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Asymmetry Model Based on Quasi Local Odds Symmetry for Square Contingency Tables

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Abstract: For the analysis of square contingency tables, the primary objective is to estimate an unknown distribution from presented data. To achieve this objective, we generally use a statistical model that fits the presented data well and has a parsimony. The recently proposed quasi local odds symmetry (QLOS) model was compared to various models that represent the structure of symmetry or asymmetry, and it provided the best fit performance compared with other models for real data. However, the QLOS model has many parameters, that is, the QLOS model is not the parsimonious model. To address this issue, this study proposes a new model that is more parsimonious than the QLOS model. The proposed model is identical to the QLOS model under the specified condition; it is the asymmetry model based on the QLOS model.Moreover, we compare the proposed model with the existing models, including the QLOS model, and show that the proposed model provides better fit performance than the existing models for real datasets.

Keywords: local odds ratio; model selection; parsimony; quasi symmetry

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1. Introduction

For analyzing two-way contingency tables, the independence model is generally used. However, for the analysis of square contingency tables with the same row and column classifications, the independence model does not generally hold. This is because many observations tend to fall in (or near) the main diagonal cells of the table. For this reason, we are interested in whether the row variable is symmetric or asymmetric with the column variable.

Consider an $r \times r$ square contingency table that has the same row and column classifications with ordinal categories. Let π_{ij} denote the cell probability that an observation will fall in the i-th row and j-th column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). For the analysis of square contingency tables, the primary objective is to estimate an unknown distribution from presented data. To achieve this objective, analysts generally use a statistical model that fits the presented data well and has a parsimony.

The symmetry (S) model proposed by Bowker [1] is defined by

$$\pi_{ij} = \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \psi_{ji}$ for $i < j$. The S model has the symmetric structure with respect to the main-diagonal cells of the table, and is the base model for analyzing square contingency tables. We consider the following log-linear model:

$$\log \pi_{ij} = \lambda + \lambda_1(i) + \lambda_2(j) + \lambda_{12}(ij) \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r. \tag{1}$$

The S model can be expressed as Equation (1) with $\lambda_1(i) = \lambda_2(i)$ and $\lambda_{12}(ij) = \lambda_{12}(ji)$ for $i = 1, \ldots, r; j = 1, \ldots, r$; see Bishop et al. [2] (p. 282).

Thereafter, various models having the symmetric or asymmetric structure have been proposed, see Tahata and Tomizawa [3]. Among them, the marginal homogeneity (MH) model [4] and the quasi-symmetry (QS) model [5] are very famous, as well as the S model.

As the marginal asymmetry models based on the MH model, for example, Tahata and Tomizawa [6] proposed the m-additional parameters marginal homogeneity (MH(m)) model, and Kurakami et al. [7] proposed the m-th generalized marginal cumulative logistic (L(m)) model, for a fixed m ($m = 1, \ldots, r-1$). The MH(1) and MH(2) models are identical to the extended marginal homogeneity model [8] and the generalized marginal homogeneity model [9], respectively. The L(1) and L(2) models are identical to the marginal cumulative logistic model [10] (p. 442) and the extended marginal cumulative logistic model [11], respectively. For the details of these marginal asymmetry models, see Tahata and Tomizawa [3].

This study deals with the models associated with the QS model proposed by Caussinus [5]. The QS is defined by

$$\pi_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \psi_{ji}$ for $i < j$. The QS model can be expressed as Equation (1) with $\lambda_{12}(ij) = \lambda_{12}(ji)$ for $i = 1, \ldots, r; j = 1, \ldots, r$. The QS model with $\{\alpha_i = \beta_i\}$ is identical to the S model. The QS model with $\{\alpha_i = \alpha^i\}$ and $\{\beta_j = \beta^j\}$ is identical to the linear diagonals-parameter symmetry (LDPS) model proposed by Agresti [12]. Thus, the LDPS model is more parsimonious than the QS model.

For a fixed k ($k = 1, \ldots, r-1$), Tahata and Tomizawa [13] considered the k-th linear asymmetry (LA$_k$) model. The LA$_k$ model is defined by

$$\pi_{ij} = \left(\prod_{l=1}^{k} \alpha_l^{i^l} \beta_l^{j^l} \right) \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \psi_{ji}$ for $i < j$. The LA$_k$ can be expressed as various models that are more parsimonious than the QS model. This is because the LA$_{r-1}$ and LA$_1$ models are identical to the QS and LDPS models, respectively. We point out that the LA$_k$ model is the asymmetry model based on the QS model.

As an extension of the QS model, Tomizawa [8] proposed the extended quasi-symmetry (EQS) model defined by

$$\pi_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \delta \psi_{ji}$ for $i < j$. The EQS model with $\delta = 1$ is identical to the QS model. The EQS model with $\{\alpha_i = \alpha^i\}$ and $\{\beta_j = \beta^j\}$ is identical to the two ratio parameter symmetry (TRPS) model proposed by Tomizawa [14]. The TRPS model with $\delta = 1$ is identical to the LDPS model. In line with the relation between the QS and LDPS models, the TRPS model is more parsimonious than the EQS model.

For a fixed k ($k = 1, \ldots, r-1$), Tahata et al. [15] proposed the extended k-th linear asymmetry (ELA$_k$) defined by

$$\pi_{ij} = \left(\prod_{l=1}^{k} \alpha_l^{i^l} \beta_l^{j^l} \right) \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \delta \psi_{ji}$ for $i < j$. The ELA$_k$ model with $\delta = 1$ is identical to the LA$_k$ model. The ELA$_{r-1}$ and ELA$_1$ models are identical to the EQS and TRPS models, respectively. We point out that the ELA$_k$ model is the asymmetry model based on the EQS model.

As an extension of the EQS model, Altun [16] recently proposed the quasi local odds symmetry (QLOS) defined by

$$\pi_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for } i = 1, \ldots, r; j = 1, \ldots, r,$$

where $\psi_{ij} = \delta \omega^{i+j} \psi_{ji}$ for $i < j$. The QLOS model can also be expressed as

$$\frac{\pi_{ik}\pi_{jl}}{\pi_{il}\pi_{jk}} = \frac{\pi_{ki}\pi_{lj}}{\pi_{li}\pi_{kj}} \quad \text{for } 1 \le i < j < k < l \le r.$$

The QLOS model has a structure of the equality of the local odds ratio between one side of the main diagonal and the corresponding other side in square contingency tables. The QLOS model was compared to various models that have the symmetric or asymmetric structure, and it provided best fit performance compared with other models for two real datasets; see Altun [16]. However, the QLOS model has many parameters ($r(r+3)/2$), that is, the QLOS model is not the parsimonious model. Thus, the QLOS model may not have a parsimony, although the QLOS model may fit the presented data. Moreover, when $r = 4$, the QLOS model can also be expressed as

$$\frac{\pi_{13}\pi_{24}}{\pi_{14}\pi_{23}} = \frac{\pi_{31}\pi_{42}}{\pi_{41}\pi_{32}}.$$

Therefore, the QLOS model is saturated on the (1, 2)th, (3, 4)th, (2, 1)th and (4, 3)th cells as well as the main diagonal of the 4×4 table. To address these issues, this study proposes a new model that is more parsimonious than the QLOS model.

The rest of this paper is organized as follows. Section 2 describes the details of a new model for square contingency tables. Section 3 introduces the goodness-of-fit test for the proposed model and the methods for choosing the best fitting model among the applied models. Section 4 shows the utility of the proposed model using real datasets. We close with concluding remarks in Section 5.

2. Proposed Model

We propose the generalized 1-th linear asymmetry (GLA$_1$) model, that is, the QLOS model with $\{\alpha_i = \alpha^i\}$ and $\{\beta_j = \beta^j\}$. The GLA$_1$ model is defined by

$$\pi_{ij} = \alpha_1^i \beta_1^j \psi_{ij} \quad \text{for } i = 1,\dots,r; j = 1,\dots,r,$$

where $\psi_{ij} = \delta\omega^{i+j}\psi_{ji}$ for $i < j$. The GLA$_1$ model can be expressed as

$$\frac{\pi_{ij}}{\pi_{ji}} = \theta_1^{j-i}\delta\omega^{i+j} \quad \text{for } i < j,$$

where $\theta_1 = \beta_1/\alpha_1$. The GLA$_1$ models with $\omega = 1$ and $\delta = \omega = 1$ are identical to the TRPS and LDPS models, respectively. It must be noted that the GLA$_1$ model is saturated on only the main diagonal of the $r \times r$ table.

Denote the row and column variables by X and Y, respectively. Under the GLA$_1$ model, (1) if the $\delta > 1$, $\theta_1 > 1$ and $\omega > 1$, then $\Pr(X \le i) > \Pr(Y \le i)$ for every $i = 1,\dots,r-1$, and (2) if the $\delta < 1$, $\theta_1 < 1$ and $\omega < 1$, then $\Pr(X \le i) < \Pr(Y \le i)$ for every $i = 1,\dots,r-1$. Thus, the parameters of the GLA$_1$ model would be useful for making inferences, such as that X is stochastically less than Y or vice versa.

For a fixed k ($k = 1,\dots,r-1$), we propose the generalized k-th linear asymmetry (GLA$_k$) model. The GLA$_k$ model is defined by

$$\pi_{ij} = \left(\prod_{l=1}^{k} \alpha_l^{i^l}\beta_l^{j^l}\right)\psi_{ij} \quad \text{for } i = 1,\dots,r; j = 1,\dots,r,$$

where $\psi_{ij} = \delta\omega^{i+j}\psi_{ji}$ for $i < j$. The GLA$_k$ models with $\omega = 1$ and $\delta = \omega = 1$ are identical to the ELA$_k$ and LA$_k$ models, respectively.

In line with the GLA$_1$, for a fixed k ($k = 1,\dots,r-1$), the GLA$_k$ model can be expressed as:

$$\frac{\pi_{ij}}{\pi_{ji}} = \left(\prod_{l=1}^{k} \theta_l^{j^l - i^l}\right)\delta\omega^{i+j} \quad \text{for } i < j,$$

where $\theta_l = \beta_l/\alpha_l$. Moreover, the GLA$_k$ model can be expressed as

$$\frac{\pi_{ij}}{\pi_{ji}} = \left(\prod_{l=0}^{k} \frac{\theta_l^{j^l}}{\theta_l^{i^l}}\right) \delta\omega^{i+j} \quad \text{for } i < j.$$

Here, we discuss the relation between the QLOS and GLA$_{r-1}$ models. The QLOS model can be expressed as

$$\frac{\pi_{ij}}{\pi_{ji}} = \frac{\gamma_j}{\gamma_i}\delta\omega^{i+j} \quad \text{for } i < j,$$

where $\gamma_i = \beta_i/\alpha_i$ and $\gamma_j = \beta_j/\alpha_j$. On the other hand, the GLA$_{r-1}$ model can be expressed as

$$\frac{\pi_{ij}}{\pi_{ji}} = \left(\prod_{l=0}^{r-1} \frac{\theta_l^{j^l}}{\theta_l^{i^l}}\right) \delta\omega^{i+j} \quad (i < j).$$

By setting γ_s to $\prod_{l=0}^{r-1} \theta_l^{s^l}$ for $s = 1, 2, \ldots, r$, the relation between $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ and $\{\theta_0, \theta_1, \ldots, \theta_{r-1}\}$ is one-to-one. Therefore, the GLA$_{r-1}$ model is identical to the QLOS model. We point out that the GLA$_k$ model is the asymmetry model based on the QLOS model.

3. Goodness-of-Fit Test and Model Selection

Let n_{ij} denote the observed frequency in the i-th row and j-th column of the $r \times r$ square contingency table ($i = 1, \ldots, r; j = 1, \ldots, r$), with $n = \sum\sum n_{ij}$. The maximum likelihood estimates (MLEs) of expected frequencies under the model can be obtained by using, for example, the Newton–Raphson method in the log-likelihood equation or the non-standard log-linear model approach by Lawal [17,18].

For example, to obtain MLEs of the expected frequencies under the GLA$_k$ model, we must maximize the Lagrangian:

$$L = \sum_{i=1}^{r}\sum_{j=1}^{r} n_{ij}\log\pi_{ij} - \phi\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\pi_{ij} - 1\right) - \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\varphi_{ij}\left(\pi_{ij} - \pi_{ji}\left(\prod_{l=1}^{k}\theta_l^{j^l - i^l}\right)\delta\omega^{i+j}\right),$$

with respect to $\{\pi_{ij}\}, \phi, \{\varphi_{ij}\}, \{\theta_l\}$ δ, and ω.

Each model can be tested for goodness-of-fit by, for example, the likelihood ratio chi-square statistic G^2 with the corresponding degrees of freedom (df). The G^2 of model M is given by

$$G^2(M) = 2\sum_{i=1}^{r}\sum_{j=1}^{r} n_{ij}\log\left(\frac{n_{ij}}{\hat{m}_{ij}}\right),$$

where $\hat{m}_{ij}$ is the MLE of expected frequency m_{ij} under model M. The number of df of the S, LA$_k$, ELA$_k$, and GLA$_k$ models are $r(r-1)/2$, $r(r-1)/2 - k$, $r(r-1)/2 - (k+1)$, and $r(r-1)/2 - (k+2)$, respectively.

Convenient methods for choosing the best-fitting model among models are to use the Akaike information criterion (AIC) and Bayesian information criterion (BIC), which are defined as

$$\begin{aligned} \text{AIC} &= -2 \times (\text{the maximum log likelihood}) + 2 \times (\text{the number of parameters}), \\ \text{BIC} &= -2 \times (\text{the maximum log likelihood}) + \log n \times (\text{the number of parameters}), \end{aligned}$$

for each model, see Akaike [19] and Schwarz [20]. The AIC and BIC give the best-fitting model as the one with the minimum AIC and minimum BIC, respectively. While most of the time, both AIC and BIC agree, it is not uncommon that AIC and BIC provide different results in choosing the best model selection for the log-linear model. The common understanding is that AIC presents the danger of overfitting and BIC, of underfitting. Bedrick and Crandall [21] provided simulation findings for model selection criteria for log-linear models.

For the analysis of contingency tables using a log-linear model or Poisson count regression, it is good to report both AIC and BIC and related indices (see [21,22]).

Since only the difference between AICs is required when two models are compared, it is possible to ignore the common constant of AIC, and we may use a modified AIC defined as

$$\mathrm{AIC}^{+} = G^2 - 2 \times (\text{the number of df}).$$

Similarly, we may use a modified BIC defined as

$$\mathrm{BIC}^{+} = G^2 - \log n \times (\text{the number of df}).$$

Thus, for the data, the model with the minimum AIC^{+} (i.e., the minimum AIC) and the minimum BIC^{+} (i.e., the minimum BIC) is the best-fitting model.

4. Application to Real Data

Table 1, taken directly from Tominaga [23] (p. 130), presents the cross-classification of the academic backgrounds of a Japanese father and his son, as examined in 1955. Note that the categories are (1) elementary school, (2) junior high school, (3) high school, and (4) university.

Table 1. Cross-classification of the academic backgrounds of a Japanese father and his son; taken from Tominaga [23] (p. 130). The two values in parentheses are the maximum likelihood estimates of expected frequencies under the GLA_1 and GLA_3 (i.e., QLOS) models, respectively.

Father's	**Son's Educational Level**				**Total**
Educational Level	**(1)**	**(2)**	**(3)**	**(4)**	
(1)	374	602	170	64	1210
	(374)	(599.21)	(169.32)	(64.58)	
	(374)	(602)	(169.15)	(64.85)	
(2)	18	255	139	71	483
	(20.80)	(255)	(145.27)	(70.53)	
	(18)	(255)	(139.85)	(70.15)	
(3)	4	23	42	55	124
	(4.68)	(16.73)	(42)	(52.10)	
	(4.85)	(22.15)	(42)	(55)	
(4)	2	6	17	53	78
	(1.42)	(6.47)	(19.90)	(53)	
	(1.15)	(6.85)	(17)	(53)	
Total	398	886	368	243	1895

Table 2 gives the values of the G^2, AIC^{+}, and BIC^{+} for the models applied to this dataset. From Table 2, the GLA_1 model is the best-fitting model. This is because the GLA_1 model has the minimum AIC^{+} and the minimum BIC^{+} among the models applied to this dataset. Table 1 also shows the MLEs of expected frequencies under the $\mathrm{GLA}_1$and GLA_3 (i.e., QLOS) models. We observe that the QLOS model is saturated on the (1, 2)th, (3, 4)th, (2, 1)th and (4, 3)th cells as well as the main diagonal of the 4×4 table (see MLEs of the expected frequencies for the QLOS model in Table 1). Therefore, the GLA_1 model may be preferred over the QLOS model when all observations on the off-diagonal cells must be used.

Under the GLA_1 model, the MLEs of θ_1, δ and ω are $\hat{\theta}_1$ =2.29, $\hat{\delta}$ =76.18 and $\hat{\omega}$ =0.55, respectively. The values of $\hat{\theta}_1^{j-i}\hat{\delta}\hat{\omega}^{i+j}$ for all $i < j$ are greater than one. Therefore, we can infer that the son's education level is higher than that of his father.

Next, we consider Table 3. Table 3, taken directly from Stuart [24], presents the cross-classification of the unaided distance vision of women aged 30–39 employed in Royal Ordnance factories in British from 1943 to 1946. Note that category is (1) best, (2) second, (3) third, and (4) worst.

Table 2. Values of the likelihood ratio chi-square statistic (G^2), the modified Akaike information criterion (AIC^+), and the modified Bayesian information criterion (BIC^+) for the models applied to Table 1.

Models	df	G^2	AIC^+	BIC^+
S	6	1151.23 *	1139.23	1105.95
LA_1 (i.e., LDPS)	5	81.62 *	71.62	43.89
LA_2	4	16.33 *	8.33	−13.86
LA_3 (i.e., QS)	3	11.82 *	5.82	−10.82
ELA_1 (i.e., TRPS)	4	53.89 *	45.89	23.70
ELA_2	3	6.28	0.28	−16.36
ELA_3 (i.e., EQS)	2	3.22	−0.78	−11.87
GLA_1 †	3	3.75	−2.25	−18.89
GLA_2 †	2	3.74	−0.26	−11.35
GLA_3 (i.e., QLOS)	1	0.85	−1.25	−6.70

The symbol ∗ implies significance at the 5% level. The symbol † indicates the proposed models.

Table 3. Cross-classification of the unaided distance vision of women aged 30–39 employed in Royal Ordnance factories in British from 1943 to 1946; taken from Stuart [24]. The two values in parentheses are the maximum likelihood estimates of expected frequencies under the LA_1 and GLA_1 models, respectively.

Right Eye Grade	Left Eye Grade (1)	(2)	(3)	(4)	Total
(1)	1520	266	124	66	1976
	(1520)	(263.37)	(133.35)	(59.12)	
	(1520)	(266.48)	(131.89)	(57.27)	
(2)	234	1,512	432	78	2256
	(236.63)	(1,512)	(418.23)	(88.53)	
	(233.52)	(1,512)	(423.15)	(87.56)	
(3)	117	362	1772	205	2456
	(107.65)	(375.77)	(1772)	(202.27)	
	(109.11)	(370.85)	(1772)	(204.64)	
(4)	36	82	179	492	789
	(42.88)	(71.47)	(181.73)	(492)	
	(44.73)	(72.44)	(179.36)	(492)	
Total	1907	2222	2507	841	7477

Table 4 gives the values of the G^2, AIC^+, and BIC^+ for the models applied to the dataset of Table 3. From Table 4, the LA_1 model is the best-fitting model. This is because the LA_1 model has the minimum AIC^+ and the minimum BIC^+ among the models applied to this dataset. Moreover, the QLOS model does not fit this dataset, although the GLA_1 model fits. Table 3 also shows the MLEs of expected frequencies under the $LA_1$and GLA_1 models.

Table 4. Values of the likelihood ratio chi-square statistic (G^2), the modified Akaike information criterion (AIC^+), and the modified Bayesian information criterion (BIC^+) for the models applied to Table 3.

Models	df	G^2	AIC^+	BIC^+
S	6	19.249 *	7.249	−34.269
LA_1 (i.e., LDPS)	5	7.280	−2.720	−37.318
LA_2	4	7.277	−0.723	−28.401
LA_3 (i.e., QS)	3	7.271	1.271	−19.488
ELA_1 (i.e., TRPS)	4	6.825	−1.175	−28.853
ELA_2	3	6.823	0.823	−19.936
ELA_3 (i.e., EQS)	2	6.823 *	2.823	−11.016
GLA_1 †	3	6.825	0.825	−19.934

Table 4. *Cont.*

Models	df	G^2	AIC$^+$	BIC$^+$
GLA$_2$ †	2	6.793 *	2.793	−11.046
GLA$_3$ (i.e., QLOS)	1	6.793 *	4.793	−2.127

The symbol ∗ implies significance at the 5% level. The symbol † indicates the proposed models.

Last, we consider Table 5. Table 5, taken directly from Bishop et al. [2] (p. 100), presents the cross-classification of the occupational status of a British father and his son. Note that the categories are (1) lowest status and (5) highest status.

Table 5. Cross-classification of the occupational status of a British father and his son; taken from Bishop et al. [2] (p. 100). The two values in parentheses are the maximum likelihood estimates of expected frequencies under the GLA$_1$ and GLA$_3$ models, respectively.

Father's	Son's Occupational Status					Total
Occupational Status	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	
(1)	50	45	8	18	8	129
	(50)	(37.51)	(9.92)	(16.96)	(5.92)	
	(50)	(43.24)	(10.58)	(17.16)	(6.46)	
(2)	28	174	84	154	55	495
	(35.49)	(174)	(86.60)	(164.96)	(53.41)	
	(29.76)	(174)	(86.52)	(156.14)	(54.84)	
(3)	11	78	110	223	96	518
	(9.08)	(75.40)	(110)	(226.54)	(94.62)	
	(8.42)	(75.49)	(110)	(216.81)	(97.90)	
(4)	14	150	185	714	447	1510
	(15.04)	(139.04)	(181.47)	(714)	(441.57)	
	(14.84)	(147.86)	(191.19)	(714)	(448.36)	
(5)	3	42	72	320	411	848
	(5.08)	(43.59)	(73.38)	(325.43)	(411)	
	(4.54)	(42.16)	(70.10)	(318.64)	(411)	
Total	106	489	459	1429	1017	3500

Table 6 gives the values of the G^2, AIC$^+$, and BIC$^+$ for the models applied to the dataset of Table 5. From Table 6, the ELA$_3$ model has the minimum AIC$^+$ among the models applied to this dataset, and the LA$_1$ model has the minimum BIC$^+$ among the models applied to this dataset. From this results, we see that the BIC presents underfitting. This is because, the LA$_1$ model does not fit this dataset. Moreover, the GLA$_1$ and GLA$_3$ models are better-fitting models than the QLOS model.

Table 6. Values of the likelihood ratio chi-square statistic (G^2), the modified Akaike information criterion (AIC$^+$), and the modified Bayesian information criterion (BIC$^+$) for the models applied to Table 5.

Models	df	G^2	AIC$^+$	BIC$^+$
S	10	37.46 *	17.46	−44.15
LA$_1$ (i.e., LDPS)	9	17.13 *	−0.87	−56.31
LA$_2$	8	11.14	−4.86	−54.14
LA$_3$	7	5.70	−8.30	−51.42
LA$_4$ (i.e., QS)	6	4.66	−7.34	−44.30
ELA$_1$ (i.e., TRPS)	8	10.02	−5.98	−55.26
ELA$_2$	7	7.60	−6.40	−49.52
ELA$_3$	6	3.48	−8.52	−45.48
ELA$_4$ (i.e., EQS)	5	2.70	−7.30	−38.10
GLA$_1$ †	7	7.87	−6.13	−49.25
GLA$_2$ †	6	7.60	−4.40	−41.36
GLA$_3$ †	5	3.29	−6.71	−37.51
GLA$_4$ (i.e., QLOS)	4	2.60	−5.40	−30.04

The symbol ∗ implies significance at the 5% level. The symbol † indicates the proposed models.

5. Concluding Remarks

This study proposed the GLA_1 model, that is, the QLOS model with $\{\alpha_i = \alpha^i\}$ and $\{\beta_j = \beta^j\}$. Moreover, in line with the relation between the QS (or EQS) and LA_k (or ELA_k) models, this study proposed the GLA_k model. Thus, the GLA_{r-1} model is identical to the QLOS model.

We compared the proposed and existing models, and showed that the GLA_1 model provided a better fit than the existing models for real dataset in Table 1. Under the GLA_1 model, we obtained the interpretation for applied to real dataset.

We observed that the QLOS model was saturated on the (1, 2)th, (3, 4)th, (2, 1)th and (4, 3)th cells as well as the main diagonal of the 4×4 table (see MLEs of the expected frequencies for the QLOS model in Table 1). Therefore, the GLA_1 model may be preferred over the QLOS model when all observations on the off-diagonal cells must be used.

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Abbreviations

The following abbreviations are used in this manuscript:

S	Symmetry
MH	Marginal homogeneity
QS	Quasi-symmetry
MH(m)	m-additional parameters marginal homogeneity
L(m)	m-th generalized marginal cumulative logistic
LDPS	Linear diagonals-parameter symmetry
LA_k	k-th linear asymmetry
EQS	Extended quasi-symmetry
TRPS	Two ratio parameter symmetry
ELA_k	Extended k-th linear asymmetry
QLOS	Quasi local odds symmetry
GLA_k	Generalized k-th linear asymmetry
MLE	Maximum likelihood estimate
df	degrees of freedom
AIC	Akaike information criterion
BIC	Bayesian information criterion

References

1. Bowker, A.H. A test for symmetry in contingency tables. *J. Am. Stat. Assoc.* **1948**, *43*, 572–574. [CrossRef] [PubMed]
2. Bishop, Y.M.M.; Fienberg, S.E.; Holland, P.W. *Discrete Multivariate Analysis: Theory and Practice*; The MIT Press: Cambridge, MA, USA, 1975.
3. Tahata, K.; Tomizawa, S. Symmetry and asymmetry models and decompositions of models for contingency tables. *SUT J. Math.* **2014**, *50*, 131–165.
4. Stuart, A. A test for homogeneity of the marginal distributions in a two-way classification. *Biometrika* **1955**, *42*, 412–416. [CrossRef]
5. Caussinus, H. Contribution à l'analyse statistique des tableaux de corrélation. *Ann. Fac. Sci. Univ. Toulouse* **1965**, *29*, 77–182. [CrossRef]
6. Tahata, K.; Tomizawa, S. Generalized marginal homogeneity model and its relation to marginal equimoments for square contingency tables with ordered categories. *Adv. Data Anal. Classif.* **2008**, *2*, 295–311. [CrossRef]
7. Kurakami, H.; Tahata, K.; Tomizawa, S. Generalized marginal cumulative logistic model for multi-way contingency tables. *SUT J. Math.* **2013**, *49*, 19–32.

8. Tomizawa, S. Three kinds of decompositions for the conditional symmetry model in a square contingency table. *J. Jpn. Stat. Soc.* **1984**, *14*, 35–42.
9. Tomizawa, S. A generalization of the marginal homogeneity model for square contingency tables with ordered categories. *J. Educ. Behav. Stat.* **1984**, *20*, 349–360. [CrossRef]
10. Agresti, A. *Categorical Data Analysis*, 2nd ed.; Wiley: New York, NY, USA, 2002.
11. Kurakami, H.; Tahata, K.; Tomizawa, S. Extension of the marginal cumulative logistic model and decompositions of marginal homogeneity for multi-way tables. *J. Stat. Adv. Theory Appl.* **2010**, *3*, 135–152.
12. Agresti, A. A simple diagonals-parameter symmetry and quasi-symmetry model. *Stat. Probab. Lett.* **1983**, *1*, 313–316. [CrossRef]
13. Tahata, K.; Tomizawa, S. Generalized linear asymmetry model and decomposition of symmetry for multiway contingency tables. *Biom. Biostat.* **2011**, *2*, 1–6. [CrossRef]
14. Tomizawa, S. Decompositions for 2-ratios-parameter symmetry model in square contingency tables with ordered categories. *Biom. J.* **1987**, *29*, 45–55. [CrossRef]
15. Tahata, K.; Naganawa, M.; Tomizawa, S. Extended linear asymmetry model and separation of symmetry for square contingency tables. *J. Jpn. Stat. Soc.* **2016**, *46*, 189–202. [CrossRef]
16. Altun, G. Quasi local odds symmetry model for square contingency table with ordinal categories. *J. Stat. Comput. Simul.* **2019**, *89*, 2899–2913. [CrossRef]
17. Lawal, H.B. Review of non-independence, asymmetry, skew-symmetry and point-symmetry models in the analysis of social mobility data. *Qual. Quant.* **2004**, *38*, 259–289. [CrossRef]
18. Lawal, H.B. Using a GLM to decompose the symmetry model in square contingency tables with ordered categories. *J. Appl. Stat.* **2004**, *31*, 279–303. [CrossRef]
19. Akaike, H. A new look at the statistical model identification. *IEEE Trans. Autom. Control* **1974**, *19*, 716–723. [CrossRef]
20. Schwarz, G. Estimating the dimension of a model. *Ann. Stat.* **1978**, *6*, 461–464. [CrossRef]
21. Bedrick, E.J.; Crandall, W.K. Model selection criteria for loglinear models. *Aust. N. Z. J. Stat.* **2010**, *52*, 439–449. [CrossRef]
22. Bolboaca, S.D.; Jäntschi, L. Comparison of quantitative structure-activity relationship model performances on carboquinone derivatives. *Sci. World J.* **2009**, *9*, 1148–1166. [CrossRef]
23. Tominaga, K. *Nippon No Kaisou Kouzou (Japanese Hierarchical Structure)*; University of Tokyo Press: Tokyo, Japan, 1979. (In Japanese)
24. Stuart, A. The estimation and comparison of strengths of association in contingency tables. *Biometrika* **1953**, *40*, 105–110. [CrossRef]