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Characterizations of $\mathcal{PR}$-Pseudo-Slant Warped Product Submanifold of Para-Kenmotsu Manifold with Slant Base

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Abstract: In this article, we study the properties of $\mathcal{PR}$-pseudo-slant submanifold of para-Kenmotsu manifold and obtain the integrability conditions for the slant distribution and anti-invariant distribution of such submanifold. We derived the necessary and sufficient conditions for a $\mathcal{PR}$-pseudo-slant submanifold of para-Kenmotsu manifold to be a $\mathcal{PR}$-pseudo-slant warped product which are in terms of warping functions and shape operator. Some examples of $\mathcal{PR}$-pseudo-slant warped products of para-Kenmotsu manifold are also illustrated in the article.

Keywords: paracontact manifold; para-Kenmotsu manifold; pseudo-slant submanifold; warped product

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1. Introduction

At the end of the twentieth century, B.Y. Chen initiated the study of slant submanifold as a generalization of $\mathcal{CR}$-submanifolds [1]. Later, A. Carriazo studied slant submanifolds in contact metric manifold as a special case of bi-slant submanifolds [2]. Thereafter, he studied pseudo-slant submanifolds under the name anti-slant [3]. The slant submanifold with pseudo-Riemannian metric was also initiated by B.Y. Chen et al. [4,5]. The authors of [6,7] studied slant submanifold of Kaehler and contact manifolds with respect to the pseudo-Riemannian metric. P. Alegre and A. Carriazo studied slant submanifolds in para-Hermitian manifold and provided detailed descriptions of such type of submanifolds in pseudo-Riemannian metric.

On the other hand, the study of warped product manifold is one of the most significant generalizations of Cartesian product of pseudo-Riemannian manifolds (or Riemannian manifolds). This fruitful generalization was initiated by R. L Bishop and B. O'Neill in 1969 (see [8]). The notion of warped products appeared in the physical and mathematical literature before 1969, for instance, semi-reducible space, which is used for warped product by Kruchkovich in 1957 [9]. It has been successfully utilized in general theory of relativity, black holes, and string theory. The warped product is defined as follows:

Assume that B and F are two pseudo-Riemannian manifolds with pseudo-Riemannian metric g_B and g_F, respectively and f is a smooth function defined by $f : B \longrightarrow (0,1)$. Then, a pseudo-Riemannian manifold $M = B \times_f F$ is said to be a warped product [8,10] if it is furnished a pseudo-Riemannian warping metric g fulfilling for any tangent vector U to M as the following:

$$g(U,U) = g(\pi_* U, \pi_* U) + (f \circ \pi)^2 g(\pi'_* U, \pi'_* U), \tag{1}$$

where $\pi : B \times F \longrightarrow B$ and $\pi' : B \times F \longrightarrow F$ are natural projections on M, and $*$ denotes the push-foreword map (or differential map). The smooth function f is called warping function. Moreover, the above relation is equivalent to

$$g = g_B + f^2 g_F. \tag{2}$$

If $f : B \longrightarrow (0, 1)$ is non-constant, then M is called a non-trivial (or proper) warped product, otherwise it is trivial. Now, consider any $U_1, U_2 \in \Gamma(TB)$ and $V_1, V_2 \in \Gamma(TF)$, then from the Proposition 3.1 of [10] (page no. 49), we obtain that

$$\nabla_{U_1} U_2 \in \Gamma(TB), \tag{3}$$

$$\nabla_{U_1} V_1 = \nabla_{V_1} U_1 = U_1(\ln f) V_1, \tag{4}$$

$$tan(\nabla_{V_1} V_2) = \nabla'_{V_1} V_2, \tag{5}$$

$$nor(\nabla_{V_1} V_2) = h^F(V_1, V_2) = -\frac{g(V_1, V_2)\nabla f}{f}. \tag{6}$$

where the symbols ∇' and h indicates are Levi–Civita connection on B and second fundamental form, respectively. By the consequence (3)–(6), we can conclude that for a warped product manifold $M = B \times_f F$, the submanifold F is a totally umbilical and the submanifold B is a totally geodesic in M.

In 1956, J.F. Nash derived a very useful theorem in Riemannian geometry known as Nash embedding theorem. The theorem states "every Riemannian manifold can be isometrically embedded in some Euclidean space" (see [11]). This theorem shows that any warped product of Riemannian (or pseudo-Riemannian) manifolds can be realized (or embedded) as a Riemannian (or pseudo-Riemannian) submanifold in Euclidean space. Due to this fact, B.Y. Chen asked a very interesting question in 2002. The question is "What can we conclude from an isometric immersion of an arbitrary warped product into a Euclidean space or into a space form with arbitrary codimension?" (see [10]). Thereafter, B.Y. Chen published the numerous articles on the $\mathcal{CR}$-warped products in K*ä*hler manifold (see [12,13]). Thereafter, several authors of [14–20] studied pseudo-slant warped product in different ambient manifolds. In 2015, A. Ali et al. derived some useful inequalities for a pseudo-slant warped product submanifold in nearly-Kenmotsu manifold [21]. Recently, the authors of [22–24] studied pseudo-slant warped product submanifold of Kenmotsu manifold and derived some characterizations and inequalities.

However, in 2014, B.Y. Chen initiated a new class of warped product called $\mathcal{PR}$-warped product and found the exact solutions of the system partial differential equations associated with $\mathcal{PR}$-warped products [25]. Recently, S.K. Srivastava and A. Sharma studied $\mathcal{PR}$-semi-invariant, $\mathcal{PR}$-pseudo-slant, and $\mathcal{PR}$-semi-slant warped product of paracosymplectic manifold in [26–29]. In the last two decades, several geometrists studied warped product submanifolds and other submanifolds in different ambient space [26–37]. Motivated by them, we analyze the geometry of $\mathcal{PR}$-pseudo-slant warped product submanifolds of para-Kenmotsu manifold which are not studied yet.

This paper is formulated as follows. The second section includes some necessary information related to para-contact and para-Kenmotsu manifold and also contains some important information about the basics of submanifolds in para-Kenmotsu manifold. Section 3 includes some useful results related to integrability of $\mathcal{PR}$-pseudo-slant submanifold in para-Kenmotsu manifold and gives examples of such submanifolds. In Section 4, we analyze the geometry of $\mathcal{PR}$-pseudo-slant warped product submanifolds in para-Kenmotsu manifold and provide some characterization results allied to shape operator and endomorphism t, and also give some examples of $\mathcal{PR}$-pseudo-slant warped product submanifold of para-Kenmotsu manifold.

2. Preliminaries

A smooth manifold $\widetilde{M}^{2n+1}$ of dimension $(2n+1)$ furnished an almost paracontact (see [26,38,39]) structure (φ, ξ, η) which includes a $(1,1)$-type tensor field φ, a vector field ξ, and a 1-form η globally defined on $\widetilde{M}^{2n+1}$ which satisfies the accompanying relation for all $U \in \Gamma(TM^{2n+1})$:

$$\varphi^2 U = U - \eta(U)\xi, \; \eta(\xi) = 1. \tag{7}$$

The tensor field φ induces an almost paracomplex structure $\mathcal{J}$ on a $2n$-dimensional horizontal distribution $\mathfrak{D}$ described as the kernel of 1-form η, i.e., $\mathfrak{D} = ker(\eta)$. The horizontal distribution $\mathfrak{D}$ can be expressed as an orthogonal direct sum of the two eigen distribution $\mathfrak{D}^+$ and $\mathfrak{D}^-$, the eigen distributions $\mathfrak{D}^+$ and $\mathfrak{D}^-$ having eigenvalue $+1$ and -1, respectively, and each has dimension n. Moreover, $\mathfrak{D}$ is invariant distribution, therefore $T\widetilde{M}^{2n+1}$ can be expressed in the following form;

$$T\widetilde{M}^{2n+1} = \mathfrak{D} \oplus \langle \xi \rangle. \tag{8}$$

If $\widetilde{M}^{2n+1}$ admits an almost paracontact structure (φ, ξ, η), then it is said to be an almost paracontact manifold [26,39]. In view of (7), we obtain

$$\eta \circ \varphi = 0, \; \varphi \circ \xi = 0 \; and \; rank(\varphi) = 2n. \tag{9}$$

An almost paracontact manifold $\widetilde{M}^{2n+1}$ is called an almost paracontact pseudo-metric manifold if it admits a pseudo-Riemannian metric of index n compatible with the triplet (φ, ξ, η) by the following relation:

$$g(\varphi U, \varphi V) = \eta(U)\eta(V) - g(U, V), \tag{10}$$

for all $U, V \in \Gamma(T\widetilde{M}^{2n+1})$; $\Gamma(T\widetilde{M}^{2n+1})$ denotes the Lie algebra on $\widetilde{M}^{2n+1}$. The dual of the unitary structural vector field ξ allied to g is η, i.e.,

$$\eta(U) = g(U, \xi). \tag{11}$$

By the utilization of (7)–(10), we attain

$$g(U, \varphi V) + g(\varphi U, V) = 0. \tag{12}$$

Definition 1. *An almost paracontact pseudo-metric manifold $\widetilde{M}^{2n+1}$ is said to be a para-Kenmotsu manifold [38] if it satisfies*

$$(\widetilde{\nabla}_U \varphi)V = \eta(V)\varphi U + g(U, \varphi V)\xi. \tag{13}$$

In the relation (13), the symbol $\widetilde{\nabla}$ indicates for the Levi–Civita connection with respect to g.

In (13) replacing V by ξ and then applying (7), we achieve that

$$\widetilde{\nabla}_U \xi = -\varphi^2 U. \tag{14}$$

Proposition 1. *On para-Kenmotsu pseudo-Riemannian manifold, the following relations holds:*

$$\eta(\widetilde{\nabla}_U \xi) = 0, \; \widetilde{\nabla}\eta = -\eta \otimes \eta + g, \tag{15}$$

$$\mathcal{L}_\xi \varphi = 0, \; \mathcal{L}_\xi \eta = 0, \mathcal{L}_\xi g = -2(g - \eta \otimes \eta), \tag{16}$$

where $\mathcal{L}$ denotes the Lie differentiation.

Geometry of Submanifolds

Let M be a m-dimensional paracompact and connected smooth pseudo-Riemannian manifold and $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold. Assume that $\psi : M \longrightarrow \widetilde{M}^{2n+1}$ is an isometric immersion. Then $\psi(M)$ is known as an isometrically immersed submanifold of a para-Kenmotsu manifold. Let us denote that ψ_* for the differential map (or push forward map) of immersion ψ is characterized by $\psi_* : T_pM \longrightarrow T_{\psi(p)}\widetilde{M}^{2n+1}$. Therefore, the induced pseudo-Riemannian metric $\mathfrak{g}$ on $\psi(M)$ is defined as follows: $\mathfrak{g}(U,V)_p = g(\psi_*U, \psi_*V)$, for all $U, V \in T_pM$. For our convenience, we use M and p in the place of $\psi(M)$ and $\psi(p)$. Now, we denote $\Gamma(TM)$ for set of all tangent vector fields on M, $\Gamma(TM^{\perp})$ for the set of all normal vector fields of M, ∇ for induced Levi–Civita connection on TM, and $\nabla^{\perp}$ for normal connection on the normal bundle $\Gamma(TM^{\perp})$. Then, Gauss and Weingarten formulas are characterized by the relation

$$\widetilde{\nabla}_U V = \nabla_U V + h(U,V), \tag{17}$$
$$\widetilde{\nabla}_U \zeta = -A_\zeta U + \nabla^{\perp}_U \zeta, \tag{18}$$

for any $U, V \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$, where A_ζ is a shape operator and h is a second fundamental form which are allied to the normal section ζ by the following relation:

$$g(h(U,V),\zeta) = g(A_\zeta U, V). \tag{19}$$

The mean curvature vector H on M is described by $H = \frac{1}{m}trace(h)$. Let $p \in M$ and $\{U_1, U_2, \cdots, U_m, U_{m+1}, \cdots, U_{2n+1}\}$ be an orthonormal basis of the $T_p\widetilde{M}^{2n+1}$ in which $\{U_1, U_2, \cdots, U_m\}$ are the tangent to M and $\{U_{m+1}, U_{m+2}, \cdots, U_{2n+1}\}$ are normal to M. Now, we set

$$h^k_{ij} = g(h(U_i, U_j), U_k), \tag{20}$$

for $i, j \in \{1, 2, \cdots, m\}$ and $k \in \{m+1, m+2, \cdots, 2n+1\}$. The norm of h is defined by the following relation:

$$\|h\| = \sqrt{\left(\sum_{i,j=1}^{m} g(h(U_i, U_j), h(U_i, U_j))\right)}. \tag{21}$$

An isometrically immersed submanifold M of a para-Kenmotsu manifold $\widetilde{M}^{2n+1}$ (φ, ξ, η, g) is said to be (see [26,39])

- *Totally geodesic* if h vanishes identically, i.e., $h \equiv 0$.
- *Umbilical* if for a normal vector field ζ, shape operator A_ζ is proportional to identity transformation.
- *Totally umbilical* if M satisfies for every $U, V \in \Gamma(TM)$

$$h(U,V) = g(U,V)H. \tag{22}$$

- *Minimal* if trace of h (or H) vanishes identically.
- *Extrinsic sphere* if M satisfies (22) and H is parallel with respect to $\nabla^{\perp}$.

From now on, we denote para-Kenmotsu manifold by $\mathcal{K}^{2n+1}$ and its pseudo-Riemannian submanifold by $\mathcal{N}$. For any $U \in \Gamma(T\mathcal{N})$, we substitute $tU = tan(\varphi U)$ and $nU = nor(\varphi U)$, where tan and nor are natural projections associated with the following direct sum:

$$T_p\mathcal{K}^{2n+1} = T_p\mathcal{N} \oplus T_p\mathcal{N}^{\perp}. \tag{23}$$

Thus, we can write

$$\varphi U = tU + nU. \tag{24}$$

Similarly, for any $\zeta \in \Gamma(T\mathcal{N}^{\perp})$, we have

$$\varphi\zeta = t^{'}\zeta + n^{'}\zeta, \tag{25}$$

where $t^{'}\zeta = tan(\varphi\zeta)$ and $n^{'}\zeta = nor(\varphi\zeta)$. In view of (12) and (22)–(25), we attain for any $U, V \in \Gamma(T\mathcal{N})$ and $\forall \zeta_1, \zeta_2 \in \Gamma(T\mathcal{N}^{\perp})$ that

$$g(n^{'}\zeta_1, \zeta_2) = -g(\zeta_1, n^{'}\zeta_2),\ g(tU, V) = -g(U, tV). \tag{26}$$

Moreover, by the consequences of Equations (12) and (24)–(25), we have

$$g(nU, \zeta) = -g(U, t^{'}\zeta). \tag{27}$$

Further, the covariant derivative of φ, t and n are characterized by, respectively,

$$(\widetilde{\nabla}_U\varphi)V = \widetilde{\nabla}_U\varphi V - \varphi\widetilde{\nabla}_U V, \tag{28}$$

$$(\nabla_U t)V = \nabla_U tV - t\nabla_U V, \tag{29}$$

$$(\nabla_U n)V = \nabla^{\perp}_U nV - n\nabla_U V, \tag{30}$$

for some $U, V \in \Gamma(T\mathcal{N})$.

Proposition 2. *Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then, we obtain*

$$(\nabla_U t)V = A_{nV}U + t^{'}h(U, V) + \eta(V)tU - g(tU, V)\xi, \tag{31}$$

$$(\nabla_U n)V = n^{'}h(U, V) + \eta(V)nU - h(U, tV), \tag{32}$$

for every $U, V \in \Gamma(T\mathcal{N})$.

Proof. By the consequence of (17)–(18), (24), (28)–(30), we arrive at

$$(\widetilde{\nabla}_U\varphi)V + A_{nV}U = -t^{'}h(U, V) + (\nabla_U t)V - n^{'}h(U, V) + h(U, tV) + (\nabla_U n)V,$$

for any $U \in \Gamma(T\mathcal{N})$. Employing (13) and (24) into the above expression, then considering tangential part and normal part of the obtained expression, we have (31) and (32), respectively. □

Proposition 3. *If ξ is normal to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$, then we acquire that*

$$(\nabla_U t)V = t^{'}h(U, V) + A_{nV}U, \tag{33}$$

$$(\nabla_U n)V = n^{'}h(U, V) + g(U, tV)\xi - h(U, tV), \tag{34}$$

for all $U, V \in \Gamma(T\mathcal{N})$.

Proof. Immediately, from (13), (17)–(18), (24), (28)–(30), we derive (33) and (34). □

Proposition 4. *Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then, we receive that*

$$(\nabla_U t^{'})\zeta = A_{n^{'}\zeta}U - g(nU, \zeta)\xi - tA_{\zeta}U, \tag{35}$$

$$(\nabla_U n^{'})\zeta = -h(U, t^{'}\zeta) - nA_{\zeta}U, \tag{36}$$

for any $U \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T\mathcal{N}^{\perp})$.

Proof. Employing (17)–(18), (25), (29), and (30) into (28), we achieve that

$$(\widetilde{\nabla}_U\varphi)\zeta = (\nabla_U n')\zeta - A_{n'\zeta}U + tA_\zeta U + nA_\zeta U + h(U, t'\zeta) + (\nabla_U t')\zeta,$$

for any $U \in \Gamma(T\mathcal{N})$. Utilizing (13) and (24) into the above expression, we achieve (35) and (36). □

Proposition 5. *If $\mathcal{N}$ is normal to ξ in $\mathcal{K}^{2n+1}$, then we achieve for any $U \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T\mathcal{N}^\perp)$ that*

$$(\nabla_U t')\zeta = A_{n'\zeta}U - tA_\zeta U + \eta(\zeta)tU, \tag{37}$$

$$(\nabla_U n')\zeta = -nA_\zeta U + \eta(\zeta)nU + g(U, t'\zeta)\xi - h(U, tV). \tag{38}$$

Proof. The process is similar to Proposition 4. □

Consider $U, \xi \in \Gamma(T\mathcal{N})$ as two vector fields; thus, by the direct application of (14) and (17)–(18), we gain

$$\nabla_U \xi = -\,\varphi^2 U,\ h(U, \xi) = 0. \tag{39}$$

If $\xi \in \Gamma(TN^\perp)$, then by the consequence of (14) and (18), we have

$$A_\xi U = U,\ \nabla^\perp_U \xi = 0. \tag{40}$$

In view of (39) and (40), we give the following remarks:

Remark 1. *Let ξ be tangent to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then relation (39) holds on $\mathcal{N}$.*

Remark 2. *Let ξ be normal to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then Equation (40) holds in $\mathcal{N}$.*

Proposition 6. *Let ξ be tangent to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then, the endomorphism t and bundle 1-form n satisfies*

$$t^2 + t'n = \mathcal{I} - \eta \otimes \xi, \tag{41}$$

$$nt + n'n = 0. \tag{42}$$

Proof. Operating φ on (24), we have

$$\varphi^2 U = \varphi(tU) + \varphi(nU).$$

Employing (7) and (24) into the above expression, we achieve

$$U - \eta(U)\xi = t^2U + ntU + t'nU + n'nU.$$

Comparing tangential and normal parts of the above expression, we obtain (41) and (42). □

In similar way, we prove the following result:

Proposition 7. *Let ξ be normal to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then, the following relations holds:*

$$tt' + t'n' = 0, \tag{43}$$

$$nt' + n'^2 = \mathcal{I}. \tag{44}$$

3. $\mathcal{PR}$-Pseudo-Slant Submanifolds

Definition 2. *Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is called a slant [40] if the quotient $\frac{g(tU,tU)}{g(\varphi U,\varphi U)} = \lambda(\theta)$ is constant for any non-zero spacelike or timelike vector $U \in T_p\mathcal{N}$ and for any $p \in \mathcal{N}$. The symbol θ is used for slant angle and $\lambda(\theta)$ for slant coefficient or function. In other words, if $\mathcal{N}$ is slant then λ does not depend on the vector field and point.*

Remark 3. *The value of $\lambda(\theta)$ can be*

(i) $\lambda = \cosh^2\theta \in [1,\infty)$ for $\frac{\|tU\|}{\|\varphi U\|} > 1$, tU is timelike or spacelike for any spacelike or timelike vector field U and $\theta > 0$.

(ii) $\lambda(\theta) = \cos^2\theta \in [0,1]$ for $\frac{\|tU\|}{\|\varphi U\|} < 1$, tU is timelike or spacelike for any spacelike or timelike vector field U and $0 \leq \theta \leq 2\pi$.

(iii) $\lambda(\theta) = -\sinh^2\theta \in (-\infty, 0]$ for tU is timelike or spacelike for any timelike or spacelike vector field U and $\theta < 0$.

Remark 4. *If $\lambda = 0$, then $\mathcal{N}$ is an anti-invariant submanifold.*

Remark 5. *If $\lambda = 1$, then $\mathcal{N}$ is an invariant submanifold.*

Example 1. *Let us consider $\widetilde{M} = \mathbb{R}^4 \times \mathbb{R}^+$ together with the the usual Cartesian coordinates (x_1, x_2, y_1, y_2, s). Then the structure (φ, ξ, η) over $\widetilde{M}$ is defined by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i},\ \varphi\left(\frac{\partial}{\partial s}\right) = 0,\ \eta = ds, \tag{45}$$

where $i, j \in \{1,2\}$ and the pseudo-Riemannian metric tensor g is defined as

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = e^{-2s},\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}\right) = -e^{-2s}, g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 1, \tag{46}$$

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = 0,\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k}\right) = 0,\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_k}\right) = 0. \tag{47}$$

Then, by simple computation, we can easily see that $\widetilde{M}$ is para-Kenmotsu manifold. Suppose M_1, M_2, and M_3 are immersed submanifolds into $\widetilde{M}$ by the immersions σ, σ', and σ'' respectively, defined by

$$\sigma(u,v,\alpha) = \left(u, \sqrt{3}v, \frac{3}{2}v, v, \alpha\right),$$
$$\sigma(u,v,\alpha) = \left(u, \frac{1}{2}v, \sqrt{2}v, v, \alpha\right),$$
$$\sigma(u,v,\alpha) = (u, 3v, 2v, v, \alpha).$$

By simple computation, we conclude that M_1, M_2, and M_3 are slant submanifolds of type I, type II, and type III of para-Kenmotsu manifold, respectively.

Theorem 1 ([40]). *Let ξ be tangent to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is slant if and only if there exists a constant $\lambda \in \mathbb{R}$ such that*

$$t^2 = \lambda(\mathcal{I} - \eta \otimes \xi). \tag{48}$$

In particular, λ is either $\cos^2\theta$ or $\cosh^2\theta$ or $-\sinh^2\theta$.

Theorem 2 ([40]). *Let $\mathcal{N}$ be a slant submanifold in $\mathcal{K}^{2n+1}$ with $\xi \in \Gamma(T\mathcal{N})$. Then, for any $U, V \in \Gamma(T\mathcal{N})$, we have*

$$g(tU, tV) = \lambda g(\varphi U, \varphi V), \tag{49}$$
$$g(nU, nV) = (1-\lambda) g(\varphi U, \varphi V). \tag{50}$$

Proposition 8. *Let $\mathcal{N}$ be a slant submanifold in $\mathcal{K}^{2n+1}$ with slant coefficient $\lambda(\theta)$ if and only if*

(i) $t'nU = (1-\lambda)U$ *and* $n\,tU = -n'nU$ *for non-lightlike tangent vector field* U *on* $\mathcal{N}$*.*
(ii) $(n')^2\zeta = \lambda\zeta$ *for non-lightlike normal vector field* ζ*.*

Proof. Assume $\mathcal{N}$ to be slant submanifold of $\mathcal{K}^{2n+1}$.

(i) Then for every $p \in \mathcal{N}$ and $U \in T\mathcal{N}$, we find

$$\begin{aligned} \varphi U =& tU + nU, \\ \varphi^2 U =& \varphi(tU + nU), \\ U - \eta(U)\xi =& t^2U + ntU + t'nU + n'nU. \end{aligned}$$

Equating tangential and normal parts and using (51), we can attain the result.

(ii) Since, $\zeta \in \Gamma\left(T\mathcal{N}^{\perp}\right)$, there exists $U \in \Gamma(T\mathcal{N})$ as $\mathcal{N}$ is slant submanifold such that $nU = \zeta$.
Now, $(n')^2\zeta = n'\,n'\,nU = -n'\,n\,tU = n\,t^2U = \lambda\zeta$.

The converse can be easily derived using the same equations. □

Definition 3. *Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is said to be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$ if its tangent bundle $T\mathcal{N}$ can orthogonally be decomposed as a direct sum of an anti-invariant distribution $\mathfrak{D}_{\perp}$ and a slant distribution $\mathfrak{D}_{\lambda}$ i.e., $T\mathcal{N} = \mathfrak{D}_{\lambda} \oplus \mathfrak{D}_{\perp} \oplus \langle\xi\rangle$, where ξ is a one-dimensional real distribution.*

Let P and Q be two orthogonal projections on the slant $\mathfrak{D}_{\lambda}$ and anti-invariant distribution $\mathfrak{D}_{\perp}$, respectively. Then, for any $U \in \Gamma(T\mathcal{N})$ can be expressed as follows:

$$U = PU + QU + \eta(U)\xi. \tag{51}$$

From (51), we have

$$P^2 = P,\; Q^2 = Q,\; PQ = QP = 0. \tag{52}$$

From (24) and (51), we obtain

$$\varphi U = tPU + nPU + tQU + nQU,$$

using the fact M is $\mathcal{PR}$-pseudo-slant, we find

$$\varphi PU = tPU + nPU + nQU,\; tQU = 0,\; tPU \in \Gamma(\mathfrak{D}_{\lambda}). \tag{53}$$

This leads to the following proposition:

Proposition 9. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then the Equation (53) holds.*

Theorem 3. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then the endomorphism n is parallel if and only if*

$$A_{\zeta}V_1 = -\frac{1}{\lambda}A_{n'\zeta}tV_1, \tag{54}$$

for all $V_1 \in \Gamma(\mathfrak{D}_\lambda)$ *and* $\zeta \in \Gamma(T\mathcal{N}^\perp)$.

Proof. Firstly, assume that the endomorphism n is parallel, then from (32), we obtain

$$n' h(V_1, V_2) - h(V_1, tV_2) - \eta(V_2) n V_1 = 0.$$

Replacing V_2 with tV_2 in the above equation, we obtain

$$n' h(V_1, tV_2) - h(V_1, t^2 V_2) = 0$$

Now, using (32) in the above equation, we have $n' h(V_1, tV_2) - \lambda h(V_1, V_2) = 0$. Now, taking inner product with $\zeta \in \Gamma(T\mathcal{N}^\perp)$ and using (19) and (26), we compute

$$g(A_\zeta V_2, V_1) = -\frac{1}{\lambda} g(A_{n'\zeta} t V_2, V_1).$$

□

Theorem 4. *Let* $\mathcal{N}$ *be a* $\mathcal{PR}$*-pseudo-slant submanifold in* $\mathcal{K}^{2n+1}$*. Then the slant distribution* $\mathfrak{D}_\lambda$ *is always integrable.*

Proof. Considering $W_1 \in \Gamma(\mathfrak{D}_\perp)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda)$, the utilization of (10) and (17) gives $g(\nabla_{V_1} V_2, W_1) = -g(\varphi \widetilde{\nabla}_{V_1} V_2, \varphi W_1) + \eta(\widetilde{\nabla}_{V_1} V_2)\eta(W_1)$. By the consequences of (14), (17), (18), and (22), the above expression takes the following form:

$$g(\nabla_{V_1} V_2, W_1) = -g(h(V_1, tV_2), nW_1) - g(\nabla^\perp_{V_1} n V_2, nW_1).$$

In the light of Equations (36) and (40), we compute

$$g(\nabla_{V_1} V_2, W_1) = -g(n' h(V_1, V_2), nW_1) - g(n\nabla_{V_1} V_2, nW_1). \tag{55}$$

By interchange V_1 and V_2 into (55), we obtain

$$g(\nabla_{V_2} V_1, W_1) = -g(n' h(V_1, V_2), nW_1) - g(n\nabla_{V_2} V_1, nW_1). \tag{56}$$

In the light of (55) and (56), we achieve $g([V_1, V_2], W_1) = -g(n[V_1, V_2], nW_1)$, now using (50), thus, we find

$$g([V_1, V_2], W_1) = (1 - \lambda)(g([V_1, V_2], W_1) - \eta([V_1, V_2])\eta(W_1)). \tag{57}$$

By the relation (57) we conclude that $\mathfrak{D}_\lambda$ is integrable. This completes the proof. □

Remark 6. *The one-dimensional real distribution of* $\mathcal{PR}$*-pseudo-slant submanifold in* $\mathcal{K}^{2n+1}$ *is always integrable.*

Theorem 5. *Let* $\mathcal{N}$ *be a* $\mathcal{PR}$*-pseudo-slant submanifold in* $\mathcal{K}^{2n+1}$*. Then, the distribution* $\mathfrak{D}_\perp$ *is integrable if and only if the shape operator satisfies*

$$A_{nW_1} W_2 = A_{nW_2} W_1, \tag{58}$$

$\forall W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. By the direct consequence of Equation (22), we obtain

$$\Phi[W_1, W_2] = t[W_1, W_2] + n[W_1, W_2] = t\widetilde{\nabla}_{W_1} W_2 - t\widetilde{\nabla}_{W_2} W_1 + n\widetilde{\nabla}_{W_1} W_2 - n\widetilde{\nabla}_{W_2} W_1.$$

Since $\mathfrak{D}_\perp$ is anti-invariant distribution then $[W_1, W_2] \in \Gamma(T\mathfrak{D}_\perp)$ if and only if $t\widetilde{\nabla}_{W_1} W_2 - t\widetilde{\nabla}_{W_2} W_1 = 0$. By the application of (29) and (53), we observe that $-(\nabla_{W_2} t)W_1 + (\nabla_{W_1} t)W_2 = 0$. In view of (31), we obtain (58). This completes the proof. □

Corollary 1. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, the distribution $\mathfrak{D}_\perp$ is integrable if and only if the endomorphism t satisfies*

$$(\nabla_{W_2} t)W_1 = (\nabla_{W_1} t)W_2, \tag{59}$$

$\forall W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$.

Lemma 1. *For a $\mathcal{PR}$-pseudo-slant submanifold $\mathcal{N}$ in $\mathcal{K}^{2n+1}$, we have*

$$g(\nabla_{V_1} V_2, W_1) = \frac{1}{\lambda} g(h(V_1, W_1), ntV_2) - g(h(V_1, tV_2), \varphi W_1), \tag{60}$$

for all $W_1 \in \Gamma(\mathfrak{D}_\perp)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle \xi \rangle)$.

Proof. By the consequence of (10) and (17), we have

$$g(\nabla_{V_1} V_2, W_1) = \eta(\bar{\nabla}_{V_1} V_2)\eta(W_1) - g(\varphi \widetilde{\nabla}_{V_1} V_2, \varphi W_1).$$

In view of (12) and (28), we obtain

$$g(\nabla_{V_1} V_2, W_1) = -g(\bar{\nabla}_{V_1} nV_2, \varphi W_1) - g(\widetilde{\nabla}_{V_1} tV_2, \varphi W_1).$$

Now using (13), (17), and (29) in the above relation,

$$g(\nabla_{V_1} V_2, W_1) = -g(h(V_1, tV_2), \varphi W_1) + g(\widetilde{\nabla}_{V_1} t' nV_2, \varphi W_1) + g(\widetilde{\nabla}_{V_1} n' nV_2, \varphi W_1)$$

The above expression reduces into the following form by the use of first part of Proposition 8 and (14):

$$g(\nabla_{V_1} V_2, W_1) = -g(h(V_1, tV_2), \varphi W_1) + (1 - \lambda) g(\nabla_{V_1} V_2, W_1) - g(\widetilde{\nabla}_{V_1} ntV_2, \varphi W_1).$$

By the virtue of (18) and (19), we have (60). □

Theorem 6. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, the distribution $\mathfrak{D}_\lambda \oplus \langle \xi \rangle$ is integrable if and only if the shape operator A satisfies*

$$g(A_{ntV_2} W_1, V_1) - g(A_{ntV_1} W_1, V_2) + g(A_{\varphi W_1} tV_1, V_2) - g(A_{\varphi W_1} V_1, tV_2) = 0, \tag{61}$$

$\forall W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$ *and* $V_1, V_2 \in \mathfrak{D}_\lambda \oplus \langle \xi \rangle$.

Proof. By the consequence of Lemma 1, we have

$$\begin{aligned} g([V_1, V_2], W_1) =& \frac{1}{\lambda}(g(h(V_1, W_1), ntV_2) - g(h(V_2, W_1), ntV_1) \\ &+ g(h(tV_1, V_2), \varphi W_1) - g(h(V_1, tV_2), \varphi W_1)) \end{aligned}$$

for every $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle \xi \rangle)$ and $W_1 \in \Gamma(\mathfrak{D}_\perp)$. In light of (19), we have

$$\begin{aligned} \lambda(g([V_1, V_2], W_1)) =& g(A_{ntV_2} W_1, V_1) - g(A_{ntV_1} W_1, V_2) \\ &+ g(A_{\varphi W_1} tV_1, V_2) - g(A_{\varphi W_1} V_1, tV_2). \end{aligned} \tag{62}$$

By the relation (62), we conclude that $\mathfrak{D}_\lambda \oplus \langle \xi \rangle$ is integrable if and only if the relation (61) holds. This completes the proof. □

Theorem 7. *Let $\mathcal{N}$ be a mixed totally geodesic $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, the distribution $\mathfrak{D}_\lambda \oplus \langle\xi\rangle$ is integrable if and only if the shape operator A satisfies*

$$A_{nW_1} tV_1 + tA_{nW_1} V_1 = 0, \tag{63}$$

$\forall W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$ *and* $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle\xi\rangle)$.

Proof. By the consequence of (10), (13), (28), and (53), we have $g([V_1, V_2], W_1) = g(\widetilde{\nabla}_{V_1} \varphi W_1, \varphi V_2) - g(\widetilde{\nabla}_{V_2} \varphi W_1, \varphi V_1)$, for every $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle\xi\rangle)$ and $W_1 \in \Gamma(\mathcal{D}_\perp)$. Now, using (17), (18), and (26) in the above expression, we have

$$\begin{aligned} g([V_1, V_2], W_1) = &- g(A_{nW_1} V_1, tV_2) + g(A_{nW_1} V_2, tV_1) \\ &+ g(\nabla^\perp_{V_1} nW_1, nV_2) - g(\nabla^\perp_{V_2} nW_1, nV_1). \end{aligned} \tag{64}$$

Furthermore, by the virtue of (13), (17), (18), (26), (28), and (53), we find

$$t\nabla_{V_1} W_1 + n\nabla_{V_1} W_1 + A_{nW_1} V_1 = \nabla^\perp_{V_1} nW_1 - t' h(V_1, W_1) - n' h(V_1, W_1). \tag{65}$$

By comparing normal components of (65), we obtain

$$\nabla^\perp_{V_1} nW_1 - n' h(V_1, W_1) = n\nabla_{V_1} W_1. \tag{66}$$

Now utilizing (65) and (66) in (64), we obtain

$$\begin{aligned} g([V_1, V_2], W_1) = &- g(A_{nW_1} V_1, tV_2) + g(A_{nW_1} V_2, tV_1) + g(n\nabla_{V_1} W_1), nV_2) \\ &+ g(n' h(V_1, W_1), nV_2) - g(n\nabla_{V_2} W_1), nV_1) - g(n' h(V_2, W_1), nV_1). \end{aligned}$$

By the application of (8), we have

$$\lambda g([V_1, V_2], W_1) = g(tA_{nW_1} V_1, V_2) + g(A_{nW_1} tV_1, V_2). \tag{67}$$

By the above expression, we conclude that $\mathfrak{D}_\lambda$ is integrable if and only if (63) holds. □

Theorem 8. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, the distribution $\mathfrak{D}_\lambda \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(A_{nW_1} V_1, tV_2) - g(A_{nW_1} tV_1, V_2) + g(\nabla^\perp_{V_1} nV_2, nW_1) - g(\nabla^\perp_{V_2} nV_1, nW_1) = 0, \tag{68}$$

for every $V_1, V_2 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle\xi\rangle)$ *and* $W_1 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. By the consequence of (17), (18), and (22), we have

$$\varphi[U, V] = t\nabla_{V_1} V_2 + n\nabla_{V_1} V_2 - t\nabla_{V_2} V_1 - n\nabla_{V_2} V_1.$$

In light of (29), (30) and (31), we observe that

$$\begin{aligned} \varphi[V_1, V_2] = &\nabla_{V_1} tV_2 + \nabla^\perp_{V_1} nV_2 - \nabla_{V_2} tV_1 - \nabla^\perp_{V_2} nV_1 + A_{nV_1} V_2 - A_{nV_2} V_1 \\ &+ \eta(V_1)\varphi V_2) - \eta(V_2)\varphi V_1 + 2g(tV_1, V_2)\xi + h(V_1, tV_2) - h(tV_1, V_2). \end{aligned} \tag{69}$$

Now, taking the inner product in the above expression with nW_1 and using (12), where $W_1 \in \Gamma(\mathfrak{D}_\perp)$;

$$\begin{aligned} g(\varphi[V_1, V_2], nW_1) = &g(h(V_1, tV_2), nV_1) - g(h(tV_1, V_2), nW_1) + g(\nabla^\perp_{V_1} nV_2, nW_1) \\ &- g(\nabla^\perp_{V_2} nV_1, nW_1). \end{aligned}$$

From using (25) and (26) in the above equation, we arrive that

$$g(t'n[V_1,V_2],W_1) = g(h(tV_1,V_2),nW_1) - g(h(V_1,tV_2),nV_1) - g(\nabla^{\perp}_{V_1}nV_2,nW_1) + g(\nabla^{\perp}_{V_2}nV_1,nW_1).$$

In light of Lemma 8, we have

$$(1-\lambda)g([V_1,V_2],W_1) = g(h(tV_1,V_2),nW_1) - g(h(V_1,tV_2),nV_1) - g(\nabla^{\perp}_{V_1}nV_2,nW_1) + g(\nabla^{\perp}_{V_2}nV_1,nW_1). \quad (70)$$

Thus, Equation (70) concludes that $\mathfrak{D}_\lambda \oplus \langle\xi\rangle$ is integrable if and only if (68) holds. □

Theorem 9. *Let $\mathcal{N}$ be a pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, the distribution $\mathfrak{D}_\perp$ is integrable if and only if it A satisfies*

$$A_{nW_1}W_2 = 0, \quad (71)$$

$\forall W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. First of all, suppose $\mathfrak{D}_\perp$ is integrable distribution, then $tW_2 = tW_1 = 0$; this implies that $\nabla_{W_2}tW_1 = \nabla_{W_1}tW_2 = 0$. Therefore, relation (31) reduces $g((\nabla_{V_1}t)W_2,W_1) = g(A_{nW_2}V_1,W_1) + g(t'h(V_1,W_2),W_1)$, for every $V_1 \in \Gamma(\mathfrak{D}_\lambda \oplus <\xi>)$; this implies that $g(A_{nW_2}V_1,W_1) = -g(t'h(V_1,W_2),W_1)$. Now, in the light of (19) and (27), the above expression turns into $g(A_{nW_2}W_1,X) = -g(A_{nW_1}W_2,V_1)$. Thus, from (58), we obtain (71).

Conversely: suppose that $\mathcal{N}$ satisfies (71), then by utilization of (19) we have $g(t'h(V_1,W_2),W_1) = 0$. Now, employing (29) and (31) into the above expression, we achieve that $g(\nabla_{W_2}W_1,V_1) = 0$, which implies that $\nabla_{W_2}W_1 \in \Gamma(\mathfrak{D}_\perp)$. This shows that $\mathfrak{D}_\perp$ is a integrable distribution. □

4. $\mathcal{PR}$-Pseudo-Slant Warped Product Submanifolds

Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then, $\mathcal{N}$ is said to be a $\mathcal{PR}$-pseudo-slant warped product if it is a warped product of type $\mathcal{N}_\perp \times_f \mathcal{N}_\lambda$ or $\mathcal{N}_\lambda \times_f \mathcal{N}_\perp$, where $\mathcal{N}_\lambda$ is slant submanifold and $\mathcal{N}_\perp$ is a anti-invariant submanifold in $\mathcal{N}$. In this paper, we only study the warped product whose base is slant, i.e., $\mathcal{N}_\lambda \times_f \mathcal{N}_\perp$.

Proposition 10. *Let $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ be a $\mathcal{PR}$-pseudo-slant submanifold warped product in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(T\mathcal{N}_\perp)$. Then $\mathcal{N}$ is a $\mathcal{PR}$-product.*

Proof. From Equation (4), we have $\nabla_{V_1}W_1 = \nabla_{W_1}V_1 = V_1(\ln f)W_1$, for $V_1 \in \Gamma(T\mathcal{N}_\lambda)$ and $W_1 \in \Gamma(T\mathcal{N}_\perp)$. Replacing by W_1 by ξ into the above expression, we have $\nabla_{V_1}\xi = V_1(\ln f)\xi$. With the help of (39), the above expression reduces into the given form $V_1(\ln f) = 0$. This completes the proof. □

Proposition 11. *There exists a non-trivial $\mathcal{PR}$-pseudo-slant submanifold warped product $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(T\mathcal{N}_\lambda)$.*

Proof. From Equation (4), we have $\nabla_{V_1}W_1 = \nabla_{W_1}V_1 = V_1(\ln f)W_1$, for $V_1 \in \Gamma(T\mathcal{N}_\lambda)$ and $W_1 \in \Gamma(T\mathcal{N}_\perp)$. Replacing by V_1 by ξ into the above expression, we have $\nabla_{W_1}\xi = \xi(\ln f)W_1$. In the light of (39), the above expression reduces into the following form $\xi(\ln f)W_1 = -W_1$. By the definition of gradient, we have

$$\frac{\nabla f}{f} = -\xi. \quad (72)$$

By the theory of differential equations we observe that Equation (72) has a solution. This shows that f is non-constant. This completes the proof. □

Remark 7. *Let $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ be $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, we have*

$$\xi(\ln f) = -1. \tag{73}$$

Now, we give some examples of $\mathcal{PR}$-pseudo-slant submanifold of type $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$.

Example 2. *Choose $\widetilde{M} = \mathbb{R}^8 \times \mathbb{R}^+$ together with the usual Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, s)$. Then the structure (φ, ξ, η) over $\widetilde{M}$ is defined by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i},\ \varphi\left(\frac{\partial}{\partial s}\right) = 0,\ \eta = ds. \tag{74}$$

where $i, j \in \{1, \cdots, 4\}$ and the pseudo-Riemannian metric tensor g is defined as

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = e^{-2s},\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}\right) = -e^{-2s}, g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 1, \tag{75}$$

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = 0,\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k}\right) = 0,\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_k}\right) = 0, \tag{76}$$

for all $k \in \{1, \cdots, 4\}$. Then by simple computation, we can easily see that $\widetilde{M}$ is para-Kenmotsu manifold. Suppose $\mathcal{N}$ is an immersed submanifold into $\widetilde{M}$ by an immersion σ which is defined by

$$x_1 = u,\ x_2 = kv\sinh\alpha,\ x_3 = \alpha^2,\ x_4 = 0,\ y_1 = v,$$
$$y_2 = kv\cosh\alpha,\ y_3 = 0,\ y_4 = \alpha^2 - 2,\ s = s,$$

for $k \in \mathbb{R}$. Thus, we can easily provide the generating set for the tangent bundle of submanifold as follows:

$$Z_\alpha = kv\cosh\alpha\frac{\partial}{\partial x_2} + 2\alpha\frac{\partial}{\partial x_3} + kv\sinh\alpha\frac{\partial}{\partial y_2} + 2\alpha\frac{\partial}{\partial y_4},$$
$$Z_u = \frac{\partial}{\partial x_1},$$
$$Z_v = k\sinh\alpha\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} + k\cosh\alpha\frac{\partial}{\partial y_2},$$
$$Z_s = \xi.$$

for $s \in \mathbb{R}$. The basis vector for $\varphi(TM)$ is given by

$$\varphi Z_\alpha = kv\sinh\alpha\frac{\partial}{\partial x_2} + 2\alpha\frac{\partial}{\partial x_4} + kv\cosh\alpha\frac{\partial}{\partial y_2} + 2\alpha\frac{\partial}{\partial y_3},$$
$$\varphi Z_u = \frac{\partial}{\partial y_1},$$
$$\varphi Z_v = \frac{\partial}{\partial x_1} + k\cosh\alpha\frac{\partial}{\partial x_2} + k\sinh\alpha\frac{\partial}{\partial y_2},$$
$$\varphi Z_s = 0.$$

By simple calculation, we obtain that the distribution $\mathcal{D}_\lambda = span\{Z_u, Z_v\}$ is slant distribution with slant function $\lambda = \frac{1}{1+k^2}$ and the distribution $\mathcal{D}_\perp = span\{Z_\alpha\}$ is anti-invariant under φ. The induced metric tensor $g_\mathcal{N}$ on $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ is given by

$$g_\mathcal{N} = ds^2 + (du^2 - (1+k^2)dv^2)e^{-2s} + e^{-2s}v^2 d\alpha^2. \tag{77}$$

The above calculation manifests that the submanifold $\mathcal{N}$ is a form of $\mathcal{PR}$-pseudo-slant warped product of type II with warping function $f = e^{-s}v$ of para-Kenmotsu manifold.

Example 3. *Choose $\widetilde{M} = \mathbb{R}^8 \times \mathbb{R}^+$ together with the usual Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, s)$. Then, the structure (φ, ξ, η) over $\widetilde{M}$ is defined by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i},\ \varphi\left(\frac{\partial}{\partial s}\right) = 0,\ \eta = ds. \tag{78}$$

where $i, j \in \{1, \cdots, 4\}$ and the pseudo-Riemannian metric tensor g is defined as

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = e^{-2s},\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}\right) = -e^{-2s}, g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 1, \tag{79}$$

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = 0,\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k}\right) = 0,\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_k}\right) = 0, \tag{80}$$

for all $k \in \{1, \cdots, 4\}$. Then, by simple computation, we can easily see that $\widetilde{M}$ is para-Kenmotsu manifold. Suppose $\mathcal{N}$ is an immersed submanifold into $\widetilde{M}$ by an immersion σ which is defined by

$$\begin{aligned} &x_1 = ku\sinh\alpha,\ x_2 = \alpha,\ x_3 = u,\ x_4 = 0,\ y_1 = ku\cosh\alpha,\\ &y_2 = 0,\ y_3 = v,\ y_4 = \alpha + 1,\ s = s, \end{aligned}$$

for $k \in \mathbb{R} \sim \{1\}$. Thus, we can easily provide the generating set for the tangent bundle of submanifold as follows:

$$\begin{aligned} Z_\alpha &= ku\cosh\alpha\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + ku\sinh\alpha\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_4},\\ Z_u &= k\sinh\alpha\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + k\cosh\alpha\frac{\partial}{\partial y_1},\\ Z_v &= \frac{\partial}{\partial y_3},\\ Z_s &= \xi. \end{aligned}$$

for $s \in \mathbb{R}$. The basis vector for $\varphi(T\mathcal{N})$ is given by

$$\begin{aligned} \varphi Z_\alpha &= ku\cosh\alpha\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + ku\sinh\alpha\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4},\\ \varphi Z_u &= k\sinh\alpha\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3} + k\cosh\alpha\frac{\partial}{\partial x_1},\\ \varphi Z_v &= \frac{\partial}{\partial x_3},\\ \varphi Z_s &= 0. \end{aligned}$$

By simple calculation, we obtain that the distribution $\mathcal{D}_\lambda = span\{Z_u, Z_v\}$ is slant distribution of with slant function $\lambda = \frac{1}{1-k^2}$ and the distribution $\mathcal{D}_\perp = span\{Z_\alpha\}$ is anti-invariant under φ. The induced metric tensor $g_\mathcal{N}$ on $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ is given by

$$g_\mathcal{N} = ds^2 + e^{-2s}((1-k^2)du^2 - dv^2) + e^{-2s}u^2 d\alpha^2. \tag{81}$$

The above calculation manifests that the submanifold $\mathcal{N}$ is a form of $\mathcal{PR}$-pseudo-slant warped product of type I if $k < 1$ and $\mathcal{PR}$-pseudo-slant warped product of type III if $k > 1$ of para-Kenmotsu manifold with warping function $f = e^{-s}u$.

Lemma 2. *For a $\mathcal{PR}$-pseudo-slant warped product submanifold $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ in $\mathcal{K}^{2n+1}$, we receive for all $V_1, V_2 \in \Gamma(T\mathcal{N}_\lambda)$ and $W_1, W_2 \in \Gamma(T\mathcal{N}_\perp)$ that*

$$g(h(V_1, V_2), nW_1) = g(h(V_1, W_1), nV_2), \tag{82}$$
$$g(h(V_1, W_1), nW_2) = g(h(V_1, W_2), nW_1). \tag{83}$$

Proof. By the consequence of (17) and (28), we have

$$g(h(V_1, V_2), nW_1) = g(\widetilde{\nabla}_{V_1} V_2, \varphi W_1) - g(\widetilde{\nabla}_{V_1} V_2, tW_1).$$

Now, applying (12) and (13) into the above expression, we achieve

$$g(h(V_1, V_2), nW_1) = -g(\widetilde{\nabla}_{V_1} tV_2, W_1) - g(\widetilde{\nabla}_{V_1} nW_1, V_2) - g(\widetilde{\nabla}_{V_1} V_2, tW_1).$$

By the utilization of (4) and (17), we obtain (82). We proceed with a similar process to prove (83). □

Lemma 3. *Let $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ be a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, we obtain for all $V_1, V_2 \in \Gamma(T\mathcal{N}_\lambda)$ and $U, V \in \Gamma(T\mathcal{N}_\perp)$ that*

$$g(h(W_1, W_1), nV_1) = g(h(V_1, W_1), nW_1) + tV_1(\ln f) g(W_1, W_1), \tag{84}$$
$$g(h(W_1, W_1), ntV_1) = g(h(tV_1, W_1), nV) + \lambda(V_1(\ln f) + \eta(V_1))(W_1, W_1). \tag{85}$$

Proof. By the consequence of (17) and (28), we have

$$g(h(W_1, W_1), nV_1) = g(\widetilde{\nabla}_{W_1} W_1, \varphi V_1) - g(\widetilde{\nabla}_{W_1} W_1, tV_1).$$

Now, applying (12) and (13) into the above expression, we achieve

$$g(h(W_1, W_1), nV_1) = -g(\widetilde{\nabla}_{W_1} \varphi W_1, V_1) - g(\widetilde{\nabla}_{W_1} W_1, tV_1).$$

By the utilization of (4), (18) and (19), we obtain (84). If we replace V_1 with tV_1 in (84), then we attain (85). □

Theorem 10. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, $\mathcal{N}$ is a $\mathcal{PR}$-pseudo-slant warped product submanifold if and only if*

$$A_{ntV_1} W_1 - A_{\varphi W_1} tV_1 = \lambda(V_1(\mu) + \eta(V_1)) W_1, \tag{86}$$

for every $V_1 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle \xi \rangle)$, $W_1 \in \Gamma(\mathfrak{D}_\perp)$ and some smooth function μ on $\mathcal{N}$ satisfies $W_2(\mu) = 0$, for every $W_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. Suppose that $\mathcal{N}$ is a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, by the virtue of (19) and (85), we easily obtain (86) by taking $\mu = \ln f$.

Conversely, suppose $\mathcal{N}$ is $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$ that satisfies (86). Then, by the application of Lemma 1 and (86), we obtain $g(\nabla_{V_1} V_2, W_1) = (V_1(\mu) + \eta(V_1))$ $g(W_1, V_2) = 0$. This shows that the distribution $\mathfrak{D}_\lambda \oplus \langle \xi \rangle$ is totally geodesic and integrable. Now, let us denote $h^\perp$ as the second fundamental form of $\mathfrak{D}_\perp$. Then, by the use of (17), we have $g(h^\perp(W_1, W_2), V_1) = g(\widetilde{\nabla}_{W_1} W_2, V_1)$. In view of (10), the above expression reduces into the following form:

$$g(h^\perp(W_1, W_2), V_1) = -g(\varphi \widetilde{\nabla}_{W_1} W_2, \varphi V_1) + \eta(V_1) g(\widetilde{\nabla}_{W_1} W_2, \xi).$$

By the consequence of (13), (14), and (28), the above expression reduces into the following form:

$$g(h^{\perp}(W_1,W_2),V_1) = -g(\widetilde{\nabla}_{W_1}\varphi W_2,\varphi V_1) + g((\widetilde{\nabla}_{W_1}\varphi)W_2,\varphi V_1) + \eta(V_1)g(W_1,W_2)$$
$$= -g(\widetilde{\nabla}_{W_1}\varphi W_2,\varphi V_1) + \eta(V_1)g(W_1,W_2).$$

Now, using (17)–(19) and (27) in the above relation, we have

$$g(h^{\perp}(W_1,W_2),V_1) = g(h(W_1,tV_1),\varphi W_2) - g(W_2,\widetilde{\nabla}_{W_1}t'nV_1) - g(W_2,\widetilde{\nabla}_{W_1}n'nV_1) + \eta(V_1)g(W_1,W_2). \tag{87}$$

In view of (86), (87), and Lemma 8, we have

$$g(h^{\perp}(W_1,W_2),V_1) = \frac{1}{\lambda}(g(h(W_1,tV_1),\varphi W_2) - g(h(W_1,W_2),ntV_1)) + \eta(V_1)g(W_1,W_2) = -V_1(\mu)g(W_1,W_2). \tag{88}$$

By definition of gradient and (88), we have

$$h^{\perp}(W_1,W_2) = -\nabla(\mu)g(W_1,W_2). \tag{89}$$

The relation (89) shows that the distribution $\mathfrak{D}_{\perp}$ is totally umbilical with mean curvature $H^{\perp} = -\nabla(\mu)$, which is parallel with respect to $\nabla^{\perp}$. By Hiepko result and the above discussion, we conclude that the $\mathcal{N} = \mathcal{N}_{\lambda} \times_f \mathcal{N}_{\perp}$ is a $\mathcal{PR}$-pseudo-slant warped product submanifold of $\mathcal{K}^{2n+1}$. This completes the proof. □

Theorem 11. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, $\mathcal{N}$ is a mixed totally geodesic $\mathcal{PR}$-pseudo-slant warped product submanifold if and only if*

$$A_{\varphi W_1}V_1 = 0, \text{ and } A_{ntV_1}W_1 = -\lambda(V_1(\mu) + \eta(V_1))W_1, \tag{90}$$

for every $V_1 \in \Gamma(\mathfrak{D}_{\lambda} \oplus \langle\xi\rangle)$, $W_1 \in \Gamma(\mathfrak{D}_{\perp})$ and some smooth function μ on $\mathcal{N}$ satisfies $W_2(\mu) = o$, for every $W_2 \in \Gamma(\mathfrak{D}_{\perp})$.

Proof. Suppose that $\mathcal{N}$ is a mixed totally geodesic $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$, then $h(V_1,W_1) = 0$, for every $V_1 \in \Gamma(T\mathcal{N}_{\lambda})$ and $W_1 \in \Gamma(T\mathcal{N}_{\perp})$. Therefore, by the virtue of (19) and (82), we achieve (90).

Conversely, suppose $\mathcal{N}$ is a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$ that satisfies (90). From Lemma 1 and (90), we have

$$g(\nabla_{V_1}V_2,W_1) = -(V_1(\mu) + \eta(X))g(W_1,V_2) = 0.$$

By this expression, we easily see that the leaves of $\mathfrak{D}_{\lambda} \oplus \langle\xi\rangle$ are totally geodesic and integrable. Let us denote $h^{\perp}$ as the second fundamental form of $\mathfrak{D}_{\perp}$. Then, by the use of (17), we have $g(h^{\perp}(W_1,W_2),V_1) = g(\widetilde{\nabla}_{W_1}W_2,V_1)$. Now, utilizing (10), (13), (14), and (28) in the above expression, we concede that

$$g(h^{\perp}(W_1,W_2),V_1) = -g(\widetilde{\nabla}_{W_1}\varphi W_2,\varphi V_1) + \eta(V_1)g(W_1,W_2).$$

By using (17)–(19), (27), and the first part part of (90) into the above relation, we receive that

$$g(h^{\perp}(W_1,W_2),V_1) = -g(W_2,\widetilde{\nabla}_{W_1}t'nV_1) - g(W_2,\widetilde{\nabla}_{W_1}n'nV_1) + \eta(V_1)g(W_1,W_2). \tag{91}$$

In view of Lemma 8, (90) and (91), we have

$$g(h^{\perp}(W_1, W_2), V_1) = V_1(\mu)g(W_1, W_2). \tag{92}$$

By definition of gradient and (92), we have

$$h^{\perp}(W_1, W_2) = \nabla(\mu)g(W_1, W_2). \tag{93}$$

The relation (93) shows that the distribution $\mathfrak{D}_{\perp}$ is totally umbilical with mean curvature $H^{\perp} = \nabla(\mu)$ which is parallel with respect to $\nabla^{\perp}$. By Hiepko result and the above discussion, we conclude that the $\mathcal{N} = \mathcal{N}_{\lambda} \times_f \mathcal{N}_{\perp}$ is a mixed totally geodesic $\mathcal{PR}$-pseudo-slant warped product submanifold of $\mathcal{K}^{2n+1}$. □

Theorem 12. *Let $\mathcal{N} = \mathcal{N}_{\lambda} \times_f \mathcal{N}_{\perp}$ be a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, $\mathcal{N}$ is locally a $\mathcal{PR}$-product if and only if*

$$A_{ntV_1}W_1 = \lambda\eta(V_1)W_1, \tag{94}$$

for every $V_1 \in \Gamma(T\mathcal{N}_{\lambda})$ and $W_1 \in \Gamma(T\mathcal{N}_{\perp})$.

Proof. By the application of Equations (10), (17), and (28), we have $g(\nabla_{W_1}V_1, W_2) = -g(\widetilde{\nabla}_{W_1}\varphi V_1,$
$\varphi W_2) + g((\widetilde{\nabla}_{W_1}\varphi)V_1, \varphi W_2)$, for every $V_1 \in \Gamma(T\mathcal{N}_{\lambda})$ and $W_1, W_2 \in \Gamma(T\mathcal{N}_{\perp})$. Now, using (10) and (27), we concede that

$$g(\nabla_{W_1}V_1, W_2) = -g(\widetilde{\nabla}_{W_1}tV_1, \varphi W_2) - \eta(V_1)g(W_1, W_2) - g(\widetilde{\nabla}_{W_1}nV_1, \varphi W_2).$$

By the consequence of (12), (13), (14), (24), and (28), the above expression relation reduces into the following form:

$$\begin{aligned} g(\nabla_{W_1}V_1, W_2) &= g(\widetilde{\nabla}_{W_1}t^2V_1, W_2) + g(\widetilde{\nabla}_{W_1}ntV_1, W_2) \\ &\quad -\eta(V_1)g(W_1, W_2) - g(\nabla^{\perp}_{W_1}nV_1, \varphi W_2). \end{aligned}$$

In light of (14), (17), (4), and Lemma 3, the above expression reduces into the following form:

$$(1-\lambda)(V_1(\ln f) - \eta(V_1))g(W_1, W_2) = g(h(W_1, W_2), ntV_1) - g(\nabla^{\perp}_{W_1}nV_1, \varphi W_2). \tag{95}$$

Interchanging W_1 and W_2 into (95), we have

$$(1-\lambda)(V_1(\ln f) - \eta(V_1))g(W_1, W_2) = g(h(W_1, W_2), ntV_1) - g(\nabla^{\perp}_{W_2}nV_1, \varphi W_1). \tag{96}$$

In view of (95) and (96), we have

$$g(\nabla^{\perp}_{W_2}nV_1, \varphi W_1) = g(\nabla^{\perp}_{W_1}nV_1, \varphi W_2). \tag{97}$$

On the other hand, by use of (13), (17), and (28), we observe that

$$\begin{aligned} g(\nabla^{\perp}_{W_1}nV_1, \varphi W_2) &= g(\varphi\widetilde{\nabla}_{W_1}V_1, \varphi W_2) - \eta(V_1)g(\varphi W_1, \varphi W_2) \\ &\quad -g(\widetilde{\nabla}_{W_1}tV_1, \varphi W_2). \end{aligned}$$

In light of (4) and (10), the above expression reduces into the following form:

$$g(\nabla^{\perp}_{W_1}nV_1, \varphi W_2) = -V_1(\ln f)g(W_1, W_2) + \eta(V_1)g(W_1, W_2) - g(\widetilde{\nabla}_{W_1}tV_1, \varphi W_2). \tag{98}$$

Again, interchanging W_1 and W_2 into (98), we have

$$g(\nabla^{\perp}_{W_2} nV_1, \varphi W_1) = -V_1(\ln f) g(W_1, W_2) + \eta(V_1) g(W_1, W_2) - g(\widetilde{\nabla}_{W_2} tV_1, \varphi W_1). \tag{99}$$

By the virtue of (98) and (99), we conclude that (97) holds if and only if

$$g(\widetilde{\nabla}_{W_2} tV_1, \varphi W_1) = 0 = -g(\widetilde{\nabla}_{W_1} tV_1, \varphi W_2). \tag{100}$$

By the utilization of (17), (24), (28), (100), and Lemma 3, we obtain

$$\lambda(V_1(\ln f) + \eta(V_1)) g(W_1, W_2)) - g(h(W_1, W_2), ntV_1) = 0. \tag{101}$$

By the above relation, we can observe that f is constant if and only if the relation (94) holds. This completes the proof. □

Lemma 4. *Let $\mathcal{N} = \mathcal{N}_\lambda \times_f \mathcal{N}_\perp$ be a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, we obtain for all $U \in \Gamma(T\mathcal{N})$, $V_1 \in \Gamma(T\mathcal{N}_\lambda)$, and $W_1 \in \Gamma(T\mathcal{N}_\perp)$ that*

$$(\nabla_U t) W_1 = - g(W_1, QU) t \nabla(\ln f), \tag{102}$$

$$(\nabla_U t) V_1 = \eta(U) A_{nV_1} \xi + \eta(V_1) tPU + g(PU, tV_1)\xi + tV_1(\ln f) QU. \tag{103}$$

$$\begin{aligned}(\nabla_U t) tV_1 =& \eta(U) A_{ntV_1} \xi + \lambda \eta(V_1) PU - \lambda \eta(V_1) g(PU, V_1)\xi \\ &+ \lambda(V_1(\ln f) + \eta(V_1)) QU.\end{aligned} \tag{104}$$

Proof. By the use of (51), we have $(\nabla_U t)W_1 = (\nabla_{PU} t)W_1 + (\nabla_{QU} t)W_1 + \eta(U)(\nabla_\xi t)W_1$. By the virtue of (4) and Definition 3, we have $(\nabla_{PU} t)W_1 = (\nabla_\xi t)W_1 = 0$. In view of (29) and (5), we observe that $(\nabla_{QU} t)W_1 = -g(W_1, QU) t\nabla(\ln f)$. By these observations, we easily concede the relation (102). By reuse of (51), we have $(\nabla_U t)V_1 = (\nabla_{PU} t)V_1 + (\nabla_{QU} t)V_1 + \eta(U)(\nabla_\xi t)V_1$. Furthermore, by the virtue of (31), we attain $(\nabla_{PU} t)V_1 = A_{nV_1} PU + t' h(PU, V_1) + \eta(V_1) tPU - g(tPU, V_1)\xi$. Since $\mathcal{N}_\lambda$ is totally geodesic, the above expression reduces into the following form:

$$(\nabla_{PU} t) V_1 = \eta(V_1) tPU - g(tPU, V_1)\xi. \tag{105}$$

By the utilization of (4) and (51), we have

$$(\nabla_{QU} t) V_1 = tV_1(\ln f) QU. \tag{106}$$

Similarly, we find

$$(\nabla_\xi t) V_1 = A_{nV_1} \xi. \tag{107}$$

By the application of (105)–(107), we achieve (103). If we replace V_1 with tV_1 in (103), we easily achieve (104). □

Theorem 13. *Let $\mathcal{N}$ be a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$. Then, $\mathcal{N}$ is a $\mathcal{PR}$-pseudo-slant warped product submanifold if and only if the endomorphism t satisfies*

$$g((\nabla_U t)V, V_1) = tV_1(\mu) g(QU, QV) + \eta(V_1) g(PU, tPV), \tag{108}$$

for every $V_1 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle \xi \rangle)$, $U, V \in \Gamma(T\mathcal{N})$, and some smooth function μ on $\mathcal{N}$ satisfies $W_2(\mu) = 0$, for every $W_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. Suppose that M is a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$, then by (51), we obtain

$$(\nabla_U t) V = (\nabla_U t) QV + (\nabla_U t) PV + \eta(V)(\nabla_U t)\xi. \tag{109}$$

By the utilization of (14), (17), (102), and (103), we achieve that

$$(\nabla_U t)V = -\eta(V)tU - g(QU, QV)t\nabla(\ln f) + \eta(U)A_{nPV}\xi$$
$$+\eta(PV)tPU + g(PU, tPV)\xi + tPV(\ln f)QU. \quad (110)$$

By taking the inner product with V_1 into (111), then using (39) and definition of gradient, we achieve

$$g((\nabla_U t)V, V_1) = tV_1(\ln f)g(QU, QV) + \eta(V_1)g(PU, tPV), \quad (111)$$

By taking $\mu = \ln f$ into (111) and using the fact that $\mathcal{N}$ is a warped product, we accomplished (108).

Conversely, assume that $\mathcal{N}$ is a $\mathcal{PR}$-pseudo-slant submanifold in $\mathcal{K}^{2n+1}$ satisfying (108). Now, replacing U with V_2 and V with W_1 in (108), we have $g((\nabla_{V_2}t)W_1, V_1) = 0$, $V_1 \in \Gamma(\mathfrak{D}_\lambda \oplus \langle\xi\rangle)$ and $W_1 \in \Gamma(\mathfrak{D}_\perp)$. In view of (26) and (29), we have $g(h^\lambda(tV_1, V_2), W_1) = 0$. This shows that $\mathfrak{D}_\lambda \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in $\mathcal{N}$. Furthermore, replacing U with W_1 and V with W_2 in (108), we have $g((\nabla_{W_1}t)W_2, V_1) = tV_1(\mu)g(W_1, W_2) + \eta(V_1)g(W_1, tV_1)$, for every $W_1, W_2 \in \Gamma(\mathfrak{D}_\perp)$. By (26) and orthogonality relation, we observe that

$$g((h^\perp(W_1, W_2), tV_1) = g(tV_1, \nabla(\ln f))g(W_1, W_2). \quad (112)$$

By the relation (112), we observe that the distribution $\mathfrak{D}_\perp$ is totally umbilical with mean curvature $H^\perp = \nabla(\mu)$. By the application of Hiepko result [41], we can conclude that M is a $\mathcal{PR}$-pseudo-slant warped product submanifold in $\mathcal{K}^{2n+1}$. This completes the proof. □

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