*_{n,p_n,q_n}(t,x) - x\Big)$$
$$= \lim_{n\to\infty} b_{n,p_n,q_n}\left(\frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} - \frac{(p_n-q_n)}{[2]_{p_n,q_n}}\frac{x}{1+a_{n,p_n,q_n}x} - \frac{a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}\right)$$
$$= \lim_{n\to\infty} b_{n,p_n,q_n}\frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} + \lim_{n\to\infty} a_{n,p_n,q_n}\frac{(q_n^n-p_n^n)}{[2]_{p_n,q_n}}\frac{x}{1+a_{n,p_n,q_n}x}$$
$$-\lim_{n\to\infty}\frac{[n]^{2\beta-1}_{p_n,q_n}x^2}{1+a_{n,p_n,q_n}x} = \frac{1}{2}.$$

For the second statement, we write

$$
\begin{aligned}
&\lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}\left((t-x)^2,x\right)\\
&=\lim_{n\to\infty} b_{n,p_n,q_n}\left\{R^*_{n,p_n,q_n}\left(t^2,x\right)-x^2-2xR^*_{n,p_n,q_n}((t-x),x)\right\}\\
&=\lim_{n\to\infty}\frac{p_n^{2n}}{[3]_{p_n,q_n}b_{n,p_n,q_n}}+\left\{\lim_{n\to\infty}\frac{\left(4q_n^3+5q_n^2p_n+3q_np_n^2\right)p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\times\lim_{n\to\infty}\frac{1}{1+a_{n,p_n,q_n}x}\right\}x\\
&+\lim_{n\to\infty} b_{n,p_n,q_n}\left\{\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\frac{1}{\left(1+a_{n,p_n,q_n}\right)^2}-1\right\}x^2\\
&-\lim_{n\to\infty} a_{n,p_n,q_n}p_n^{n-1}\times\lim_{n\to\infty}\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\times\lim_{n\to\infty}\frac{x^2}{\left(1+a_{n,p_n,q_n}x\right)^2}-x,
\end{aligned}
$$

now, if we substitute the following limits in the previous equality

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{p_n^{2n}}{[3]_{p_n,q_n}b_{n,p_n,q_n}}=0,\ \lim_{n\to\infty} a_{n,p_n,q_n}p_n^{n-1}=0,\ \lim_{n\to\infty}\frac{1}{\left(1+a_{n,p_n,q_n}x\right)}=1,\\
&\lim_{n\to\infty}\frac{\left(4q_n^3+5q_n^2p_n+3q_np_n^2\right)p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n}}=1,\ \lim_{n\to\infty}\frac{1}{\left(1+a_{n,p_n,q_n}x\right)}=\lim_{n\to\infty}\frac{1}{\left(1+a_{n,p_n,q_n}x\right)^2}=1,\\
&\lim_{n\to\infty} b_{n,p_n,q_n}\left\{\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}-1\right\}=0,\\
&\lim_{n\to\infty} a_{n,p_n,q_n}\left(q_n^n-p_n^n\right)\left(\frac{3q_n^3-p_n^3-p_n^2q_n-p_nq_n^2}{q_n^4-p_n^4+p_nq_n^3-q_np_n^3}\right)=0,
\end{aligned}
$$

we obtain $\lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}\left((t-x)^2,x\right)=x$, which proves the lemma. □

4. Local Approximation Theorem

In this section, we establish local approximation theorem for the new Kantorovich-type (p,q)-analogue of the Balázs–Szabados operators. Let $C_B[0,\infty)$ be the space of all the real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$. We consider the Peetre's K-functional (see [20]).

$$K_2(f,\delta) := \inf\left\{\|f-g\|+\delta\|g''\| : g\in C_B^2[0,\infty)\right\},\ \delta\geq 0,$$

where

$$C_B^2[0,\infty) := \left\{g\in C_B[0,\infty) : g',g''\in C_B[0,\infty)\right\}$$

From the known result given in [20], there exists an absolute constant $C_0>0$ such that

$$K_2(f,\delta)\leq C_0\omega_2\left(f,\sqrt{\delta}\right), \tag{10}$$

where $\omega_2\left(f,\sqrt{\delta}\right)=\sup_{0\leq h\leq\sqrt{\delta}}\ \sup_{x\pm h\in[0,\infty)}|f(x-h)-2f(x)+f(x+h)|$ is the second modulus of smoothness of $f\in C_B[0,\infty)$. Moreover, we let

$$\omega(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)}|f(x+h)-f(x)|.$$

First main result on the local approximation of the operators $R^*_{n,p_n,q_n}(f,x)$ is stated in the following theorem.

Theorem 1. *There exists an absolute constant $C > 0$ such that*

$$\left|R^*_{n,p_n,q_n}(f,x) - f(x)\right| \le C\,\omega_2\left(f, \sqrt{\eta_n(x)}\right) + \omega(f, \lambda_n(x))$$

where

$$f \in C_B[0,\infty),\ 0 \le x < \infty,\quad 0 < q_n < p_n < 1,$$

$$\eta_n(x) = \left\{\tfrac{A_1}{b_n}\phi_n(p_n,q_n)(1+x)^2\right\}$$

$$+\tfrac{1}{b_{n,p_n,q_n}}\left\{\tfrac{1}{b_{n,p_n,q_n}} + \tfrac{(p_n^n-q_n^n)^2}{b_{n,p_n,q_n}}\left(\tfrac{1}{p_n+q_n} + \tfrac{1}{p_n-q_n}\left(a_{n,p_n,q_n}x\right)\right)^2\right\}$$

and

$$\lambda_n(x) = \frac{p^n}{[2]_{p,q}b_n} + \left(\frac{(p-q)}{[2]_{p,q}} + a_n x\right)\frac{x}{1+a_n x}.$$

Proof. Let

$$\widetilde{R}^*_{n,p_n,q_n}(f,x) = R^*_{n,p_n,q_n}(f,x) + f(x) - f\left(\gamma_n + \alpha\frac{x}{1+a_{n,p_n,q_n}x}\right)$$

where $f \in C_B[0,\infty),\ \gamma_n = \frac{p_n^n}{[2]_{p_n,q_n}b_{n,p_n,q_n}},\ \alpha = \frac{2q_n}{[2]_{p_n,q_n}}$.

By using the Taylor's formula, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds, g \in C_B^2[0,\infty),$$

then, we have

$$\widetilde{R}^*_{n,p_n,q_n}(g,x) = g(x) + R^*_{n,p_n,q_n}\left(\int_x^t (t-s)g''(s)ds, x\right)$$
$$-\int_x^{\gamma_n+\alpha\frac{x}{1+a_{n,p_n,q_n}x}}\left(\gamma_n + \alpha\tfrac{x}{1+a_{n,p_n,q_n}x} - s\right)g''(s)ds$$

Hence,

$$\left|\widetilde{R}^*_{n,p_n,q_n}(g,x) - g(x)\right| \le R^*_{n,p_n,q_n}\left(\left|\int_x^t |t-s|\,|g''(s)|\,ds\right|, x\right)$$
$$+\left|\int_x^{\gamma_n+\alpha\frac{x}{1+a_{n,p_n,q_n}x}}\left|\gamma_n + \alpha\tfrac{x}{1+a_{n,p_n,q_n}x} - s\right|\,|g''(s)|\,ds\right| \tag{11}$$

$$\le \|g''\|\,R^*_{n,p_n,q_n}\left((t-x)^2, x\right) + \|g''\|\left(\gamma_n + \alpha\tfrac{x}{1+a_{n,p_n,q_n}x} - x\right)^2$$
$$= \|g''\|\,R_{n,p,q}\left((t-x)^2, x\right) + \|g''\|\left(R_{n,p,q}((t-x),x)\right)^2$$
$$\le \|g''\|\left\{\tfrac{A_1}{b_n}\phi_n(p,q)(1+x)^2\right\}$$
$$+\|g''\|\tfrac{1}{b_n}\left\{\tfrac{1}{b_n} + \tfrac{(p^n-q^n)^2}{b_n}\left(\tfrac{1}{p+q} + \tfrac{1}{p-q}(a_n x)\right)^2\right\} \tag{12}$$
$$= \|g''\|\,\eta_n(x).$$

Using (12) and the uniform boundedness of $\widetilde{R}^*_{n,p_n,q_n}$ we get

$$\begin{aligned}\left|R^*_{n,p_n,q_n}(f,x)-f(x)\right| &\le \left|\widetilde{R}^*_{n,p_n,q_n}((f-g),x)\right|+\left|\widetilde{R}^*_{n,p_n,q_n}(g,x)-g(x)\right| \\ &+|f(x)-g(x)|+\left|f\left(\gamma_n+\alpha\tfrac{x}{1+a_{n,p_n,q_n}x}\right)-f(x)\right| \\ &\le 4\|f-g\|+\|g''\|\eta_n(x)+\omega\left(f,\left|\gamma_n+\alpha\tfrac{x}{1+a_{n,p_n,q_n}x}-x\right|\right).\end{aligned}$$

If we take the infimum on the right-hand side overall, $g\in C_B^2[0,\infty)$, we obtain

$$\left|R_{n,p,q}(f,x)-f(x)\right|\le 4K_2(f;\eta_n(x))+\omega(f,\lambda_n(x)),$$

which together with (10) gives the proof of the theorem. □

Corollary 1. *Let $0<q_n<p_n\le 1,\ q_n\to 1$ as $n\to\infty$. Then for each $f\in C[0,\infty)$ the sequence $\left\{R^*_{n,p_n,q_n}(f,x)\right\}$ converges to f uniformly on $[0,a]$, $a>0$.*

One of the main problems in approximation theory is estimating the rate of convergence for sequences of positive linear operators. Voronovskaja-type formulas are one of the most important tools for studying their asymptotic behavior. Now, we give a Voronovskaja-type theorem for the new Kantorovich-type (p,q)-analogue of the Balázs–Szabados operators.

Theorem 2. *Assume that $0<q_n<p_n\le 1,\ q_n\to 1,\ q_n{}^n\to 1$ as $n\to\infty$ and let $a>0$, $0<\beta<\frac{1}{2}$. For any $f\in C_B^2[0,\infty)$ the following equality holds:*

$$\lim_{n\to\infty} b_{n,p_n,q_n}\left(R^*_{n,p_n,q_n}(f,x)-f(x)\right)=\frac{1}{2}f'(x)+\frac{1}{2}x f''(x),$$

uniformly on $[0,a]$.

Proof. Suppose that $f\in C_B^2[0,\infty)$ and $x\in[0,\infty)$ is fixed. By using Taylor's formula, we write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{13}$$

where the function $r(t,x)$ is the Peano form of the remainder $r(t,x)\in C_B[0,\infty)$ and $\lim\limits_{t\to x} r(t,x)=0$. Applying R^*_{n,p_n,q_n} to (13) we obtain

$$\begin{aligned}&b_{n,p_n,q_n}\left(R^*_{n,p_n,q_n}(f,x)-f(x)\right)\\ &=f'(x)b_{n,p_n,q_n}R^*_{n,p_n,q_n}((t-x),x)+\tfrac{1}{2}f''(x)b_{n,p_n,q_n}R^*_{n,p_n,q_n}\left((t-x)^2,x\right)\\ &+b_{n,p_n,q_n}R^*_{n,p_n,q_n}\left(r(t,x)(t-x)^2,x\right).\end{aligned} \tag{14}$$

By using Cauchy–Schwartz inequality, we get

$$R^*_{n,p_n,q_n}\left(r(t,x)(t-x)^2,x\right)\le\sqrt{R^*_{n,p_n,q_n}(r^2(t,x),x)}\sqrt{R^*_{n,p_n,q_n}\left((t-x)^4,x\right)}. \tag{15}$$

We observe that $r^2(x,x)=0$, $r^2(.,x)\in C_B[0,\infty)$. Then by the well-known Korovkin-type result, which is given in Corollary 1, it follows that

$$\lim_{n\to\infty}R^*_{n,p_n,q_n}\left(r^2(t,x),x\right)=r^2(x,x)=0, \tag{16}$$

uniformly for $x \in [0, a]$. Now by (15), (16), and Lemma 3, we get immediately

$$\lim_{n\to\infty} b_{n,p_n,q_n} R^*_{n,p_n,q_n}\left(r(t,x)(t-x)^2, x\right)$$
$$\leq \lim_{n\to\infty} \sqrt{R^*_{n,p_n,q_n}(r^2(t,x),x)} \times \lim_{n\to\infty} b_{n,p_n,q_n} \sqrt{R^*_{n,p_n,q_n}\left((t-x)^4, x\right)} = 0. \tag{17}$$

Then, substituting the limits given in Lemma 4 and using (17) in Equation (14), we get the desired result. □

5. Weighted Approximation

Let $B_\sigma(\mathbb{R}^+)$ be a weighted space of functions $f(x)$ defined on $\mathbb{R}^+ = [0, \infty)$ and satisfy the inequality $|f(x)| \leq L_f\, \sigma(x)$, where $\sigma(x)$ represents a weighted function that is continuously increasing on $\mathbb{R}^+ = [0, \infty)$, $\sigma(x) \geq 1$ and L_f represents a positive constant depending on f. The norm of each function f that belongs to $B_\sigma(\mathbb{R}^+)$ is given by $\|f\|_\sigma = \sup\limits_{x\in\mathbb{R}^+} \frac{|f(x)|}{\sigma(x)}$. We consider the following spaces:

$$C_\sigma[0,\infty) = \{f : f \in B_\sigma[0,\infty) \text{ and } f \text{ is continuous } \},$$
$$C^*_\sigma[0,\infty) = \left\{f : f \in C_\sigma[0,\infty) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\sigma(x)} < \infty \right\}.$$

Remark 3. *Let $\sigma(x)$ be a weighted function such that $\sigma(x) \geq 1$ and the inequality $\left|R^*_{n,p,q}(\sigma, x)\right| \leq L\sigma(x)$, $L > 0$, is satisfied.* Then we can say that the sequence of positive linear operators $\left(R^*_{n,p,q}\right)_{n\geq 1}$ *acts from $C_\sigma[0,\infty)$* to *$B_\sigma[0,\infty)$ (see [21]).*

Theorem 3. *Assume that $q = q_n$, $p = p_n$ are sequences such that $0 < q_n < p_n \leq 1$ and $q_n \to 1$* as *$n \to \infty$. Then for each $f \in C^*_\sigma[0,\infty)$, we have $\lim\limits_{n\to\infty} \|R^*_{n,p_n,q_n}(f,x) - f(x)\|_\sigma = 0$,* where *$\sigma(x) = 1 + x^2$.*

Proof. By using the Korovkin Theorem for weighted approximation which is given in [22], it is sufficient to show that

$$\lim_{n\to\infty} \|R^*_{n,p_n,q_n}(t^m, x) - x^m\|_\sigma = 0, \text{ for } m = 0,\ 1,\ 2. \tag{18}$$

Since $R^*_{n,p_n,q_n}(1,x) = 1$, (18) holds for $m = 0$. Now by Lemma 2, we have

$$R^*_{n,p_n,q_n}(t,x) - x = \frac{p_n^n}{[2]_{p_n,q_n} b_n} + \frac{2q_n}{[2]_{p_n,q_n}}\left(\frac{x}{1+a_{n,p_n,q_n}x}\right) - x$$
$$= \frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} - \frac{(p_n - q_n)}{[2]_{p_n,q_n}} \frac{x}{1+a_{n,p_n,q_n}x} - \frac{a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}.$$

Then, we obtain

$$\|R^*_{n,p_n,q_n}(t,x) - x\|_\sigma$$
$$\leq \sup_{0\leq x<\infty} \frac{1}{1+x^2}\left\{\frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} + \frac{(p_n-q_n)}{[2]_{p_n,q_n}} \frac{x}{1+a_{n,p_n,q_n}x} + \frac{a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}\right\}$$
$$\leq \frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} \sup_{0\leq x<\infty} \frac{1}{1+x^2} + \frac{(p_n-q_n)}{[2]_{p_n,q_n}} \sup_{0\leq x<\infty} \frac{x}{1+x^2(1+a_{n,p_n,q_n}x)}$$
$$+ a_{n,p_n,q_n} \sup_{0\leq x<\infty} \frac{x^2}{1+x^2(1+a_{n,p_n,q_n}x)},$$

now by taking limit overall the last inequality, we have

$$\lim_{n\to\infty}\|R^*_{n,p_n,q_n}(t,x)-x\|_\sigma \le \lim_{n\to\infty}\frac{p_n^n}{[2]_{p_n,q_n}b_{n,p_n,q_n}}+\lim_{n\to\infty}\frac{(p_n-q_n)}{[2]_{p_n,q_n}}+\lim_{n\to\infty}a_{n,p_n,q_n}=0.$$

Again, by using Lemma 2, we have

$$\begin{aligned}R^*_{n,p_n,q_n}(t^2,x)-x^2 &= \frac{p_n^{2n}}{[3]_{p_n,q_n}b_n^2}+\frac{(4q_n^3+5q_n^2p_n+3q_np_n^2)p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n}b_n}\left(\frac{x}{1+a_{n,p_n,q_n}x}\right)\\&+\frac{q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\left(\frac{x}{1+a_{n,p_n,q_n}x}\right)^2-x^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|R^*_{n,p_n,q_n}(t^2,x)-x^2\|_\sigma &\le \frac{p_n^{2n}}{[3]_{p_n,q_n}b_n^2}\times\sup_{0\le x<\infty}\frac{1}{1+x^2}\\&+\frac{(4q_n^3+5q_n^2p_n+3q_np_n^2)p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n}b_n}\times\sup_{0\le x<\infty}\frac{x}{(1+a_{n,p_n,q_n}x)(1+x^2)}\\&+\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\times\sup_{0\le x<\infty}\frac{x^2}{(1+a_{n,p_n,q_n}x)^2(1+x^2)}\\&+\left\{1-\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\right\}\times\sup_{0\le x<\infty}\frac{x^2}{(1+a_{n,p_n,q_n}x)^2(1+x^2)}.\end{aligned}$$

Now by taking limit overall the last inequality, we get

$$\begin{aligned}&\lim_{n\to\infty}\|R^*_{n,p_n,q_n}(t^2,x)-x^2\|_\sigma\\&\le \lim_{n\to\infty}\frac{p_n^{2n}}{[3]_{p_n,q_n}b_n^2}+\lim_{n\to\infty}\frac{(4q_n^3+5q_n^2p_n+3q_np_n^2)p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n}b_n}\times\lim_{n\to\infty}\frac{1}{(1+a_{n,p_n,q_n}x)}\\&+\lim_{n\to\infty}\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\times\lim_{n\to\infty}\frac{1}{(1+a_{n,p_n,q_n}x)^2}\\&+\lim_{n\to\infty}\left\{1-\frac{4q_n^3+q_n^2p_n+q_np_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}}\right\}\times\lim_{n\to\infty}\frac{1}{(1+a_{n,p_n,q_n}x)^2}=0.\end{aligned}$$

Therefore, we obtain the desired result $\lim_{n\to\infty}\|R^*_{n,p_n,q_n}(t^2,x)-x^2\|_\sigma=0.$ □

6. Conclusions

By using the notion of (p,q)-integers, we introduced a new Kantorovich-type (p,q)-analogue of the Balázs–Szabados operators. The new operators have an advantage compared with the previous analogues; they are capable of approximating integrable functions. In the case $p=1$ these polynomials reduce to the new Kantorovich-type q-analogue of the Balázs–Szabados operators which are defined by Hamal and Sabancigil in [14]. We established the moments of the operators with the help of the recurrence formula. We studied the local approximation properties of these new operators in terms of modulus of continuity and proved a Voronovskaja-type theorem. Lastly, we examined the weighted approximation properties of the operators.

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