symmetry MDPI

Article

Certain Subclasses of Analytic Functions Associated with Generalized Telephone Numbers

Gangadharan Murugusundaramoorthy [†] and Kaliappan Vijaya [*,†]

School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, Tamilnadu, India; gms@vit.ac.in or gmsmoorthy@yahoo.com
* Correspondence: kvijaya@vit.ac.in
† These authors contributed equally to this work.

Abstract: The goal of this article is to contemplate coefficient estimates for a new class of analytic functions f associated with generalized telephone numbers to originate certain initial Taylor coefficient estimates and Fekete–Szegö inequality for f in the new function class. Comparable results have been attained for the function f^{-1}. Further application of our outcomes to certain functions demarcated by convolution products with certain normalized analytic functions in the open unit disk are specified, and we obtain Fekete–Szegö variations for this new function class defined over Poisson and Borel distribution series.

Keywords: analytic functions; convex functions; starlike functions; subordination; Hadamard product; Fekete–Szegö inequality; Poisson distribution series; Borel distribution series

MSC: 30C45; 30C50; 30C80

check for updates

Citation: Murugusundaramoorthy, G.; Vijaya, K. Certain Subclasses of Analytic Functions Associated with Generalized Telephone Numbers. *Symmetry* **2022**, *14*, 1053. https://doi.org/10.3390/sym14051053

Academic Editor: Alexander Zaslavski

Received: 10 April 2022
Accepted: 9 May 2022
Published: 20 May 2022

1. Introduction and Preliminaries

1.1. *Telephonic Numbers:*

The conventional telephone numbers (or involution numbers) are quantified by the recurrence relation

$$\mathfrak{T}(n) = \mathfrak{T}(n-1) + (n-1)\mathfrak{T}(n-2) \quad for \quad n \geq 2$$

with initial conditions

$$\mathfrak{T}(0) = \mathfrak{T}(1) = 1.$$

In the year 1800, Heinrich August Rothe associated these numbers over symmetric groups and figured out that $\mathfrak{T}(n)$ is the number of involutions (self-inverse permutations) in the symmetric group (see, for example, [1,2]). Meanwhile, involutions resemble a standard Young tableaux, so it is correct that the nth involution number is also the number of the Young tableaux on the set $1, 2, \ldots, n$ (for details, see [3]). According to John Riordan, the above recurrence relation yields the number of construction patterns in a telephone system with n subscribers (see [4]). Wlochand Wolowiec-Musial [5] familiarized generalized telephone numbers $\mathfrak{T}(\ell, n)$ defined for integers $n \geq 0$ and $\ell \geq 1$ by the ensuing recursion:

$$\mathfrak{T}(\ell, n) = \ell T(\ell, n-1) + (n-1)\mathfrak{T}(\ell, n-2)$$

with initial conditions

$$\mathfrak{T}(\ell, 0) = 1, \mathfrak{T}(\ell, 1) = \ell,$$

and deliberate few properties. Lately, Bednarz and Wolowiec-Musial [6] gave accessible generalization of telephone numbers by

$$\mathfrak{T}_{\ell}(n) = \mathfrak{T}_{\ell}(n-1) + \ell(n-1)\mathfrak{T}_{\ell}(n-2)$$

for integers $n \geq 2$ and $\ell \geq 1$ with initial conditions

$$\mathfrak{T}_\ell(0) = \mathfrak{T}_\ell(1) = 1$$

They presented the generating function, matrix generators and direct formula for these numbers. In addition, they demonstrated a few properties of these numbers related with congruence. Most recently, Deniz [7] derived the exponential generating function (or the summation formula) for $\mathfrak{T}_\ell(n)$ as follows:

$$e^{(x+\ell\frac{x}{2})} = \sum_{n=0}^{\infty} \mathfrak{T}_\ell(n) \frac{x^n}{n!} \quad (\ell \geq 1)$$

As we can observe, if $\ell = 1$, then we obtain $\mathfrak{T}_\ell(n) \equiv \mathfrak{T}(n)$ classical telephone numbers. Clearly, $\mathfrak{T}_\ell(n)$ is for some values of n as

1. $\mathfrak{T}_\ell(0) = \mathfrak{T}_\ell(1) = 1;$
2. $\mathfrak{T}_\ell(2) = 1 + \ell;$
3. $\mathfrak{T}_\ell(3) = 1 + 3\ell;$
4. $\mathfrak{T}_\ell(4) = 1 + 6\ell + 3\ell^2;$
5. $\mathfrak{T}_\ell(5) = 1 + 10\ell + 15\ell^2;$
6. $\mathfrak{T}_\ell(6) = 1 + 15\ell + 45\ell^2 + 15\ell^3.$

We now consider the function

$$\begin{aligned} \Psi(\varsigma) : &= e^{(\varsigma+\ell\frac{\varsigma^2}{2})} \\ &= 1 + \varsigma + \frac{\varsigma^2}{2} + \frac{1+\ell}{6}\varsigma^3 + \frac{1+3\ell}{24}\varsigma^4 + \frac{3\ell^2+6\ell+1}{120}\varsigma^5 + \cdots \end{aligned}$$

with its domain of definition as the open unit disk

$$\Delta := \{\varsigma : \varsigma \in \mathbb{C} \quad \text{and} \quad |\varsigma| < 1\}$$

(for details, see Deniz [7]). Congruently, $\Psi(\varsigma)$ is an analytic in Δ, such that

1. $\Re\,(\Psi) > 0$ in Δ;
2. $\Psi(0) = 1$, $\Psi'(0) > 0$;
3. Ψ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

1.2. The Analytic Functions

Let $\mathfrak{A}$ signify the collection of functions f that are holomorphic in Δ and have the following normalized form:

$$f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n, \quad \varsigma \in \Delta, \tag{1}$$

with $f(0) = 0 = f'(0) - 1$. Further, let $\mathfrak{S}$ be the sub-collection of the set $\mathfrak{A}$ comprising univalent functions. A function $f \in \mathfrak{S}$ is supposed to be *starlike* in Δ if

$$\Re\Big(\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\Big) > 0, \quad (\varsigma \in \Delta) \tag{2}$$

and a function $f \in \mathfrak{S}$ is supposed to be *convex* in Δ if

$$\Re\Big(\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)}\Big) > 0, \quad (\varsigma \in \Delta) \tag{3}$$

holds. We correspondingly symbolize these function classes by $\mathfrak{S}^*$ and $\mathfrak{C}$sustaining Principles (2) and (3).

For $\mathrm{f}_1, \mathrm{f}_2 \in \mathfrak{A}$ and is analytic in Δ, so we say that f_1 is subordinate to f_2 if there exists a Schwarz function $w(\varsigma)$ that is holomorphic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(\varsigma)| < 1 \quad (\varsigma \in \Delta),$$

such that

$$\mathrm{f}_1(\varsigma) = \mathrm{f}_2(\omega(\varsigma)) \quad (\varsigma \in \Delta).$$

This subordination is symbolically written as follows:

$$\mathrm{f}_1 \prec \mathrm{f}_2 \quad \text{or} \quad \mathrm{f}_1(\varsigma) \prec \mathrm{f}_2(\varsigma) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, then the following equivalence holds true

$$\mathrm{f}_1(0) = \mathrm{f}_2(0) \quad \textit{and} \quad \mathrm{f}_1(\Delta) \subset \mathrm{f}_2(\Delta).$$

Raina and Sokół [8] have defined a function class

$$\mathfrak{S}^\star(\psi) = \{f \in \mathfrak{A} : \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \psi(\varsigma) =: \varsigma + \sqrt{1+\varsigma^2}, \;\; \varsigma \in \Delta, \}$$

and discussed some coefficient inequalities for $f \in \mathfrak{S}^\star(\psi)$. Further in view of the Alexander result between the class $f \in \mathfrak{C}(\psi) \Leftrightarrow \varsigma f'(\varsigma) \in \mathfrak{S}^*(\psi)$, Sokół and Thomas [9] improved the results for $f \in \mathfrak{C}(\psi)$. In recent years, several authors studied many families of holomorphic and univalent functions by varying the function ψ, as illustrated below:

1. For $\psi = \frac{1+A\varsigma}{1+B\varsigma}$ $(-1 \leq B < A \leq 1)$, we obtain the class $\mathcal{S}^*(A, B)$, see [10];
2. For different values of A and B, the class $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1-2\alpha, -1)$ is shown in [11];
3. For $\psi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{\varsigma}}{1-\sqrt{\varsigma}}\right)^2$, the class was defined and studied in [12];
4. For $\psi = \sqrt{1+\varsigma}$, the class is denoted by $\mathcal{S}^*_L$, details can be seen in [13] and further considered in [14];
5. If $\psi = 1 + \frac{4}{3}\varsigma + \frac{2}{3}\varsigma^2$, then such class denoted by $\mathcal{S}^*_C$ was introduced in [15] and further considered by [16];
6. For $\psi = e^\varsigma$, the class $\mathcal{S}^*_e$ was defined and dicussed in [17,18];
7. For $\psi = \cosh(\varsigma)$, the class is represented by $\mathcal{S}^*_{\cosh}$, see [19];
8. For $\psi = 1 + \sin(\varsigma)$, the class is signified by $\mathcal{S}^*_{\sin}$, see [20].

We recall a generalized function class due to Guo and Liu [21], as given below. For $\vartheta \geqq 0$, $\kappa \geqq 0$ and $0 \leqq \wp < 1$ and $f \in \mathfrak{A}$, we let $f \in Q(\vartheta, \kappa, \wp)$ if

$$\Re\left\{\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\left(\frac{f(\varsigma)}{z}\right)^\vartheta + \kappa\left[1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)} - \frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \vartheta\left(\frac{zf'(\varsigma)}{f(\varsigma)} - 1\right)\right]\right\} > \wp$$

holds. Inspired fundamentally by the aforementioned works (see [8,22–24]) on coefficient estimate problems, the Fekete–Szegö problem on certain subclasses of analytic functions (refers [25–31]), and Murugusundaramoorthy [32,33], in this paper, we describe the new class $\mathfrak{W}^\kappa_\vartheta(\Psi, \ell)$ given in Definition 1, which amalgamates the various subclasses of $\mathfrak{S}^*$ and $\mathfrak{C}$ in connotation with generalized telephonic numbers. We shall find estimations of the first few coefficients of f given by Equation (1) belonging to $\mathfrak{W}^\kappa_\vartheta(\Psi, \ell)$, and we show the Fekete–Szegö inequality and similarly for $f^{-1} \in \mathfrak{W}^\kappa_\vartheta(\Psi, \ell)$. Furthermore, we confer a certain application of our outcomes to functions defined over Borel and Poisson distribution series.

Now, we define the new family $\mathfrak{W}^\kappa_\vartheta(\Psi, \ell)$:

Definition 1. *For $\vartheta \geqq 0$, $\kappa \geqq 0$; $f \in \mathfrak{A}$ is in the class $\mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$ if*

$$\left\{ \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \left(\frac{f(\varsigma)}{\varsigma} \right)^{\vartheta} + \kappa \left[\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - \frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \vartheta \left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)} - 1 \right) \right] \right\}$$
$$\prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})} =: \Psi(\varsigma);\ \varsigma = re^{i\theta} \in \Delta. \tag{4}$$

By specializing the parameters ϑ and κ, we state the following new subclasses of $\mathfrak{S}$, as illustrated below, which are not discussed sofar for functions associated with telephonic numbers:

Remark 1.

1. $\mathfrak{W}^{0}_{0}(\Psi, \ell) \equiv \mathfrak{S}^*(\Psi, \ell) = \left\{ f \in \mathfrak{A} : \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})}; \varsigma = re^{i\theta} \in \Delta. \right\}$
2. $\mathfrak{W}^{\kappa}_{0}(\Psi, \ell) \equiv \mathfrak{W}_{\kappa}(\Psi, \ell) = \left\{ f \in \mathfrak{A} : \left[\kappa \frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} + (1-\kappa) \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \right] \prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})}; \varsigma = re^{i\theta} \in \Delta \right\}$
3. $\mathfrak{W}^{1}_{0}(\Psi, \ell) \equiv \mathfrak{C}(\Psi, \ell) = \left\{ f \in \mathfrak{A} : \frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} \prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})}; \varsigma = re^{i\theta} \in \Delta \right\}$
4. $\mathfrak{W}^{0}_{\vartheta}(\Psi, \ell) \equiv \mathfrak{B}_{\vartheta}(\Psi, \ell) = \left\{ f \in \mathfrak{A} : \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \left(\frac{f(\varsigma)}{z} \right)^{\vartheta} \prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})}; \varsigma = re^{i\theta} \in \Delta \right\}$
5. $\mathfrak{W}^{0}_{1}(\Psi, \ell) \equiv \mathfrak{R}(\Psi, \ell) = \left\{ f \in \mathfrak{A} : f'(\varsigma) \prec e^{(\varsigma + \ell \frac{\varsigma^2}{2})}; \varsigma = re^{i\theta} \in \Delta \right\}$

2. Coefficient Estimates and Fekete–Szegö Inequality

To prove our main result, we need the following:

Let $\mathfrak{P}$ be the family of functions holomorphic in Δ with $\mathsf{p}(0) = 1$ and

$$\Re(\mathsf{p}(\varsigma)) > 0 \qquad (\varsigma \in \Delta),$$

is given by

$$\mathsf{p}(\varsigma) = 1 + c_1 z + c_2 \varsigma^2 + \cdots. \tag{5}$$

Lemma 1 ([34]). *If $p(\varsigma) \in \mathfrak{P}$ and assumed as in Equation (5), then*

$$|c_2 - v c_1^2| \leqq \begin{cases} -4v + 2, & \text{if} \quad v \leqq 0, \\ 2, & \text{if} \quad 0 \leqq v \leqq 1, \\ 4v - 2, & \text{if} \quad v \geqq 1. \end{cases}$$

when $v < 0$ or $v > 1$, the equality holds true for $p_1(\varsigma) = \dfrac{1+\varsigma}{1-\varsigma}$ or one of its rotations. In addition, the equality holds if $p_2(\varsigma) = \dfrac{1+\varsigma^2}{1-\varsigma^2}$, when $0 < v < 1$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_3(\varsigma) = \left(\frac{1}{2} + \frac{1}{2}\eta \right) \frac{1+\varsigma}{1-\varsigma} + \left(\frac{1}{2} - \frac{1}{2}\eta \right) \frac{1-\varsigma}{1+\varsigma} \quad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Lemma 2 ([35]). *If $p(\varsigma) \in \mathfrak{P}$ and assumed as in Equation (5), then*

$$| \, c_n \, | \leq 2, (n \geq 1) \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 3 ([36]). *If $p(\varsigma) \in \mathfrak{P}$ and assumed as in Equation (5), and $vs. \in \mathbb{C}$, then*

$$|c_2 - vs.c_1^2| \leqq 2\max\{1, |2v - 1|\} \tag{6}$$

and is sharp for the functions

$$p_1(\varsigma) = \frac{1+\varsigma}{1-\varsigma}, \quad p_2(\varsigma) = \frac{1+\varsigma^2}{1-\varsigma^2}.$$

Lemma 4 ([37]). *If $p(\varsigma) \in \mathfrak{P}$ and is assumed as in Equation (5), then for any $\hbar \in \mathbb{C}$,*

$$\left|c_2 - \hbar\frac{c_1^2}{2}\right| \leq \max(2, 2|\hbar - 1|) = \begin{cases} 2, & 0 \leq \hbar \leq 2; \\ 2|\hbar - 1|, & \textit{elsewhere}. \end{cases}$$

the result is sharp for $p_1(\varsigma) = \frac{1+\varsigma}{1-\varsigma}$ or $p_2(\varsigma) = \frac{1+\varsigma^2}{1-\varsigma^2}$.

Theorem 1. *Let $\vartheta \geqq 0$ and $\kappa \geqq 0$ and f be assumed by Equation (1). If $f \in \mathfrak{W}_\vartheta^\kappa(\Psi, \ell)$, then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{(1+\vartheta)(1+\kappa)}, \\ |a_3| &\leqq \frac{1}{(\vartheta+2)(1+2\kappa)}\max\{1, \frac{1}{2}|\left(\frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right) - 1 - \ell|\}. \end{aligned}$$

These results are sharp.

Proof. Express the function $\mathsf{p}(\varsigma)$ by

$$\mathsf{p}(\varsigma) : \;=\; \frac{1+w(\varsigma)}{1-w(\varsigma)} = 1 + c_1\varsigma + c_2\varsigma^2 + \cdots.$$

It is informal to see that

$$\begin{aligned} w(\varsigma) &= \frac{\mathsf{p}(\varsigma)-1}{\mathsf{p}(\varsigma)+1} \\ &= \frac{1}{2}\left[c_1\varsigma + \left(c_2 - \frac{c_1^2}{2}\right)\varsigma^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\varsigma^3 + \cdots\right], \end{aligned} \tag{7}$$

where $w(\varsigma), w(0) = 0; |w(\varsigma)| < 1$ in Δ is a Schwarz function and regular in Δ. We perceive that $\Re(\mathsf{p}(\varsigma)) > 0$ and $\mathsf{p}(0) = 1$; thus

$$\begin{aligned} \Psi(w(\varsigma)) &= e^{\left(\frac{\mathsf{p}(\varsigma)-1}{\mathsf{p}(\varsigma)+1} + \ell\frac{[\frac{\mathsf{p}(\varsigma)-1}{\mathsf{p}(\varsigma)+1}]^2}{2}\right)} \\ &= 1 + \frac{c_1}{2}\varsigma + \left(\frac{c_2}{2} + \frac{(\ell-1)c_1^2}{8}\right)\varsigma^2 + \left(\frac{c_3}{2} + (\ell-1)\frac{c_1c_2}{4} + \frac{(1-3\ell)}{48}c_1^3\right)\varsigma^3 + \ldots.. \end{aligned} \tag{8}$$

If $f \in \mathfrak{W}_{\vartheta}^{\kappa}(\Psi, \ell)$, then there is a Schwarz function $w(\varsigma)$ such that

$$\begin{aligned} &\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\left(\frac{f(\varsigma)}{\varsigma}\right)^{\vartheta} + \kappa\left[\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - \frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \vartheta\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)} - 1\right)\right] \\ &= \Psi(w(\varsigma)) \\ &= e^{(w(\varsigma)+\kappa\frac{[w(\varsigma)]^2}{2})}. \end{aligned} \tag{9}$$

For assumed $f(\varsigma)$ as in Equation (1), a computation confirms that

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = 1 + a_2\varsigma + (2a_3 - a_2^2)\varsigma^2 + (3a_4 + a_2^3 - 3a_3a_2)\varsigma^3 + \cdots .$$

Similarly, we have

$$1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)} = 1 + 2a_2\varsigma + (6a_3 - 4a_2^2)\varsigma^2 + \cdots .$$

Let us describe the function $\mathfrak{W}(\varsigma)$ by

$$\mathfrak{W}(\varsigma) : \;=\; \frac{\varsigma f'(\varsigma)}{f(\varsigma)}\left(\frac{f(\varsigma)}{\varsigma}\right)^{\vartheta} + \kappa\left[\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - \frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \vartheta\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)} - 1\right)\right] \tag{10}$$

An easy computation shows that

$$\begin{aligned} \mathfrak{W}(\varsigma) &= 1 + (1+\vartheta)(1+\kappa)a_2\varsigma + (\vartheta+2)(1+2\kappa)a_3\varsigma^2 \\ &\quad + \left(\frac{\vartheta^2+\vartheta}{2} - (\vartheta+3)\kappa - 1\right)a_2^2\varsigma^2 + \cdots \\ &= 1 + b_1\varsigma + b_2\varsigma^2 + \cdots . \end{aligned} \tag{11}$$

Now by Equations (8) and (11),

$$b_1 = \frac{c_1}{2} \quad \text{and} \quad b_2 = \frac{c_2}{2} + \frac{(\ell-1)c_1^2}{8}. \tag{12}$$

In view of Equations (11) and (12), we see that

$$b_1 = (1+\vartheta)(1+\kappa)a_2, \tag{13}$$

$$b_2 = (\vartheta+2)(1+2\kappa)a_3 + \left(\frac{\vartheta^2+\vartheta}{2} - (\vartheta+3)\kappa - 1\right)a_2^2 \tag{14}$$

or equivalently, we have

$$a_2 = \frac{c_1}{2(1+\vartheta)(1+\kappa)}, \tag{15}$$

$$\begin{aligned} a_3 &= \frac{1}{(\vartheta+2)(1+2\kappa)}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left[1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right]\right) \\ &= \frac{1}{2(\vartheta+2)(1+2\kappa)}\left(c_2 - \frac{c_1^2}{4}\left[1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right]\right). \end{aligned} \tag{16}$$

Therefore, by Lemma 2, we have

$$a_2 \leq \frac{1}{(1+\vartheta)(1+\kappa)}$$

and by means of Approximation (6) specified in Lemma 3, we have

$$\begin{aligned} |a_3| &\leqq \frac{1}{(2+\vartheta)(1+2\kappa)} \max\{1, |2 \times \frac{1}{4}\left(1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right) - 1|\} \\ &= \frac{1}{(2+\vartheta)(1+2\kappa)} \max\{1, \frac{1}{2}|\left(\frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right) - 1 - \ell|\}. \end{aligned}$$

To show that the bounds are sharp, we express $K_{\Psi_n}(\varsigma)$, $\Psi_n = \Psi(\varsigma^{n-1})(n \geqq 2)$ with $K_{\Psi_n}(0) = 0 = [K_{\Psi_n}]'(0) - 1$, by

$$\frac{\varsigma K'_{\Psi_n}(\varsigma)}{K_{\Psi_n}(\varsigma)}\left(\frac{K_{\Psi_n}(\varsigma)}{\varsigma}\right)^{\vartheta} + \kappa\left[\frac{(\varsigma K'_{\Psi_n}(\varsigma))'}{K'_{\Psi_n}(\varsigma)} - \frac{\varsigma K'_{\Psi_n}(\varsigma)}{K_{\Psi_n}(\varsigma)} + \vartheta\left(\frac{\varsigma K'_{\Psi_n}(\varsigma)}{K_{\Psi_n}(\varsigma)} - 1\right)\right] = \Psi(\varsigma^{n-1}).$$

Evidently, $K_{\Psi_n} \in \mathfrak{W}_{\vartheta,\kappa}(\Psi, \ell)$. We inscribe $K_\Psi := K_{\Psi_2}$. That is, when n = 2, we obtain $\Psi(\varsigma) = e^{\varsigma + \frac{\ell\varsigma^2}{2}}$. Thus,

$$\begin{aligned} f(\varsigma) &= \int_0^{\varsigma} \Psi(t)dt \\ &= \int_0^{z} e^{t+\frac{\ell t^2}{2}} dt \\ &= \varsigma + \frac{\varsigma^2}{2} + \frac{1+\ell}{6}\varsigma^3 + \frac{1+3\ell}{24}\varsigma^4 + \frac{3\ell^2+6\ell+1}{120}\varsigma^5 + \cdots . \end{aligned}$$

Here, we have $b_1 = 1$ and $b_2 = 1/2$. By means of Equations (13) and (12), we obtain

$$|a_2| = \frac{1}{(1+\vartheta)(1+\kappa)}$$

and by using Equations (12) and (14), we have

$$c_2 + \frac{(\ell-1)c_1^2}{4} = (\vartheta+2)(1+2\kappa)a_3 + \left(\frac{\vartheta^2+\vartheta}{2} - (\vartheta+3)\kappa - 1\right)a_2^2.$$

Substituting for $a_2 = \frac{1}{(1+\vartheta)(1+\kappa)}$ modest computation and taking the modulus yields

$$|a_3| = \frac{1}{2(2+\vartheta)(1+2\kappa)}\left|\frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2} - \ell - 1\right|.$$

□

Remark 2. *Let* $\vartheta = 0$ *and* $\kappa \geqq 0$*. If* $f \in \mathfrak{W}_\kappa(\Psi, \ell)$ *is given by Equation (1), then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{1+\kappa}, \\ |a_3| &\leqq \frac{1}{2(1+2\kappa)} \max\{1, |\frac{\kappa^2+8\kappa+3}{2(1+\kappa)^2} + \ell|\} = \frac{1}{2(1+2\kappa)}\left(\frac{\kappa^2+8\kappa+3}{2(1+2\kappa)(1+\kappa)^2} + \ell\right). \end{aligned}$$

Remark 3. *Fixing* $\vartheta = 0$ *and* $\kappa = 0$*. If* $f \in \mathfrak{W}_0^0(\Psi, \ell) \equiv \mathfrak{S}^*(\Psi, \ell)$ *given by Equation (1), then*

$$|a_2| \leqq 1, \qquad \textit{and} \qquad |a_3| \leqq \frac{1}{2}\max\{1, |\frac{3}{2} + \ell|\} = \frac{1}{2}(\frac{3}{2} + \ell).$$

Remark 4. *Assuming* $\vartheta = 0$ *and* $\kappa = 1$, *and if* $f \in \mathfrak{W}_0^1(\Psi,\ell) \equiv \mathfrak{C}(\Psi,\ell)$ *is given by Equation* (1), *then*

$$|a_2| \leqq \frac{1}{2}, \qquad \textit{and} \qquad |a_3| \leqq \frac{1}{6}\max\{1, |\frac{1}{2}+\ell|\} = \frac{1}{6}(\frac{1}{2}+\ell).$$

Remark 5. *Taking* $\kappa = 0$, *and if* $f \in \mathfrak{W}_\vartheta^0(\Psi,\ell) \equiv \mathfrak{B}_\vartheta(\Psi,\ell)$ *given by* (1), *then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{1+\vartheta}, \\ |a_3| &\leqq \frac{1}{\vartheta+2}\max\{1, \frac{1}{2}|\frac{\vartheta^2+\vartheta-2}{(1+\vartheta)^2} - 1 - \ell|\} = \frac{1}{2(1+\vartheta)}\left(\frac{\vartheta+3}{(1+\vartheta)^2} + \ell\right). \end{aligned}$$

Remark 6. *Let* $\vartheta = 1$ *and* $\kappa = 0$. *If* $f(\varsigma)$ *given by* (1) *belongs to* $\mathfrak{W}_0^0(\Psi,\ell) = \mathfrak{R}(\Psi,\ell)$, *then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{1+\kappa}, \\ |a_3| &\leqq \frac{1}{3}\max\{1, \frac{1}{2}|1+\ell|\} = \frac{1}{6}(1+\ell). \end{aligned}$$

Theorem 2. *Let* $0 \leqq \mu \leqq 1$, $\vartheta \geqq 0$, $\kappa \geqq 0$ *and if* $f \in \mathfrak{W}_\vartheta^\kappa(\Psi,\ell)$ *be given in* (1), *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2\mathbf{q}}\left(1+\ell-\dfrac{\aleph}{\mathbf{w}^2}\right), & \textit{if} \quad \mu \leqq \sigma_1, \\ \dfrac{1}{\mathbf{q}}, & \textit{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{2\mathbf{q}}\left(-1-\ell+\dfrac{\aleph}{\mathbf{w}^2}\right), & \textit{if} \quad \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 = \frac{(\ell-1)\mathbf{w}^2 + 2(\vartheta+3)\kappa - \mathbf{m}}{2\mathbf{q}}; \sigma_2 = \frac{(3+\ell)\mathbf{w}^2 + 2(\vartheta+3)\kappa - \mathbf{m}}{2\mathbf{q}};$$

$$\aleph := \mathbf{m} - 2(\vartheta+3)\kappa + 2\mu\mathbf{q}, \tag{17}$$

$$\mathbf{m} := \vartheta^2 + \vartheta - 2, \tag{18}$$

$$\mathbf{q} := (\vartheta+2)(1+2\kappa), \tag{19}$$

and

$$\mathbf{w} := (1+\vartheta)(1+\kappa). \tag{20}$$

Proof. Using Equations (15) and (16), we obtain

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{1}{(\vartheta+2)(1+2\kappa)}\left(\frac{c_2}{2}-\frac{c_1^2}{8}\times\right.\\ &\qquad \left.\left[1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa+2\mu(\vartheta+2)(1+2\kappa)}{((1+\vartheta)(1+\kappa))^2}\right]\right)\\ &= \frac{1}{2(\vartheta+2)(1+2\kappa)}\left(c_2-\frac{c_1^2}{4}\right.\\ &\qquad \left.\left[1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa+2\mu(\vartheta+2)(1+2\kappa)}{((1+\vartheta)(1+\kappa))^2}\right]\right)\\ &= \frac{1}{2(\vartheta+2)(1+2\kappa)}\left(c_2-vc_1^2\right) \end{aligned}$$

where

$$\begin{aligned} v: &= \frac{1}{4}\left[1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa+2\mu(\vartheta+2)(1+2\kappa)}{((1+\vartheta)(1+\kappa))^2}\right]\\ &= \frac{1}{4}\left[1-\ell+\frac{\mathbf{m}-2(\vartheta+3)\kappa+2\mu\mathbf{q}}{\mathbf{w}^2}\right]. \end{aligned}$$

To illustrate that the bounds are sharp, we express F_η and G_η $(0\leqq\eta\leqq 1)$, correspondingly, with $F_\eta(0)=0=F_\eta'(0)-1$ and $G_\eta(0)=0=G_\eta'(0)-1$ by

$$\begin{aligned} \frac{\varsigma F_\eta'(\varsigma)}{F_\eta(\varsigma)}\left(\frac{F_\eta(\varsigma)}{\varsigma}\right)^\vartheta &+ \kappa\left[\frac{(\varsigma F_\eta'(\varsigma))'}{F_\eta'(\varsigma)}-\frac{\varsigma F_\eta'(\varsigma)}{F_\eta(\varsigma)}+\vartheta\left(\frac{\varsigma F_\eta'(\varsigma)}{F_\eta(\varsigma)}-1\right)\right]\\ &= \Upsilon\left(\frac{\varsigma(\eta+\varsigma)}{1+\eta\varsigma}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{\varsigma G_\eta'(\varsigma)}{G_\eta(\varsigma)}\left(\frac{G_\eta(\varsigma)}{\varsigma}\right)^\vartheta &+ \kappa\left[\frac{(\varsigma G_\eta'(\varsigma))'}{G_\eta'(\varsigma)}-\frac{\varsigma G_\eta'(\varsigma)}{G_\eta(\varsigma)}+\vartheta\left(\frac{\varsigma G_\eta'(\varsigma)}{G_\eta(\varsigma)}-1\right)\right]\\ &= \Upsilon\left(-\frac{\varsigma(\eta+\varsigma)}{1+\eta\varsigma}\right), \end{aligned}$$

respectively.

Clearly, the functions $K_{\Upsilon_n}=\Psi(\varsigma^{n-1}), F_\eta, G_\eta\in\mathfrak{W}_\vartheta^\kappa(\Psi,\ell)$. In addition, we write $K_{\Upsilon_2}:=\Psi(\varsigma)=e^{(\varsigma+\ell\frac{\varsigma^2}{2})}$.

If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if f is K_{Υ_2} or one of its rotations. The equality holds if and only if f is $K_{\Upsilon_3}=\Psi(\varsigma^2)=e^{(\varsigma^2+\ell\frac{\varsigma^4}{2})}$, i.e., $\Psi(\varsigma^2)=1+\varsigma^2+\frac{(1+\ell)\varsigma^4}{2}+\cdots$, when $\sigma_1<\mu<\sigma_2$ or one of its rotations. Here, we have $b_1=0$; $b_2=1$; thus, by substituting these values into Equations (13) and (14), we obtain sharp results.

The equivalence is true if and only if f is F_η when $\mu=\sigma_1$ or one of its turnings. If $\mu=\sigma_2$, the equivalence holds if and only if f is G_η or one of its rotation. □

By Lemma 3, we state the following:

Theorem 3. *Let* $0 \leqq \vartheta \leqq 1$, *and* $0 \leqq \kappa \leqq 1$. *For any* $\mu \in \mathbb{C}$, *and if* $f \in \mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$, *then*

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{(\vartheta+2)(1+2\kappa)} \\ &\times \max\left\{1, \frac{1}{2}\left|-1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa+2\mu(\vartheta+2)(1+2\kappa)}{((1+\vartheta)(1+\kappa))^2}\right|\right\} \\ &\leq \frac{1}{\mathbf{q}}\max\left\{1, \frac{1}{2}\left|-1-\ell+\frac{\mathbf{m}-2(\vartheta+3)\kappa+2\mu\mathbf{q}}{\mathbf{w}^2}\right|\right\} \end{aligned}$$

holds.

3. Coefficient Inequalities for the Function f^{-1}

Theorem 4. *If* $f \in \mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$ *and* $f^{-1}(\varpi) = \varpi + \sum\limits_{n=2}^{\infty} d_n \varpi^n$ *is the inverse function of* f *with* $|\varpi| < r_0; r_0 \geq \frac{1}{4}$ *the radius of the Kőebe domain of the class* $f \in \mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$, *then*

$$|d_2| \leq \frac{1}{2(1+\vartheta)(1+\kappa)}$$
$$|d_2| \leq \frac{1}{2\mathbf{q}} max\left\{1, \left|\frac{-(1+\ell)\mathbf{w}^2+4\mathbf{q}+\mathbf{m}-2(\vartheta+3)\kappa}{2\mathbf{w}^2}\right|\right\}.$$

For any complex number μ, *we have*

$$|d_3 - \mu d_2^2| \leq \frac{1}{\mathbf{q}} max\left\{1, \left|\frac{-(1+\ell)\mathbf{w}^2+\mathbf{m}-2(\vartheta+3)\kappa+(4-2\mu)\mathbf{q}}{2\mathbf{w}^2}\right|\right\}. \tag{21}$$

Proof. As

$$f^{-1}(\varpi) = \varpi + \sum_{n=2}^{\infty} d_n \varpi^n, \tag{22}$$

it can be understood that

$$f^{-1}(f(\varsigma)) = f\{f^{-1}(\varsigma)\} = \varsigma. \tag{23}$$

From Equations (1) and (23), we note that

$$f^{-1}(\varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n) = \varsigma. \tag{24}$$

From Equations (23) and (24), one can obtain

$$\varsigma + (a_2 + d_2)\varsigma^2 + (a_3 + 2a_2 d_2 + d_3)\varsigma^3 + \dots\dots\dots = \varsigma. \tag{25}$$

By associating the coefficients of ς and ς^2 from Relation (25), we understand that

$$d_2 = -a_2 \tag{26}$$
$$d_3 = 2a_2^2 - a_3. \tag{27}$$

From Relations (15), (16), (26), and (27)

$$d_2 = -\frac{c_1}{2(1+\vartheta)(1+\kappa)}; \tag{28}$$

$$\begin{aligned} d_3 &= \frac{c_1^2}{2[(1+\vartheta)(1+\kappa)]^2} - \frac{1}{2(\vartheta+2)(1+2\kappa)} \\ &\times \left(c_2 - \frac{c_1^2}{4}\left[1-\ell+\frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa))^2}\right]\right); \\ &= -\frac{1}{2\mathbf{q}}\left(c_2 - \frac{(1-\ell)\mathbf{w}^2+4\mathbf{q}+\mathbf{m}-2(\vartheta+3)\kappa}{4\mathbf{w}^2}c_1^2\right). \end{aligned} \tag{29}$$

For any complex number μ, consider

$$d_3 - \mu d_2^2 = -\frac{1}{2\mathbf{q}}\left(c_2 - \frac{(1-\ell)\mathbf{w}^2+\mathbf{m}-2(\vartheta+3)\kappa+(4-2\mu)\mathbf{q}}{4\mathbf{w}^2}c_1^2\right) \tag{30}$$

Taking the modulus and by using Lemma 3 on the right-hand side of Equation (30), one can acquire the result as in Equation (21). Hence, this concludes the proof. □

Remark 7. *By appropriately specifying ϑ, κ in Theorem 4, one can easily deduce the results for the function classes listed in Remark 1, which are new and have not been discussed so far in association with generalized telephone numbers.*

4. Application to Functions Defined by Certain Distributions Based on Convolution

The distributions, such as Binomial, Poisson, Pascal, logarithm, and hypergeometric, and their applications to the class of univalent functions have been intensively studied by various researchers in different prospectives.

Let $\varphi(\varsigma) = \varsigma + \sum_{n=2}^{\infty} \varphi_n \varsigma^n, \quad (\varphi_n > 0)$ and $f \in \mathfrak{A}$, then

$$\mathcal{F}(\varsigma) = (f * \varphi)(\varsigma) = \varsigma + \sum_{n=2}^{\infty} \varphi_n a_n \varsigma^n.$$

We define the class $\mathfrak{W}^{\varphi}_{\vartheta,\kappa}(\Psi, \ell)$ in the following way:

$$\mathfrak{W}^{\varphi}_{\vartheta,\kappa} := \{f \in \mathfrak{A} \quad \text{and} \quad \mathcal{F}(\varsigma) \in \mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)\} \tag{31}$$

where $\mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$ is as given in Definition 1.

In this section, for $f \in \mathfrak{W}^{\varphi}_{\vartheta,\kappa}(\Psi, \ell)$, we obtain a_2 and a_3 corresponding to $f \in \mathfrak{W}^{\kappa}_{\vartheta}(\Psi, \ell)$. Applying Theorem 2 to the function

$$\mathcal{F}(\varsigma) = (f * \varphi)(\varsigma) = \varsigma + \varphi_2 a_2 \varsigma^2 + \varphi_3 a_3 \varsigma^3 + \cdots \tag{32}$$

we obtain the following Theorems 5 and 6:

Theorem 5. *Let $0 \leqq \vartheta \leqq 1$, and $0 \leqq \kappa \leqq 1$. If $f \in \mathfrak{W}^{\varphi}_{\vartheta,\kappa}(\Psi, \ell)$, then for $\mu \in \mathbb{C}$, we have*

$$\begin{aligned} |a_3 - \mu a_2^2| \quad &= \frac{2}{(\vartheta+2)(1+2\kappa)\varphi_3} \\ & max\left\{1, \frac{1}{2}\left|-1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa}{((1+\vartheta)(1+\kappa))^2}+\frac{2\mu(\vartheta+2)(1+2\kappa)\varphi_3}{((1+\vartheta)(1+\kappa)\varphi_2)^2}\right|\right\}. \end{aligned}$$

Proof. For $f(\varsigma) \in \mathfrak{W}^{\varphi}_{\vartheta,\kappa}(\Psi, \ell)$, then by Equations (31) and (32), we have

$$\begin{aligned} p(\varsigma): \quad &= \quad \frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)}\left(\frac{\mathcal{F}(\varsigma)}{\varsigma}\right)^{\vartheta} + \kappa\left[\frac{\varsigma(\mathcal{F}'(\varsigma))'}{\mathcal{F}'(\varsigma)} - \frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} + \vartheta\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} - 1\right)\right] \\ &= \quad 1 + b_1\varsigma + b_2\varsigma^2 + \cdots. \end{aligned} \tag{33}$$

Proceeding online similar to Theorem 1, we obtain

$$p(\varsigma) = 1 + (1+\vartheta)(1+\kappa)\varphi_2 a_2 \varsigma + (\vartheta+2)(1+2\kappa)\varphi_3 a_3 \varsigma^2 + \left(\frac{\vartheta^2+\vartheta}{2} - (\vartheta+3)\kappa - 1\right)\varphi_2^2 a_2^2 \varsigma^2 + \cdots . \tag{34}$$

From Equations (13)–(16) and from Equation (34), we obtain

$$a_2 = \frac{c_1}{2(1+\vartheta)(1+\kappa)\varphi_2} \tag{35}$$

$$a_3 = \frac{1}{(\vartheta+2)(1+2\kappa)\varphi_3}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left(1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa)\varphi_2)^2}\right)\right) \tag{36}$$

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(\vartheta+2)(1+2\kappa)\varphi_3}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left(1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa)\varphi_2)^2}\right)\right) \\ &\quad -\mu\frac{c_1^2}{4(1+\vartheta)^2(1+\kappa)^2\varphi_2^2} \\ &= \frac{1}{(\vartheta+2)(1+2\kappa)\varphi_3}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left(1 - \ell + \frac{\vartheta^2+\vartheta-2(\vartheta+3)\kappa-2}{((1+\vartheta)(1+\kappa)\varphi_2)^2} - \mu\frac{2(\vartheta+2)(1+2\kappa)\varphi_3}{(1+\vartheta)^2(1+\kappa)^2\varphi_2^2}\right)\right). \end{aligned}$$

Thus, our result follows by taking the absolute value and applying Lemma 3. The equality holds for the functions given as

$$\frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)}\left(\frac{\mathcal{F}(\varsigma)}{z}\right)^{\vartheta} + \kappa\left[\frac{(\varsigma\mathcal{F}'(\varsigma))'}{\mathcal{F}'(\varsigma)} - \frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} + \vartheta\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} - 1\right)\right] = \Psi(\varsigma)$$

and

$$\frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)}\left(\frac{\mathcal{F}(\varsigma)}{\varsigma}\right)^{\vartheta} + \kappa\left[\frac{(\varsigma\mathcal{F}'(\varsigma))'}{\mathcal{F}'(\varsigma)} - \frac{z\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} + \vartheta\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} - 1\right)\right] = \Psi(\varsigma^2)$$

□

Theorem 6. *Let $\vartheta \geqq 0$, $\kappa \geqq 0$, $0 \leqq \mu \leqq 1$ and $\varphi_n > 0$ and f be given by Equation (1). If $f \in \mathfrak{W}^{\varphi}_{\vartheta,\kappa}(\Psi, \ell)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2\mathbf{q}\varphi_3}\left(1 + \ell - \dfrac{\aleph_2}{2\mathbf{w}^2}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{1}{\mathbf{q}\varphi_3}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{2\mathbf{q}\varphi_3}\left(-1 - \ell + \dfrac{\aleph_2}{2\mathbf{w}^2}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{\varphi_2^2}{\varphi_3}\left[\frac{(\ell-1)\mathbf{w}^2+2(\vartheta+3)\kappa-\mathbf{m}}{2\mathbf{q}}\right], \qquad \sigma_2 = \frac{\varphi_2^2}{\varphi_3}\left[\frac{(3+\ell)\mathbf{w}^2+2(\vartheta+3)\kappa-\mathbf{m}}{2\mathbf{q}}\right],$$

$$\aleph_2 := \mathbf{m}-2(\vartheta+3)\kappa+2\mu\frac{\varphi_3}{\varphi_2^2}\mathbf{q},$$

and $\mathbf{m}, \mathbf{q}, \mathbf{w}$ *are as defined in Equations (18)–(20).*

Proof. Using Equations (35) and (36) and following the steps in Theorems 2 and 5, we obtain the desired result. □

Suppose $\mathcal{X}$ is the Poisson distributed variable, and $\mathcal{X}$ takes the values $0,1,2,3,\cdots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, $\cdots$ correspondingly, where $m>0$. Thus,

$$P(\mathcal{X}=r)=\frac{e^{-m}m^r}{r!},\ r=0,1,2,3,\cdots.$$

In [38], Porwal presented a power series whose coefficients are probabilities of the Poisson distribution

$$\mathcal{K}(m,\varsigma)=\varsigma+\sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{(n-1)!}\varsigma^n, \qquad \varsigma\in\Delta,$$

where $m>0$. Through a ratio test, the radius of convergence of the above series is infinity. By means of the Hadamard product, Porwal [38] (see also, [22,39,40] presented a new linear operator $\mathcal{I}^m(\varsigma):\mathcal{A}\to\mathcal{A}$ defined by

$$\begin{aligned}\mathcal{I}^m f=\mathcal{K}(m,\varsigma)*f(\varsigma) &= \varsigma+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n\varsigma^n,\\ &= \varsigma+\sum_{n=2}^{\infty}\psi_m a_n\varsigma^n,\\ &= =\varsigma+\psi_2a_2\varsigma^2+\psi_3a_3\varsigma^3+\cdots \qquad \varsigma\in\Delta,\end{aligned}$$

where $\psi_n=\frac{m^{n-1}}{(n-1)!}e^{-m}$. Precisely,

$$\psi_2=me^{-m} \tag{37}$$

and

$$\psi_3=\frac{m^2}{2}e^{-m}. \tag{38}$$

From Equations (37) and (38), by taking $\varphi_2=me^{-m}$ and $\varphi_3=\frac{m^2}{2}e^{-m}$ in Theorems 5 and 6, we state the results in Theorem 7 and 8.

Theorem 7. *Let* $0\leqq\kappa\leqq1$, *and* $0\leqq\vartheta\leqq1$. *For* $\mu\in\mathbb{C}$ *and if* $f\in\mathfrak{W}^m_{\vartheta,\kappa}(q)$, *then,*

$$\begin{aligned}&|a_3-\mu a_2^2|\\ &=\frac{4}{(\vartheta+2)(1+2\kappa)m^2e^{-m}}\times\\ &\max\left\{1,\frac{1}{2}\left|-1-\ell+\frac{\vartheta^2+\vartheta-2-2(\vartheta+3)\kappa}{((1+\vartheta)(1+\kappa))^2}+\frac{\mu(\vartheta+2)(1+2\kappa)}{((1+\vartheta)(1+\kappa))^2e^{-m}}\right|\right\}.\end{aligned}$$

Theorem 8. *Let $0 \leqq \mu \leqq 1, \vartheta \geqq 0, \kappa \geqq 0$ and $\psi_n > 0$. If $f \in \mathfrak{W}^m_{\vartheta,\kappa}(q)$ be as given in Equation (1), then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{\mathbf{q}m^2e^{-m}}\left(1+\ell-\dfrac{\aleph_2}{2\mathbf{w}^2}\right), & \text{if} \quad \mu \leqq \sigma_1, \\ \dfrac{2}{\mathbf{q}m^2e^{-m}}, & \text{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{\mathbf{q}m^2e^{-m}}\left(-1-\ell+\dfrac{\aleph_2}{2\mathbf{w}^2}\right), & \text{if} \quad \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := e^{-m}\left[\frac{(\ell-1)\mathbf{w}^2+2(\vartheta+3)\kappa-\mathbf{m}}{2\mathbf{q}}\right], \qquad \sigma_2 = e^{-m}\left[\frac{(3+\ell)\mathbf{w}^2+2(\vartheta+3)\kappa-\mathbf{m}}{2\mathbf{q}}\right],$$

$$\aleph_2 := \mathbf{m} - 2(\vartheta+3)\kappa + \frac{\mu}{e^{-m}}\mathbf{q},$$

and $\mathbf{m}$, $\mathbf{q}$, and $\mathbf{w}$ are as defined in Equations (18)–(20).

Recently, there has been an interest in the application of a Borel distribution to the results for various subclasses of analytic and univalent functions (see [41–43]). Now, we state certain functional inequalities for $f \in \mathfrak{W}^\kappa_\vartheta(\Psi, \ell)$.

A discrete random variable X is said to follow a Borel distribution with parameter ϱ if its probability mass function $p(x)$ is given by

$$p(x=r) = \frac{(\varrho r)^{r-1}e^{-\varrho r}}{r!} \qquad r = 1,2,3,\cdots. \tag{39}$$

Recently, for $0 \leq \varrho \leq 1$, Wanas and Khuttar [44] presented a power series

$$\mathfrak{B}(\varrho,\varsigma) = \varsigma + \sum_{n=2}^{\infty} \frac{(\varrho(n-1))^{n-2}e^{-\varrho(n-1)}}{(n-1)!}\varsigma^n \quad (\varsigma \in \Delta) \tag{40}$$

whose coefficients are the probabilities of the Borel distribution. It has be shown that the radius of convergence of the above series is infinity through a ratio test.

Let us introduce a linear operator$L_\varrho : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\begin{aligned} L_\varrho f(\varsigma) = \mathfrak{B}(\varrho,\varsigma) * f(\varsigma) &= \varsigma + \sum_{n=2}^{\infty} \frac{(\varrho(n-1))^{n-2}e^{-\varrho(n-1)}}{(n-1)!}a_n\varsigma^n \\ &= \varsigma + \sum_{n=2}^{\infty} \Lambda_n a_n \varsigma^n \\ &= \varsigma + \Lambda_2 a_2 \varsigma^2 + \Lambda_3 a_3 \varsigma^3 + \cdots, \end{aligned} \tag{41}$$

where $\Lambda_n = \Lambda_n(\varrho) = \frac{(\varrho(n-1))^{n-2}e^{-\varrho(n-1)}}{(n-1)!}$. In particular, by fixing $n=2$ and $n=3$, we have

$$\Lambda_2 = e^{-\varrho} \quad \Lambda_3 = \varrho e^{-2\varrho}.$$

Now, by taking

$$\varphi_2 = \Lambda_2 = e^{-\varrho} \quad \varphi_3 = \Lambda_3 = \varrho e^{-2\varrho}.$$

one can easily state the results associated with the Borel distribution.

Remark 8. *By suitably fixing the values for ϑ, κ in Theorems 7 and 8, we derive the results for the functionals listed in Remark 1 by subordinating with telephonic numbers that are related with Poisson (or Borel) distribution series by convolution, which yields new results not yet discussed.*

5. Conclusions

In the current paper, we mainly obtain the upper bounds of the initial Taylors coefficients of the generalized function class $\mathfrak{W}_{\vartheta}^{\kappa}(\Upsilon, \ell)$ defined Δ and obtained initial coefficient estimates for $f \in \mathfrak{W}_{\vartheta}^{\kappa}(\Upsilon, \ell)$. Furthermore, we find the Fekete–Szegö inequalities for function in $f \in \mathfrak{W}_{\vartheta}^{\kappa}(\Upsilon, \ell)$. Several consequences of the results are also pointed out by suitably fixing the parameters in Theorem 1 as a remark. In Theorem 2, we deduced the Fekete–Szegö functional for these function classes. Comparable results have been attained for the function f^{-1}. We acquire Fekete–Szegö variations for certain subclasses of functions defined over Poisson and Borel distribution series. Then, we applied our outcomes to certain functions demarcated by convolution products. Furthermore, motivating further research on this subject, we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications and developments in the area of mathematical analysis based on hypergeometric functions [45]. In conclusion, we choose to reiterate an important observation, which was presented in the recently published review-cum-expository review article by Srivastava ([31], p. 340), new $q-$ analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called $(q)-$ analogues (with $0 < |q| \leq 1$) by applying some obvious parametric variations.

Author Contributions: Conceptualization, G.M. and K.V.; methodology, G.M. and K.V.; validation, G.M. and K.V.; formal analysis, G.M. and K.V.; investigation, G.M. and K.V.; resources, G.M. and K.V.; writing—original draft preparation, G.M. and K.V.; writing—review and editing, G.M. and K.V.; supervision, G.M. and K.V.; project administration, G.M. and K.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors are grateful to the reviewers of this article who gave insightful comments and advice that allowed us to revise and improve the content of the paper.

Conflicts of Interest: I declare that I do not have any commercial or associative interest that represent conflicts of interest in connection with this manuscript. There are no professional or other personal interest that can inappropriately influence our submitted work.

References

1. Chowla, S.; Herstein, I.N.; Moore, W.K. On recursions connected with symmetric groups I. *Can. J. Math.* **1951**, *3*, 328–334. [CrossRef]
2. Knuth, D.E. *The Art of Computer Programming*; Addison-Wesley: Boston, MA, USA, 1973; Volume 3.
3. Beissinger, J.S. Similar constructions for Young tableaux and involutions,and their applications to shiftable tableaux. *Discret. Math.* **1987**, *67*, 149–163. [CrossRef]
4. Riordan, J. *Introduction to Combinatorial Analysis*; Dover: Mineola, TX, USA, 2002.
5. Włoch, A.; Wołowiec-Musiał, M. On generalized telephone number, their interpretations and matrix generators. *Util. Math.* **2017**, *10*, 531–539.
6. Bednarz, U.; Wolowiec-Musial, M. On a new generalization of telephone numbers. *Turk. J. Math.* **2019**, *43*, 1595–1603. [CrossRef]
7. Deniz, E. Sharp coefficient bounds for starlike functions associated with generalized telephone numbers. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 1525–1542. [CrossRef]
8. Raina, R.K.; Sokół, J. On coeffcient estimates for a certain class of starlike functions. *Hacet. J. Math. Stat.* **2015**, *44*, 1427–1433.
9. Sokół, J.; Thomas, D.K. Further results on a class starlike functions related to the Bernoulli lemniscate. *Houst. J. Math.* **2018**, *44*, 8–95.
10. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*, 159–177. [CrossRef]

11. Robertson, M.S. Certain classes of starlike functions. *Mich. Math. J.* **1985**, *32*, 135–140. [CrossRef]
12. Ronning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
13. Sokół, J. Radius problem in the class $\mathcal{SL}^*$. *Appl. Math. Comput.* **2009**, *214*, 569–573. [CrossRef]
14. Mohsin, M.; Malik, S.N. Upper bound of third hankel determinant for class of analytic functions related with lemniscate of bernoulli. *J. Inequal. Appl.* **2013**, 412. [CrossRef]
15. Sharma, K.; Jain, N.K.; Ravichandran,V. Starlike functions associated with a cardioid. *Afr. Mat.* **2016**, *27*, 923–939. [CrossRef]
16. Shi, L.; Izaz, A.; Arif, M.; Cho, N.E.; Hussain, S.; Hassan, K. A study of third hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain. *Mathematics* **2019**, *7*, 418. [CrossRef]
17. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
18. Shi, L.; Srivastava, H.M.; Arif, M.; Hussain, S.; Khan, H. An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. *Symmetry* **2019**, *11*, 14. [CrossRef]
19. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikeness associated with cosine hyperbolic function. *Mathematics* **2020**, *8*, 16. [CrossRef]
20. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran,V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
21. Guo, D.; Liu, M-S. On certain subclass of Bazilevič functions. *J. Inequal. Pure Appl. Math.* **2007**, *8*, 12.
22. El-Deeb, S.M.; Bulboacă, T. Fekete-Szegö inequalities for certain class of analytic functions connected with q-anlogue of Bessel function. *J. Egyptian. Math. Soc.* **2019**, 1–11. [CrossRef]
23. Murugusundaramoorthy, G.; Bulboacă, T. Hankel determinants for new subclasses of analytic functions related to a shell shaped region. *Mathematics* **2020**, *8*, 1041. [CrossRef]
24. Sharma, R.B.; Haripriya, M. On a class of a-convex functions subordinate to a shell shaped region. *J. Anal.* **2016**, *25*, 93–105. [CrossRef]
25. Kowalczyk, B.; Lecko, A.; Srivastava, H.M. A note on the Fekete-Szegö problem for close-to-convex functions with respect to convex functions. *Publ. Inst. Math.* **2017**, *101*, 143–149. [CrossRef]
26. Srivastava, H.M.; Mishra, A.K.; Das, M.K. The Fekete-Szegö problem for a subclass of close-to-convex functions. *Complex Var. Theory Appl.* **2001**, *44*, 145–163. [CrossRef]
27. Srivastava, H.M.; Khan, N.; Darus, M.; Khan, S.; Ahmad, Q. Z.; Hussain,S. Fekete-Szegö type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points. *Mathematics* **2020**, *8*, 842. [CrossRef]
28. Srivastava, H.M.; Raza, N.; Abu Jarad, E.S.A.; Srivastava, G.; AbuJarad, M.H. Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* **2019**, *113*, 3563–3584. [CrossRef]
29. Srivastava, H.M.; Hussain, S.; Raziq, A.; Raza, M. The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination. *Carpathian J. Math.* **2018**, *34*, 103–113. [CrossRef]
30. Srivastava, H.M.; Mostafa, A.O.; Aouf, M.K.; Zayed, H.M. Basic and fractional q-calculus and associated Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order. *Miskolc Math. Notes* **2019**, *20*, 489–509. [CrossRef]
31. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44* , 327–344. [CrossRef]
32. Murugusundaramoorthy, G. Fekete-Szegö inequalities for certain subclasses of analytic functions related with leaf-like domain. *Montes Taurus. J. Pure Appl. Math.* **2021**, *3*, 305–316.
33. Murugusundaramoorthy, G. Fekete-Szegö inequality for certain subclasses of analytic functions related with crescent-shaped domain and application of Poison distribution series. *J. Math. Exten.* **2021**, *15*, 1–20. [CrossRef]
34. Ma, W.; Minda, D. A Unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; Int. Press: Cambridge, UK, 1994; pp. 157–169.
35. Grenander, U.; Szegö,G. *Toeplitz Forms and Their Applications, California Monographs in Mathematical Sciences*; University of California Press: Berkeley, CA, USA, 1958.
36. Libera, R.J.; Zlotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Amer. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]
37. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
38. Porwal, S. An application of a Poisson distribution series on certain analytic functions. *J. Complex Anal.* **2014**, *2014*, 984135. [CrossRef]
39. Murugusundaramoorthy, G. Subclasses of starlike and convex functions involving Poisson distribution series. *Afr. Mat.* **2017**, *28*, 1357–1366. [CrossRef]
40. Murugusundaramoorthy, G.; Vijaya, K.; Porwal, S. Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series. *Hacet. J. Math. Stat.* 2016, *45*, 1101–1107. [CrossRef]
41. El-Deeb, S.M.; Murugusundaramoorty, G.; Alburaikan, A. A bi-Bazilevi˘c functions based on the Mittag-Leffler-Type Borel distribution associated with Legendre polynomials. *J. Math. Comput. Sci.* **2021**, *24*, 235–245. [CrossRef]

42. El-Deeb, S.M.; Murugusundaramoorthy, G. Applications on a subclass of β-uniformly starlike functions connected with -Borel distribution. *Asian Eur. J. Math.* **2021**, 2250158. [CrossRef]
43. Srivastava, H.M.; Murugusundaramoorty, G.; El-Deeb, S.M., Faber polynomial coefficient estimates of bi-close-to-convex functions connected with Borel distribution of the Mittag-Leffler-type. *J. Nonlinear Var. Anal.* **2021**, *5*, 103–118.
44. Wanas, A.K.; Khuttar, J.A. Applications of Borel distribution series on analytic functions. *Earthline J. Math. Sci.* **2020**, **4**, 71–82. [CrossRef]
45. Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. *Canad. J. Math.* **1987**, *39*, 1057–1077. [CrossRef]