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Article

A Generalization of Group-Graded Modules

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Abstract: In this article, we generalize the concept of group-graded modules by introducing the concept of $\mathfrak{G}$-weak graded R-modules, which are R-modules graded by a set $\mathfrak{G}$ of left coset representatives, where R is a $\mathfrak{G}$-weak graded ring. Moreover, we prove some properties of these modules. Finally, results related to $\mathfrak{G}$-weak graded fields and their vector spaces are deduced. Many considerable examples are provided with more emphasis on the symmetric group S_3 and the dihedral group D_6, which gives the group of symmetries of a regular hexagon.

Keywords: graded rings; graded modules; strongly graded modules; left coset representatives

MSC: 16W50; 13A02; 16D25

1. Introduction

Rings and modules that can be graded by groups were intensively studied especially in connection with Clifford's theory; see [1–3].

Many results about group-graded rings and modules were generalized by using semigroups or monoids for grading instead of groups, leading to more general constructions, as we can see in [4–8].

The group-graded fields and their graded vector spaces, as well as their properties have been investigated by many mathematicians; see for example [9–11].

There are other generalizations of graded rings and modules in the literature, for example the semi-graded rings and semi-graded modules (see [12]), which are justified by considering the non-commutative algebraic geometry for quantum algebras.

Many ways have been used to investigate the properties of these rings and modules. In [13], Cohen and Montgomery introduced an interesting way using duality theorems; see also [14]. Moreover, some mathematicians introduced categorical methods to study these graded rings such as the study of separable functors introduced in [15,16]. Most of these methods have been introduced for the case when the grading group is finite. However, more additional investigations have been performed considering the infinite case; see for example [17].

In [18], Beggs considered a fixed set $\mathfrak{G}$ and defined a binary operation "$*$" on it, which is not associative in a trivial way, though we can deduce the associativity by applying a "cocycle" f. The elements of $\mathfrak{G}$ are left coset representatives for a subgroup H of a finite group X. It was shown that the results are not affected by the choice of the representatives. These data were used to construct non-trivially associated tensor categories and non-trivially associated modular categories (see [18,19]).

In [20], the concept of the rings that can be graded by groups and their modules was generalized using the set $\mathfrak{G}$ mentioned above. It was shown that many results related to these graded rings and their modules could be carried on in the new setting. These induced graded rings were given the name "$\mathfrak{G}$-weak graded rings".

It is natural to ask if the properties of the original finite groups have an affect on the induced graded rings and modules. However, this is still an active area for researchers.

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In [21,22], $\mathfrak{G}$-weak graded rings were deeply investigated and additional properties were derived supported by many illustrative examples.

In this article, we discuss important properties of $\mathfrak{G}$-weak graded R-modules. Throughout this article, many examples are provided, in particular a counter example showing that there is a field having a non-homogeneous unit, although it is a non-trivially X-graded field ($X \neq \mathbb{Z}$). The symmetric group S_3 and the dihedral group D_6, which gives the group of symmetries of a regular hexagon, are considered among these examples.

The importance of this work, besides using a set with a binary operation satisfying specific properties for grading instead of using a group or even a semigroup, is associating this grading with a factorization of a given group, which may lead to a quantization of the classical results of group-graded modules.

As this work is based on [18], which considered X to be finite, and on [1], we assumed, unless otherwise stated, that all groups are finite, the rings are unity, and the modules are unital.

2. Preliminaries

In this section, we include the definition of the binary operation $*$, the cocycle f, and the actions $\triangleright$ and $\triangleleft$ that are used intensively in this work. For more explanation and related results, the reader is referred to [18].

Definition 1 ([18])**.** *Let s, t be elements in $\mathfrak{G}$. Then, $f(s,t)$ in H and $s * t$ in $\mathfrak{G}$ are determined by $st = f(s,t)(s*t)$ in X. Furthermore, the action $\triangleright : \mathfrak{G} \times H \to H$ and the coaction $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ are determined by $su = (s \triangleright u)(s \triangleleft u)$, where $s, s \triangleleft u$ are elements in $\mathfrak{G}$ and $u, s \triangleright u$ are elements in H. These factorizations are unique.*

The binary operation "$*$" ensures the right division property and the left identity for each $s \in \mathfrak{G}$. In the case that $e \in \mathfrak{G}$, then $e_{\mathfrak{G}} = e$.

In what follows, whenever $\mathfrak{G}$ and H are mentioned, we mean the set and the subgroup defined above.

3. The Category of $\mathfrak{G}$-Weak Graded Modules

Definition 2 ([20])**.** *Let R be a $\mathfrak{G}$-weak graded ring. Then, a $\mathfrak{G}$-weak graded left R-module M is a left R-module satisfying:*

$$M = \bigoplus_{s \in \mathfrak{G}} M_s \qquad \text{(as abelian groups)} \tag{1}$$

and

$$R_s M_t \subseteq M_{s*t} \quad \forall\, s,t \in \mathfrak{G}. \tag{2}$$

If the relation (2) is replaced by

$$R_s M_t = M_{s*t} \quad \forall\, s,t \in \mathfrak{G}, \tag{3}$$

then M is termed a strongly (or a fully) $\mathfrak{G}$-weak graded left R-module.

Definition 3. *The elements of $\cup_{s\in\mathfrak{G}} M_s$ are termed weak graded or $\mathfrak{G}$-homogeneous elements of M. A non-zero element $m \in M_s$ is termed a $\mathfrak{G}$-homogeneous element of grade s, and we write $\langle m \rangle = s$.*

Remark 1. *Every element $m \in M$ has decomposition $m = \sum_{s\in\mathfrak{G}} m_s$, which is unique with $m_s \in M_s$, $\forall s \in \mathfrak{G}$. These $\{m_s\}_{s\in\mathfrak{G}}$ are named the $\mathfrak{G}$-homogeneous components of m. It can be noted that the sum is finite (i.e., almost all m_s are zero).*

$\mathfrak{G}$-GrR-Mod denotes the category of the class of $\mathfrak{G}$-weak graded left R-modules. The morphisms in $\mathfrak{G}$-GrR-Mod are $\mathfrak{G}$-weak graded-preserving morphisms, i.e., if φ :

$M \longrightarrow N$ in $\mathfrak{G}$-GrR-Mod, then $\varphi(m) \in N_s$ for all $m \in M_s$ and $s \in \mathfrak{G}$, where M and N are $\mathfrak{G}$-weak graded left R-modules. These morphisms are denoted by $\mathfrak{G}$-$GrHom_R(M, N)$.

Definition 4. *Let M and N be two $\mathfrak{G}$-weak graded R-modules. Then, we define an additive subgroup $\mathfrak{G}$-$GrHom_R(M, N)_t$ of $\mathfrak{G}$-$GrHom_R(M, N)$, for $t \in \mathfrak{G}$, by*

$$\mathfrak{G}-GrHom_R(M, N)_t = \{\phi \in \mathfrak{G}-GrHom_R(M, N) \ : \ \phi(M_s) \subseteq N_{s*t}, \textit{ for all } s \in \mathfrak{G}\}.$$

Proposition 1. *$\mathfrak{G}$-$GrHom_R(M, N)$ is a subgroup of $Hom_R(M, N)$ additively.*

Proof. Let $\phi, \varphi \in \mathfrak{G}$-$GrHom_R(M, N)$. Hence, for all $s \in \mathfrak{G}$, we have

$$\phi(M_s) \subseteq N_s \qquad \text{and} \qquad \varphi(M_s) \subseteq N_s.$$

Thus,

$$(\phi + \varphi)(M_s) = \phi(M_s) + \varphi(M_s) \subseteq N_s + N_s = N_s.$$

Knowing that ϕ and φ are homomorphisms and that N_s is an additive abelian subgroup of N completes the proof. □

Theorem 1. *Let M and N be $\mathfrak{G}$-weak graded left R-modules. Then,*

$$\mathfrak{G}-GrHom_R(M, N) = \bigoplus_{s \in \mathfrak{G}} \mathfrak{G}-GrHom_R(M, N)_s\,,$$

as additive subgroups.

Proof. Let $\varphi_s \in \mathfrak{G}-GrHom_R(M, N)_s$ for $s \in \mathfrak{G}$ s.t. all φ_s, except a finite number, are zero maps, and let

$$\sum_{s \in \mathfrak{G}} \varphi_s = 0. \tag{4}$$

Now, we show that φ_s is a zero map. By Definition 4, we have $\varphi_s(m) \in N_{t*s}$ for all $m \in M_t$ and an arbitrary fixed element $t \in \mathfrak{G}$. Hence, considering (4), we have:

$$(\sum_{s \in \mathfrak{G}} \varphi_s)(m) = \sum_{s \in \mathfrak{G}} \varphi_s(m) = 0_N.$$

This is the unique expansion of 0_N in the direct sum $N = \bigoplus_{s \in \mathfrak{G}} N_{t*s}$ or equivalently $N = \bigoplus_{s \in \mathfrak{G}} N_s$. Thus, $\varphi_s(m) = 0_N$ for each $s \in \mathfrak{G}$, which means that φ_s is a zero map on M_t for $t \in \mathfrak{G}$. Since t is an arbitrary element and $M = \bigoplus_{t \in \mathfrak{G}} M_t$, it follows that φ_s is a zero map on M, which guarantees that the equality in the theorem is satisfied. □

Proposition 2. *Let $e_{\mathfrak{G}}$ be a two-sided identity in $\mathfrak{G}$. Then, the left multiplication by any weak graded unit $x \in R_s$ for any $s \in \mathfrak{G}$ is an isomorphism in $R_{e_{\mathfrak{G}}}$-Mod. Moreover, $xR_{e_{\mathfrak{G}}} = R_s$.*

Proof. Since x is a unit, it immediately follows that the left multiplication by x is an $R_{e_{\mathfrak{G}}}$-isomorphism of $R_{e_{\mathfrak{G}}}$ onto $xR_{e_{\mathfrak{G}}}$. Furthermore, [22], Theorem 4, yields

$$xR_{e_{\mathfrak{G}}} \subseteq R_s R_{e_{\mathfrak{G}}} \subseteq R_{s*e_{\mathfrak{G}}} = R_s.$$

On the other hand,

$$R_s = xx^{-1}R_s \subseteq xR_{s^L}R_s \subseteq xR_{e_{\mathfrak{G}}},$$

which means $R_s = xR_{e_{\mathfrak{G}}}$, as required. □

Proposition 3. *Let M be a $\mathfrak{G}$-weak graded left R-module. Then, M_s is a left $R_{e_{\mathfrak{G}}}$-submodule of M for each $s \in \mathfrak{G}$.*

Proof. By Definition 2, M_s is an abelian subgroup of M for each $s \in \mathfrak{G}$. Furthermore, by using relation (2) of Definition 2, we obtain $R_{e_{\mathfrak{G}}} M_s \subseteq M_{e_{\mathfrak{G}} * s} = M_s$, which means that M_s is a left $R_{e_{\mathfrak{G}}}$-module. Knowing that $M_s \subseteq M$ completes the proof. □

Definition 5. *A non-empty subset N of a $\mathfrak{G}$-weak graded R-module M is termed a $\mathfrak{G}$-weak graded submodule of M if N itself is a $\mathfrak{G}$-weak graded R-module.*

Proposition 4. *Let N be a non-trivial sub-module of a $\mathfrak{G}$-weak graded left R-module M. If N contains all of its components that are $\mathfrak{G}$-homogeneous, then N is a $\mathfrak{G}$-weak graded sub-module.*

Proof. Let $n = \sum_{s\in\mathfrak{G}} n_s$ such that $n_s \in N$ for all $n \in N$, $n_s \in N_s$, and $s \in \mathfrak{G}$. Now, we prove that $N = \bigoplus_{s\in\mathfrak{G}} N_s$. It is clear that $N = \sum_{s\in\mathfrak{G}} N_s$. Since N is a submodule of $M = \bigoplus_{s\in\mathfrak{G}} M_s$, hence $N_s = N \cap M_s$ for each $s \in \mathfrak{G}$. Furthermore, as M is a $\mathfrak{G}$-weak graded left R-module, we obtain $M_s \bigcap (\sum_{t\in\mathfrak{G}} M_t) = \{0\}$ for all $s \in \mathfrak{G}$ with $s \neq t$. Thus, $N_s \bigcap (\sum_{t\in\mathfrak{G}} N_t) = \{0\}$. Therefore, $N = \bigoplus_{s\in\mathfrak{G}} N_s$, as required.Next, to prove that $R_s N_t \subseteq N_{s*t}$, let $N_t = N \cap M_t$. Then, for $s \in \mathfrak{G}$, we have

$$R_s N_t = R_s(N \cap M_t) = R_s N \cap R_s M_t \subseteq N \cap R_s M_t \subseteq N \cap M_{s*t} = N_{s*t},$$

which completes the proof. □

Proposition 5. *If M is a $\mathfrak{G}$-weak graded R-module and N is a non-trivial $\mathfrak{G}$-weak graded sub-module of M, then M/N is a $\mathfrak{G}$-weak graded R-module, where for each $s \in \mathfrak{G}$, we have*

$$(M/N)_s = (M_s + N)/N = \{m + N \quad : \quad m \in M_s\},$$

for each $s \in \mathfrak{G}$.

Proof. First, for all $m \in M$, we have $m = \sum_{s\in\mathfrak{G}} m_s$, where $m_s \in M_s$, which implies

$$m + N = (\sum_{s\in\mathfrak{G}} m_s) + N = \sum_{s\in\mathfrak{G}} (m_s + N).$$

Hence,

$$M/N = \sum_{s\in\mathfrak{G}} (M/N)_s.$$

Suppose that

$$\sum_{s\in\mathfrak{G}} (m_s + N) = 0 + N \tag{5}$$

for $m_s \in M_s$. We have to show that $m_s + N = 0 + N$ for all $s \in \mathfrak{G}$. From (5), clearly, $\sum_{s\in\mathfrak{G}} m_s \in N$. Since N is a $\mathfrak{G}$-weak graded submodule, then $m_s \in N$ for each m_s. Therefore, $m_s + N = 0 + N$, $\forall s \in \mathfrak{G}$, as required. Thus,

$$M/N = \bigoplus_{s\in\mathfrak{G}} (M/N)_s.$$

Next, for the inclusion property, we have:

$$R_s(M/N)_t = R_s(M_t + N)/N = (R_s M_t + N)/N \subseteq (M_{s*t} + N)/N = (M/N)_{s*t}.$$

Therefore, M/N is a $\mathfrak{G}$-weak graded R-module, as required. □

Example 1. *Consider the ring $R = M_2(\mathbb{R})$. Let $X = D_6 = < x, y : x^6 = y^2 = 1, xy = yx^5 >$ be the dihedral group of symmetries of a regular hexagon. The group generators are given by a counterclockwise rotation through $\dfrac{\pi}{3}$ radians and reflection in a line joining the midpoints of two opposite edges where x denotes the rotation and y denotes the reflection. If we choose*

$H = \{1, x^3, y, x^3y\}$ and $\mathfrak{G} = \{1, x, x^5\}$, then R is a $\mathfrak{G}$-weak graded ring (see [22], Example 3.1). Define

$$M = M_{2\times 3}(\mathbb{R}) = \left\{ \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \end{bmatrix} : \quad r_i \in \mathbb{R}, \quad 1 \leqslant i \leqslant 6 \right\}.$$

Then, M is a $\mathfrak{G}$-weak graded R-module with

$$M = M_1 \oplus M_x \oplus M_{x^5},$$

where, $M_1 = \left\{ \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} \right\}$, $M_x = \left\{ \begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} \right\}$ *and* $M_{x^5} = \left\{ \begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} \right\}$.

Furthermore, $R_sM_t \subseteq M_{s*t}$, $\forall s, t \in \mathfrak{G}$, as follows:

1. $R_1M_1 \subseteq M_{1*1}$ *as for all* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in R_1$ *and* $\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} \in M_1$, *we have*

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} = \begin{bmatrix} ar_1 & 0 & 0 \\ 0 & dr_5 & 0 \end{bmatrix} \in M_1 = M_{1*1}.$$

2. $R_1M_x \subseteq M_{1*x}$ *as for all* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in R_1$ *and* $\begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} \in M_x$, *we have*

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a\,r_3 \\ dr_4 & 0 & 0 \end{bmatrix} \in M_x = M_{1*x}.$$

3. $R_1M_{x^5} \subseteq M_{1*x^5}$ *as for all* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in R_1$ *and* $\begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} \in M_{x^5}$, *we have*

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} = \begin{bmatrix} 0 & ar_2 & 0 \\ 0 & 0 & dr_6 \end{bmatrix} \in M_{x^5} = M_{1*x^5}.$$

4. $R_xM_1 \subseteq M_{x*1}$ *as for all* $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in R_x$ *and* $\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} \in M_1$, *we have*

$$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ cr_1 & 0 & 0 \end{bmatrix} \in M_x = M_{x*1}.$$

5. $R_xM_x \subseteq M_{1*x}$ *as for all* $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in R_x$ *and* $\begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} \in M_x$, *we have*

$$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & cr_3 \end{bmatrix} \in M_{x^5} = M_{x*x}.$$

6. $R_xM_{x^5} \subseteq M_{x*x^5}$ *as for all* $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in R_x$ *and* $\begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} \in M_{x^5}$, *we have*

$$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & cr_2 & 0 \end{bmatrix} \in M_1 = M_{x*x^5}.$$

7. $R_{x^5}M_1 \subseteq M_{x^5*1}$ *as for all* $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in R_{x^5}$ *and* $\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} \in M_1$, *we have*

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & br_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{x^5} = M_{x^5*1}.$$

8. $R_{x^5}M_x \subseteq M_{x^5*x}$ *as for all* $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in R_{x^5}$ *and* $\begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} \in M_x$, *we have*

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} br_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_1 = M_{x^5*x}.$$

9. $R_{x^5}M_{x^5} \subseteq M_{x^5*x^5}$ *as for all* $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in R_{x^5}$ *and* $\begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} \in M_{x^5}$, *we have*

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & b\,r_6 \\ 0 & 0 & 0 \end{bmatrix} \in M_x = M_{x^5*x^5}.$$

Thus, M is a non-trivial $\mathfrak{G}$-weak graded left R-module.

Proposition 6. *Let M be a strongly $\mathfrak{G}$-weak graded R-module. Then, $M = 0$ if and only if $M_t = 0$ for some $t \in \mathfrak{G}$.*

Proof. Suppose that $M_t = 0$ for some $t \in \mathfrak{G}$, then, for all $e_{\mathfrak{G}} \neq s \in \mathfrak{G}$, we have

$$0 = R_s M_t = M_{s*t} = M_p$$

where $p = s * t$. Thus, the right division property on $\mathfrak{G}$ implies $M_p = 0$ for all $p \in \mathfrak{G}$, as required.

The converse is even more obvious. □

4. $\mathfrak{G}$-Weak Graded Fields

Definition 6. *We call a $\mathfrak{G}$-weak graded ring that is not the zero ring a $\mathfrak{G}$-weak graded field if each $\mathfrak{G}$-homogeneous non-zero element has an inverse.*

It can be noted that a $\mathfrak{G}$-weak graded field is not necessarily a field, as we show in the next example:

Example 2. *Let $(\mathbb{H}, +, \cdot)$ be the ring of real quaternions. Let $X = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, $H = \{(0,0), (0,1)\}$, and $\mathfrak{G} = \{(0,1), (1,1)\}$. Then, the $*$ operation (Table 1) , the cocycle f (Table 2), and the actions $\triangleright$ and $\triangleleft$ (Table 3) are calculated as follows.*

Table 1. The binary operation $*$.

$*$	$(0,1)$	$(1,1)$
$(0,1)$	$(0,1)$	$(1,1)$
$(1,1)$	$(1,1)$	$(0,1)$

Table 2. The cocycle f.

f	$(0,1)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,1)$
$(1,1)$	$(0,1)$	$(0,1)$

Table 3. The actions $s \triangleright u$ and $s \triangleleft u$.

$s \triangleright u$	$(0,0)$	$(0,1)$	$s \triangleleft u$	$(0,0)$	$(0,1)$
$(0,1)$	$(0,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
$(1,1)$	$(0,0)$	$(0,1)$	$(1,1)$	$(1,1)$	$(1,1)$

Thus, $\mathbb{H} = \mathbb{C}_{(0,1)} \oplus \mathbb{C}_{(1,1)}$, where $\mathbb{C}_{(0,1)} = \mathbb{R} \oplus \mathbb{R}i$ and $\mathbb{C}_{(1,1)} = \mathbb{R}j \oplus \mathbb{R}k$. Now, the following calculations are needed to show that the inclusion property is satisfied:

1. $\mathbb{C}_{(0,1)}\mathbb{C}_{(0,1)} \subseteq \mathbb{C}_{(0,1)*(0,1)} = \mathbb{C}_{(0,1)}$ *as for all* $(r_1 + r'_1 i), (r_2 + r'_2 i) \in \mathbb{C}_{(0,1)}$, *we have*
$$(r_1 + r'_1 i)(r_2 + r'_2 i) = (r_1 r_2 - r'_1 r'_2) + (r_1 r'_2 + r'_1 r_2)i \in \mathbb{R} \oplus \mathbb{R}i = \mathbb{C}_{(0,1)}.$$
2. $\mathbb{C}_{(0,1)}\mathbb{C}_{(1,1)} \subseteq \mathbb{C}_{(0,1)*(1,1)} = \mathbb{C}_{(1,1)}$ *as for all* $(r + r'i) \in \mathbb{C}_{(0,1)}$ *and* $(r_1 j + r'_1 k) \in \mathbb{C}_{(1,1)}$, *we have*
$$(r + r'i)(r_1 j + r'_1 k) = (rr_1 - r'r'_1)j + (rr'_1 + r'r_1)k \in \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C}_{(1,1)}.$$
3. $\mathbb{C}_{(1,1)}\mathbb{C}_{(0,1)} \subseteq \mathbb{C}_{(1,1)*(0,1)} = \mathbb{C}_{(1,1)}$ *as for all* $(r + r'i) \in \mathbb{C}_{(0,1)}$ *and* $(r_1 j + r'_1 k) \in \mathbb{C}_{(1,1)}$, *we have*
$$(r_1 j + r'_1 k)(r + r'i) = (r_1 r + r'_1 r')j + (r'_1 r - r_1 r')k \in \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C}_{(1,1)}.$$
4. $\mathbb{C}_{(1,1)}\mathbb{C}_{(1,1)} \subseteq \mathbb{C}_{(1,1)*(1,1)} = \mathbb{C}_{(0,1)}$ *as for all* $(r_1 j + r'_1 k), (r_2 j + r'_2 k) \in \mathbb{C}_{(1,1)}$, *we have*
$$(r_1 j + r'_1 k)(r_2 j + r'_2 k) = (-r_1 r_2 - r'_1 r'_2) + (r_1 r'_2 - r'_1 r_2)i \in \mathbb{R} \oplus \mathbb{R}i = \mathbb{C}_{(0,1)}.$$

Thus, $\mathbb{H}$ *is a strongly* $\mathfrak{G}$*-weak graded ring. Moreover, as every element in* $\mathbb{H}$ *is invertible, hence every* $\mathfrak{G}$*-homogeneous element is invertible. Consequently,* $\mathbb{H}$ *is a* $\mathfrak{G}$*-weak graded field.*

Now, we give an example of a $\mathfrak{G}$-weak graded field, which is a field.

Example 3. *Let* X *be the dihedral group of symmetries of a regular hexagon* $D_6 = \{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$, $H = \{1, x^2, x^4, y, x^2y, x^4y\}$ *and* $\mathfrak{G} = \{1, x\}$. *Then, the* $*$ *operation (Table 4), the cocycle* f *(Table 5), and the actions* $\triangleright$ *and* $\triangleleft$ *(Table 6) are calculated as follows.*

Table 4. The binary operation $*$.

$*$	1	x
1	1	x
x	x	1

Table 5. The cocycle f.

f	1	x
1	1	1
x	1	x^2

Table 6. The actions $s \triangleright u$ and $s \triangleleft u$.

$s \triangleright u$	1	x^2	x^4	y	x^2y	x^4y	$s \triangleleft u$	1	x^2	x^4	y	x^2y	x^4y
1	1	x^2	x^4	y	x^2y	x^4y	1	1	1	1	1	1	1
x	1	x^2	x^4	x^2y	x^4y	y	x	x	x	x	x	x	x

Now, we consider the field $F = (\mathbb{R}^2, +, \cdot)$ *with product:*
$$(a, b)\ (a_1, b_1) = (aa_1 - bb_1, a\, b_1 + ba_1),$$
for all $(a, b), (a_1, b_1) \in F$. *Hence,* F *is a* $\mathfrak{G}$*-weak graded with* $F = F_1 \oplus F_x$, *where*
$$F_1 = \{(a, 0) : \quad a \in \mathbb{R}\} \quad and \quad F_x = \{(0, b) : \quad b \in \mathbb{R}\},$$
are additive subgroups. In addition, the inclusion property is satisfied as follows:
1. $F_1F_1 = \{(a_1, 0)(a_2, 0) : \quad a_1, a_2 \in \mathbb{R}\} = \{(a_1a_2, 0) : \quad a_1a_2 \in \mathbb{R}\} = F_1 = F_{1*1}$.
2. $F_1F_x = \{(a, 0)(0, b) : \quad a, b \in \mathbb{R}\} = \{(0, ab) : \quad ab \in \mathbb{R}\} = F_x = F_{1*x}$.
3. $F_xF_1 = \{(0, b)(a, 0) : \quad a, b \in \mathbb{R}\} = \{(0, ba) : \quad ba \in \mathbb{R}\} = F_x = F_{x*1}$.
4. $F_xF_x = \{(0, b_1)(0, b_2) : \quad b_1, b_2 \in \mathbb{R}\} = \{(-b_1b_2, 0) : \ -b_1b_2 \in \mathbb{R}\} = F_1 = F_{x*x}$.

Therefore, F is a strongly $\mathfrak{G}$-weak graded field.

Definition 7. *Let R be a $\mathfrak{G}$-weak graded ring. An ideal I of R is termed a $\mathfrak{G}$-homogeneous ideal of R if I satisfies the condition: if $x \in I$ and $x = \sum_{s\in\mathfrak{G}} x_s$ with $x_s \in R_s$, then each $x_s \in I$.*

Theorem 2. *Let R be a $\mathfrak{G}$-weak graded ring. Then, R is a $\mathfrak{G}$-weak graded field if and only if $\{0\}$ is a maximal $\mathfrak{G}$-homogeneous ideal of R.*

Proof. First, let R be a $\mathfrak{G}$-weak graded field, and let I be a maximal $\mathfrak{G}$-homogeneous ideal of R. Suppose that $I \neq \{0\}$, then there is at least $0 \neq x \in I$ with $x = \sum_{s\in\mathfrak{G}} x_s$, which implies $0 \neq x_s \in I$ for some $\mathfrak{G}$-homogeneous component x_s of x. Hence, as R is a graded field, $x_s^{-1} \in R$, which implies $x_s^{-1}x_s = 1_R \in I$. Thus, $I = R$. However, by the maximality of I, $I \neq R$, which means that $I = \{0\}$.

On the other hand, let $I = \{0\}$ be the maximal $\mathfrak{G}$-homogeneous ideal of R. Then, for any non-zero $\mathfrak{G}$-homogeneous element $x \in R$, we have

$$< x > = \{rx : \quad r \in R\}.$$

Hence, by the maximality of I, $< x > = R$, which implies $1_R \in < x >$. This means that there exist $r' \in R$ such that $r'x = 1_R \in < x >$, which implies $r' = x^{-1}$. Thus, every $\mathfrak{G}$-homogeneous element ($\neq 0$) has an inverse. Therefore, R is a $\mathfrak{G}$-weak graded field. □

It is known that every field is an integral domain, which is commutative [23]. We give here a counterexample, which shows that there is a field that has a non-homogeneous unit, although it is a non-trivially X-graded field where X is a finite group. This may not coincide with some of what was mentioned in [10,11,24].

Example 4. *Consider the field $F = (\mathbb{R}^2, +, \cdot)$, where $\mathbb{R}^2 = \{(a,b) : \ a,b \in \mathbb{R}\}$ with $(a_1,b_1)(a_2,b_2) = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$, for all $(a_1,b_1),(a_2,b_2) \in \mathbb{R}^2$. Hence, $F = F_0 \oplus F_1$ is a $\mathbb{Z}_2$-graded field with:*

$$F_0 = \{(a,0) : \quad a \in \mathbb{R}\} \quad \textit{and} \quad F_1 = \{(0,b) : \quad b \in \mathbb{R}\},$$

where F_0 and F_1 are additive subgroups. The inclusion property is satisfied as follows:

1. $F_0F_0 = \{(a_1,0)(a_2,0) : \quad a_1,a_2 \in \mathbb{R}\} = \{(a_1a_2,0) : \quad a_1a_2 \in \mathbb{R}\} = F_0 = F_{0+0}$.
2. $F_0F_1 = \{(a,0)(0,b) : \quad a,b \in \mathbb{R}\} = \{(0,ab) : \quad ab \in \mathbb{R}\} = F_1 = F_{0+1}$.
3. $F_1F_0 = \{(0,b)(a,0) : \quad a,b \in \mathbb{R}\} = \{(0,ba) : \quad ba \in \mathbb{R}\} = F_1 = F_{1+0}$.
4. $F_1F_1 = \{(0,b_1)(0,b_2) : b_1,b_2 \in \mathbb{R}\} = \{(-b_1b_2,0) : -b_1b_2 \in \mathbb{R}\} = F_0 = F_{1+1}$.

Thus, F is a strongly $\mathbb{Z}_2$-graded field. Moreover, it can be noted that $(1,1)$ is a unit as

$$(1,1)(\frac{1}{2}, -\frac{1}{2}) = (\frac{1}{2}, -\frac{1}{2})(1,1) = (1,0) = 1_R,$$

but, obviously, $(1,1)$ is not a homogeneous element.

As $\mathfrak{G}$-weak graded vector spaces are no more than modules over fields, all the results mentioned for $\mathfrak{G}$-weak graded modules are applicable here.

Example 5. *Let F be a field that is $\mathfrak{G}$-weak graded. Then, F is a $\mathfrak{G}$-weak graded vector space over itself by putting $V_s = F_s$ for all $s \in \mathfrak{G}$.*

Example 6. *Let X be the symmetric group $S_3 = \{e,(12),(13),(23),(123),(132)\}$, $H = \{e,(12)\}$, and $\mathfrak{G} = \{(12),(13),(23)\}$. Then, the $*$ operation (Table 7), the cocycle f (Table 8), and the actions $\triangleright$ and $\triangleleft$ (Table 9) are calculated as follows.*

Table 7. The binary operation $*$.

$*$	(12)	(13)	(23)
(12)	(12)	(13)	(23)
(13)	(23)	(12)	(13)
(23)	(13)	(23)	(12)

Table 8. The cocycle f.

f	(12)	(13)	(23)
(12)	(12)	(12)	(12)
(13)	(12)	(12)	(12)
(23)	(12)	(12)	(12)

Table 9. The actions $s \triangleright u$ and $s \triangleleft u$.

$s \triangleright u$	e	(12)	$s \triangleleft u$	e	(12)
(12)	e	(12)	(12)	(12)	(12)
(13)	e	(12)	(13)	(13)	(23)
(23)	e	(12)	(23)	(23)	(13)

If we choose F to be the field $\mathbb{R}$ of real numbers, then F is $\mathfrak{G}$-weak graded by putting $F_{e_{\mathfrak{G}}} = F$. Consider the vector space $\mathbb{R}^3 = \{(a,b,c) : a,b,c \in \mathbb{R}\}$, and define

$$V_{(12)} = \{(a, a, a) \quad : \quad a \in \mathbb{R}\},$$

$$V_{(13)} = \{(0, b, 0) \quad : \quad b \in \mathbb{R}\}$$

and

$$V_{(23)} = \{(0, 0, c) \quad : \quad c \in \mathbb{R}\}.$$

Hence, $\mathbb{R}^3 = V_{(12)} \oplus V_{(13)} \oplus V_{(23)}$. Moreover, the property (3) is satisfied for all $s \in \mathfrak{G}$. Therefore, $V = \mathbb{R}^3$ is a strongly $\mathfrak{G}$-weak graded vector space.

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