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A q-Difference Equation and Fourier Series Expansions of q-Lidstone Polynomials

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Abstract: In this paper, we present the q-Lidstone polynomials which are q-Bernoulli polynomials generated by the third Jackson q-Bessel function, based on the Green's function of a certain q-difference equation. Also, we provide the q-Fourier series expansions of these polynomials and derive some results related to them.

Keywords: q-difference equation; q-Lidstone polynomials; Green's function; q-Fourier series

MSC: 05A30; 39A13; 41A58; 30E20; 35J08; 42A16

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1. Introduction

In 1929, Lidstone [1] introduced a generalization of Taylor's series that approximates an entire function $f(z)$ of exponential type less than π in a neighborhood of two points instead of one:

$$f(z) = \sum_{n=0}^{\infty} A_n(z) f^{2n}(1) + \sum_{n=0}^{\infty} A_n(1-z) f^{2n}(0),$$

where the set $\{A_n(z)\}_n$ called Lidstone polynomials. In [2], Whittaker proved that

$$A_n(z) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}(\frac{z+1}{2}),$$

where $B_n(x)$ is the Bernoulli polynomial of degree n, which may be defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Recently, Ismail and Mansour [3] introduced a q analog of the Lidstone expansion theorem where they expand a class of entire functions of q-exponential growth in terms of Jackson q-derivatives of even degree at 0 and 1. See also [4–6] for some results and applications to the q-Lidstone theorem.

In [7], the authors constructed another formula of q-Lidstone expansion theorem by using the symmetric q-difference operator δ_q (see Section 2), that is

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} \widetilde{A}_n(z) - \frac{\delta_q^{2n} f(0)}{\delta_q z^{2n}} \widetilde{B}_n(z) \right], \tag{1}$$

where $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ are the q-Lidstone polynomials defined by the generating functions

$$\begin{aligned} \frac{exp_q(zw) - exp_q(-zw)}{exp_q(w) - exp_q(-w)} &= \sum_{n=0}^{\infty} \widetilde{A}_n(z)\, w^{2n}, \\ \frac{exp_q(zw)exp_q(-w) - exp_q(-zw)exp_q(w)}{exp_q(w) - exp_q(-w)} &= \sum_{n=0}^{\infty} \widetilde{B}_n(z)\, w^{2n}. \end{aligned} \tag{2}$$

Moreover, it turns out that

$$\widetilde{B}_n(z) = \frac{2^{2n+1}}{[2n+1]_q!}\widetilde{B}_{2n+1}(z/2;q), \tag{3}$$

where $\widetilde{B}_n(z;q)$ are q-Bernoulli polynomials generated by

$$\frac{w\, exp_q(zw)\, exp_q(\frac{-w}{2})}{exp_q(\frac{w}{2}) - exp_q(\frac{-w}{2})} = \sum_{n=0}^{\infty} \widetilde{B}_n(z;q)\frac{w^n}{[n]_q!}, \tag{4}$$

and the function $exp_q(.)$ is the q-exponential function which has the series representation

$$exp_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{4}}}{[n]_q!} z^n; \quad z \in \mathbb{C}. \tag{5}$$

In this paper, we assume that q is a positive number less than one and the set A_q^* is defined by

$$A_q^* := \{q^n : n \in \mathbb{N}_0\} \cup \{0\},$$

where $\mathbb{N}_0 := \{0,1,2,\ldots\}$. We present the q-Lidstone polynomials $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ based on the Green's function of a q-boundary value problem

$$\begin{cases} \frac{\delta_q^{2n} f(z)}{\delta_q z^{2n}} = \phi(z), \\ \\ \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} = a_k, \ \frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} = b_k \ (k = 0,1,\ldots,n-1), \end{cases} \tag{6}$$

where f and ϕ are assumed to be continuous functions on A_q^*. Also, we introduce the q-Fourier series expansions of these functions and derive some results related to them. For other recent contributions on this area, one may refer to [8–10].

This article is organized as follows: In the next section, we present some background on q-analysis which we need in our investigations. In Section 3, we establish the existence of a solution for the system (51). In Section 4, we introduce the q-Fourier series expansions of some functions. As an application, in Section 5, we define q-Lidstone polynomials based on the Green's function of the system (51), and we provide the q-Fourier series expansions of these polynomials. Moreover, relying on the obtained q-Fourier series, we derive a close approximation to $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ for large n.

2. Preliminaries

Recall that the q-derivative D_q of the function f is defined by

$$D_q f(z) := \frac{f(z) - f(qz)}{z - qz}, \text{ for } z \neq 0, \tag{7}$$

and the q-derivative at zero is defined to be $f'(0)$ if it exists, see [11]. The q-shifted fractional $(a;q)_n$ of $a \in \mathbb{C}$ is defined by

$$(a;q)_0 := 1 \text{ and } (a;q)_n := \prod_{j=0}^{n}(1 - aq^j), \text{ for } n \in \mathbb{N},$$

and the q-number factorial $[n]_q!$ is defined for $q \neq 1$ by

$$[n]_q! = \prod_{j=0}^{n}[j]_q, \quad [j]_q = \frac{1 - q^j}{1 - q}.$$

Jackson [12] introduced the following integral, as a right inverse of the q-derivative (7), by

$$\int_a^b f(t)\,d_qt := \int_0^b f(t)d_qt - \int_0^a f(t)\,d_qt \quad (a,b \in \mathbb{C}),$$

where

$$\int_0^z f(t)\,d_qt := (1-q)\sum_{n=0}^{\infty} zq^n f(zq^n),$$

provided that the series converges at $z = a$ and $z = b$. We can interchange the order of double q-integral by

$$\int_0^z \int_0^v f(t)\,d_qtd_qv = \int_0^z \int_{qt}^z f(t)\,d_qvd_qt = \int_0^z (z-qt)\,f(t)\,d_qt. \tag{8}$$

The symmetric q-difference operator δ_q which is acting on a function f defined by

$$\frac{\delta_q f(z)}{\delta_q z} := \frac{f(q^{\frac{1}{2}}z) - f(q^{\frac{-1}{2}}z)}{z(q^{\frac{1}{2}} - q^{\frac{-1}{2}})}, \text{ for } z \neq 0. \tag{9}$$

(see [11,13]). From (7) and (9), it follows

$$\frac{\delta_q f(z)}{\delta_q z} := D_q f(\frac{z}{\sqrt{q}}).$$

Therefore, we have

$$\int_0^a \frac{\delta_q f(z)}{\delta_q z}\,d_qz = q^{\frac{1}{2}}\,[f(q^{-\frac{1}{2}}\,a) - f(0)]. \tag{10}$$

A function f defined on A_q^* is called q-regular at zero if it satisfies

$$\lim_{n\to\infty} f(xq^n) = f(0), \quad \text{for all } x \in A_{q,t}^*.$$

The q-integration by parts rule on A_q^* (see [13]) is

$$\int_0^a g(q^{-\frac{1}{2}}t)\frac{\delta_q f(t)}{\delta_q t}\,d_qt = q^{\frac{1}{2}}\,(gf)(q^{-\frac{1}{2}}\,t)\Big|_0^a - \int_0^a f(q^{\frac{1}{2}}t)\,\frac{\delta_q g(t)}{\delta_q t}\,d_qt, \tag{11}$$

where f and g are complex valued q-regular functions at zero.

We will use a q-exponential function $exp_q(.)$ defined in (5) and the q-linear sine and cosine, $S_q(z)$ and $C_q(z)$, which defined by

$$\begin{aligned} S_q(z) &:= \frac{exp_q(iz) - exp_q(-iz)}{2i} = \sum_{n=0}^{\infty}(-1)^n\,\frac{q^{n(n+\frac{1}{2})}}{[2n+1]_q!}z^{2n+1}, \\ C_q(z) &:= \frac{exp_q(iz) + exp_q(-iz)}{2} = \sum_{n=0}^{\infty}(-1)^n\,\frac{q^{n(n-\frac{1}{2})}}{[2n]_q!}z^{2n}. \end{aligned} \tag{12}$$

They can be written in terms of the third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$ [14,15] as follows

$$\begin{aligned} S_q(z) &:= q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}z^{1/2}\,J_{1/2}^{(3)}(q^{-1/4}z;q^2), \\ C_q(z) &:= q^{-3/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}z^{1/2}\,J_{-1/2}^{(3)}(q^{-3/4}z;q^2). \end{aligned} \tag{13}$$

These functions satisfy

$$\frac{\delta_q C_q(wz)}{\delta_q z} = -w\, S_q(wz), \quad \frac{\delta_q S_q(wz)}{\delta_q z} = w\, C_q(wz), \tag{14}$$

see [11,13]. We denote to the derivative of $S_q(z)$ by $S'_q(z)$ and we assume that $\{w_k : k \in \mathbb{N}$ with $w_1 < w_2 < w_3 < \ldots\}$ is the set of positive zeroes of $S_q(z)$.

3. Existence Solutions of q-Differential System

In this section, we construct the solution of the q-differential system (51). Let $C_q^n(A_q^*)$ denote the space of all continues functions with continuous q-derivatives up to order $n-1$ on A_q^* with values in $\mathbb{R}$.

Lemma 1. *Let f, $\phi \in C_q^2(A_q^*)$. Then, the solution of the q-differential equation*

$$\frac{\delta_q^2 f(z)}{\delta_q z^2} - \phi(z) = 0, \tag{15}$$

subject to the boundary conditions $f(0) = f(1) = 0$ is equivalent to the basic Fredholm q-integral equation

$$f(z) = \int_0^1 \widetilde{G}(z, qt)\phi(qt)\, d_q t, \tag{16}$$

where $\widetilde{G}(z,t)$ is the Green's function defined on A_q^* by

$$\widetilde{G}(z,t) := \begin{cases} \sqrt{q}z(t-1), & z < t; \\ \sqrt{q}t\,(z-1), & t < z. \end{cases} \tag{17}$$

Proof. The q-differential Equation (15) can be written as

$$D_q^2 f(z) - \sqrt{q}\phi(qz) = 0 \quad (z \in A_q^*). \tag{18}$$

By taking double q-integral for (18) and using (8), we obtain

$$f(z) = c_0 + c_1 z + \sqrt{q}\int_0^z (z - qt)\phi(qt)\, d_q t, \tag{19}$$

where c_0 and c_1 are arbitrary constant. Using the boundary conditions, we get $c_0 = 0$ and

$$c_1 = -\sqrt{q}\int_0^1 (1 - qt)\phi(qt)\, d_q t.$$

Substituting in (19), we have

$$f(z) = -\sqrt{q}z\int_0^1 (1-qt)\phi(qt)\, d_q t + \sqrt{q}\int_0^z (z-qt)\phi(qt)\, d_q t,$$

and then we obtain (16). □

Remark 1. *By induction on n, one can verify that if $f, \phi \in C_q^{2n}(A_q^*)$, then the function*

$$f(z) = \int_0^1 \widetilde{G}_n(z, qt)\, \phi(qt)\, d_q t, \tag{20}$$

is the solution of the q-boundary value problem

$$\begin{cases} \frac{\delta_q^{2n} f(z)}{\delta_q z^{2n}} = \phi(z), \\ \\ \frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} = \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} = 0 \ (k = 0, 1, \ldots, n-1), \end{cases} \tag{21}$$

where $\widetilde{G}_1(z,t)$ *is the Green's function defined as in* (17) *and*

$$\begin{aligned} \widetilde{G}_n(z,qt) &= \int_0^1 \widetilde{G}(z,qw)\, \widetilde{G}_{n-1}(qw,qt)\, d_qw \\ &= \int_0^1 \widetilde{G}_{n-1}(z,qw)\, \widetilde{G}(qw,qt)\, d_qw \quad (n = 2, 3, \ldots). \end{aligned} \tag{22}$$

Theorem 1. *If* $f(z)$ *and* $\phi(z)$ *are functions of class* $C_q^{2n}(A_q^*)$*, then any solution of the system*

$$\begin{cases} \frac{\delta_q^{2n} f(z)}{\delta_q z^{2n}} = \phi(z), \\ \\ \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} = a_k, \ \frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} = b_k \ (k = 0, 1, \ldots, n-1) \end{cases} \tag{23}$$

is given by

$$\begin{aligned} f(z) &= a_0(z-1) + \sum_{k=1}^{n-1} a_k \int_0^1 (qt-1)\, \widetilde{G}_k(z,qt)\, d_qt + b_0 z \\ &+ \sum_{k=1}^{n-1} b_k \int_0^1 (qt)\, \widetilde{G}_k(z,qt)\, d_qt + \int_0^1 \widetilde{G}_n(z,qt)\phi(qt)\, d_qt, \end{aligned} \tag{24}$$

where the functions $\widetilde{G}_n(z,qt)$ $(n \in \mathbb{N})$ *defined as in* (17) *and* (22).

Proof. From (17), (22) and Equation (23) we get

$$\begin{aligned} R_n(z) &= \int_0^1 \widetilde{G}_n(z,qt)\phi(qt)\, d_qt \\ &= \int_0^1 \widetilde{G}_{n-1}(z,qw) \int_0^1 \widetilde{G}(qw,qt)\, \frac{\delta_q^{2n} f(qt)}{\delta_q z^{2n}}\, d_qt\, d_qw \\ &= \int_0^1 \widetilde{G}_{n-1}(z,qw) \Big[\sqrt{q}(qw-1) \int_0^{qw} (qt)\, \frac{\delta_q^{2n} f(qt)}{\delta_q z^{2n}}\, d_qt \\ &+ q\sqrt{q}w \int_{qw}^1 (qt-1)\, \frac{\delta_q^{2n} f(qt)}{\delta_q z^{2n}}\, d_qt \Big]\, d_qw. \end{aligned} \tag{25}$$

Using the rule (11), after some simplifications, we obtain

$$\begin{aligned} R_n(z) &= \frac{\delta_q^{2n-2} f(0)}{\delta_q z^{2n-2}} \int_0^1 (qw-1)\, \widetilde{G}_{n-1}(z,qw)\, d_qw - \\ &\frac{\delta_q^{2n-2} f(1)}{\delta_q z^{2n-2}} \int_0^1 (qw)\, \widetilde{G}_{n-1}(z,qw)\, d_qw + \int_0^1 \widetilde{G}_{n-1}(z,qw)\, \frac{\delta_q^{2n-2} f(qw)}{\delta_q z^{2n-2}}\, d_qw. \end{aligned} \tag{26}$$

Repeating the q-integration by parts on the last q-integral of Equation (26) $(n-1)$ times, we get

$$\begin{aligned} R_n(z) = \sum_{k=1}^{n-1} \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} \int_0^1 (qw-1)\, \widetilde{G}_k(z,qt)\, d_q w - \\ \sum_{k=1}^{n-1} \frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} \int_0^1 (qw)\, \widetilde{G}_k(z,qw)\, d_q w + \int_0^1 \widetilde{G}(z,qw) \frac{\delta_q^2 f(qw)}{\delta_q z^2}\, d_q w. \end{aligned} \tag{27}$$

Computing the last integral of (27), we get

$$\begin{aligned} &\int_0^1 \widetilde{G}(z,qw) \frac{\delta_q^2 f(qw)}{\delta_q z^2}\, d_q w \\ =&\sqrt{q}(1-z) \int_0^z (-qw) \frac{\delta_q^2 f(qw)}{\delta_q z^2}\, d_q w - \sqrt{q} z \int_z^1 (1-qw) \frac{\delta_q^2 f(qw)}{\delta_q z^2}\, d_q w \\ =&a_0(1-z) + b_0 z - f(z). \end{aligned} \tag{28}$$

Now, by substituting (28) in (27), we obtain the required result. □

4. Certain q-Fourier Expansions

In this section, we consider the q-trigonometric functions $C_q(z)$ and $S_q(z)$ which are defined in (12). Our aim is to obtain the q-Fourier expansions of certain q-integral transforms involving the Green's functions $\widetilde{G}_n(z,qt)$ defined in Section 3.

Recall that the q-Fourier series expansion for $f(x) = 1$ and $g(x) = x$ are given [13,16] by

$$\begin{aligned} 1 =& 2 \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2} w_k)}{w_k C_q(q^{1/2} w_k) S_q'(w_k)} S_q(q w_k x), \quad x \in A_q^*, \\ x =& -\frac{1}{q} \sum_{k=1}^{\infty} \frac{2}{w_k S_q'(w_k)} S_q(q w_k x), \end{aligned} \tag{29}$$

where $\{w_k : k \in \mathbb{N}\}$ is the set of positive zeroes of $S_q(z)$.

Lemma 2. *Let $z \in A_q^*$. Then*

$$\int_0^1 \widetilde{G}(z,qt) S_q(q w_k t)\, d_q t = -\frac{1}{w_k^2} S_q(w_k z).$$

Proof. From (17), we get

$$\begin{aligned} \int_0^1 \widetilde{G}(z,qt) S_q(q w_k t)\, d_q t = \sqrt{q}(1-z) \int_0^z (-qt)\, S_q(q w_k t)\, d_q t \\ -\sqrt{q}\, z \int_z^1 (1-qt)\, S_q(q w_k t)\, d_q t. \end{aligned} \tag{30}$$

Using q-integration by parts (11), we obtain

$$\int_0^z (-qt)\, S_q(q w_k t)\, d_q t = \frac{z}{\sqrt{q}\, w_k} C_q\!\left(\frac{w_k z}{\sqrt{q}}\right) - \frac{1}{\sqrt{q}\, w_k^2} S_q(w_k z), \tag{31}$$

$$\int_z^1 (1-qt)\, S_q(q w_k t)\, d_q t = \frac{(1-z)}{\sqrt{q}\, w_k} C_q\!\left(\frac{w_k z}{\sqrt{q}}\right) + \frac{1}{\sqrt{q}\, w_k^2} S_q(w_k z). \tag{32}$$

Substituting from (31) and (32) into (30), we have the required result. □

Lemma 3. *For $z \in A_q^*$, the following q-Fourier series expansion holds:*

$$\int_0^1 \widetilde{G}(z,qt)\, d_qt = -\sum_{k=1}^{\infty} \frac{L_k}{w_k^2}\, S_q(w_k z), \tag{33}$$

where

$$L_k := \frac{2 - 2C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}.$$

Proof. According to (29), we have

$$1 = 2\sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}\, S_q(qw_k t), \quad t \in A_q^*. \tag{34}$$

Multiplying (34) by $\widetilde{G}(z,qt)$, and integrating with respect to t from zero to unity, we get

$$\begin{gathered} \int_0^1 \widetilde{G}(z,qt)\, d_qt = \\ 2\sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} \int_0^1 \widetilde{G}(z,qt) S_q(w_k qt)\, d_qt. \end{gathered} \tag{35}$$

By setting $L_k := \dfrac{2 - 2C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}$ and using Lemma 2, we obtain the result. □

Theorem 2. *For $z \in A_q^*$, the following q-Fourier series expansion holds:*

$$\int_0^1 \widetilde{G}_n(z,qt)\, d_qt = (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n}}\, S_q(w_k z). \tag{36}$$

Proof. We prove the result by mathematical induction with respect to n. We first observe that for $n = 1$, the Formula (36) reduces to the formula in Lemma 3; that is, Equation (36) is true for $n = 1$.

Next, assume that (36) is true for some $n \geq 2$. Then

$$\begin{aligned} \int_0^1 \widetilde{G}_{n+1}(z,qt)\, d_qt &= \int_0^1 \int_0^1 \widetilde{G}(z,qy) \widetilde{G}_n(qy,qt)\, d_qy\, d_qt \\ &= \int_0^1 \widetilde{G}(z,qy) \Big[\int_0^1 \widetilde{G}_n(qy,qt)\, d_qt \Big]\, d_qy \\ &= (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n}} \int_0^1 \widetilde{G}(z,qy) S_q(w_k qy)\, d_qy \\ &= (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n}} \Big[\frac{-1}{w_k^2} S_q(w_k z) \Big] \\ &= (-1)^{n+1} \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2(n+1)}}\, S_q(w_k z). \end{aligned}$$

□

Lemma 4. *For $z \in A_q^*$, the following q-Fourier series expansion holds:*

$$\int_0^1 (qt)\, \widetilde{G}(z,qt)\, d_qt = 2\sum_{k=1}^{\infty} \frac{1}{w_k^3 S_q'(w_k)}\, S_q(w_k z), \quad k \in \mathbb{N}. \tag{37}$$

Proof. Consider the function $g(t) = t$. From (29), we have

$$t = -\frac{1}{q}\sum_{k=1}^{\infty}\frac{2}{w_k S_q'(w_k)}\, S_q(qw_k t), \quad 0 < t < 1. \tag{38}$$

Hence, the proof can be performed by using (38) similar to the proof of Lemma 3. □

Theorem 3. *For $z \in A_q^*$, the following q-Fourier series expansion holds:*

$$\int_0^1 (qt)\, \widetilde{G}_n(z, qt)\, d_q t = (-1)^n \sum_{k=1}^{\infty}\frac{2}{w_k^{2n+1}\, S_q'(w_k)}\, S_q(w_k z). \tag{39}$$

Proof. The proof can be performed by induction similar to the proof of Theorem 2. So, we omit it. □

5. Fourier Series Expansions of the q-Lidstone Polynomials

The Fourier expansion of special polynomials has been studied by some mathematicians; see [17–20]. In this section, we consider the q-Lidstone polynomials $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ defined in (2). We define these polynomials by using the Green's functions $\widetilde{G}_n(z, qt)$ defined in (17) and (22) and then, we introduce the q-Fourier Series Expansions for them.
We begin with the following result from [7]:

Lemma 5. *For $n \in \mathbb{N}$, the q-polynomials $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ satisfy the q-difference equations*

$$\frac{\delta_q^2\, \widetilde{A}_n(z)}{\delta_q z^2} = \widetilde{A}_{n-1}(z) \text{ and } \frac{\delta_q^2\, \widetilde{B}_n(z)}{\delta_q z^2} = \widetilde{B}_{n-1}(z),$$

with the boundary conditions $\widetilde{A}_n(0) = \widetilde{A}_n(1) = 0 = \widetilde{B}_n(0) = \widetilde{B}_n(1) = 0$. Moreover,

$$\widetilde{A}_0(z) = z, \quad \widetilde{B}_0(z) = z - 1.$$

We have the following:

Proposition 1. *The q-Lidstone polynomials $\widetilde{A}_n$ and $\widetilde{B}_n$ can be expressed as $\widetilde{A}_0(z) = z$, $\widetilde{B}_0(z) = z - 1$, and for $n \in \mathbb{N}$*

$$\widetilde{A}_n(z) = q\int_0^1 t\, \widetilde{G}_n(z, qt)\, d_q t, \tag{40}$$

$$\widetilde{B}_n(z) = \int_0^1 (qt - 1)\, \widetilde{G}_n(z, qt)\, d_q t, \tag{41}$$

where

$$\widetilde{G}(z,t) := \widetilde{G}_1(z,t) = \begin{cases} \sqrt{q}z(t-1), & 0 \le z < t \le 1; \\ \sqrt{q}t\,(z-1), & 0 \le t < z \le 1. \end{cases}$$
$$\widetilde{G}_n(z, qt) = \int_0^1 \widetilde{G}(z, qw)\, \widetilde{G}_{n-1}(qw, qt)\, d_q w \quad (n = 2, 3, \ldots). \tag{42}$$

Proof. We use the induction on n. By Lemma 5, we have

$$\begin{cases} \frac{\delta_q^2\, \widetilde{A}_n(z)}{\delta_q z^2} = \widetilde{A}_{n-1}(z) \quad (n \in \mathbb{N}), \\ \widetilde{A}_n(0) = \widetilde{A}_n(1) = 0. \end{cases} \tag{43}$$

So, if $n = 1$ we get the q-boundary value problem

$$\begin{cases} \frac{\delta_q^2 \widetilde{A}_1(z)}{\delta_q z^2} = z \quad (z \in A_q^*), \\ \\ \widetilde{A}_1(0) = \widetilde{A}_1(1) = 0. \end{cases} \tag{44}$$

According to Lemma 1, we have the result.

Next, assume that (40) is true for $n \geq 1$. According to Remark (1), the solution $\widetilde{A}_{n+1}(z)$ of the q-boundary value problem

$$\begin{cases} \frac{\delta_q^2 \widetilde{A}_{n+1}(z)}{\delta_q z^2} = \widetilde{A}_n(z), \\ \\ \widetilde{A}_{n+1}(0) = \widetilde{A}_{n+1}(1) = 0, \end{cases} \tag{45}$$

is given by

$$\begin{aligned} \widetilde{A}_{n+1}(z) &= \int_0^1 \widetilde{G}(z, qw)\widetilde{A}_n(qw)\, d_q w \\ &= \int_0^1 \widetilde{G}(z, qw) \Big[\int_0^1 qt\, \widetilde{G}_n(qw, qt)\, d_q t \Big]\, d_q w \\ &= \int_0^1 \Big[\int_0^1 qt\, \widetilde{G}(z, qw)\widetilde{G}_n(qw, qt)\, d_q w \Big]\, d_q t \\ &= \int_0^1 qt\, \widetilde{G}_{n+1}(z, qt)\, d_q t. \end{aligned}$$

Similarly, we can prove Equation (41). Finally, by induction on n $(n \geq 2)$ again, it is easy to see that

$$\widetilde{G}_n(z, qt) = \int_0^1 \widetilde{G}_{n-1}(z, qw)\widetilde{G}(qw, qt)\, d_q w.$$

□

The following result offers the explicit representation of the interpolating q-Lidstone polynomials and the associated error function $R_n(z)$.

Theorem 4. *Let $0 < q < 1$ and $f \in C_q^2(A_q^*)$. Then*

$$f(z) = \sum_{k=0}^{n-1} \Big[\frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} \widetilde{A}_k(z) + \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} \widetilde{B}_k(z) \Big] + R_n(z), \tag{46}$$

where

$$R_n(z) = \int_0^1 \widetilde{G}_n(z, qt)\, \frac{\delta_q^{2n} f(qz)}{\delta_q z^{2n}}\, d_q t.$$

Proof. The proof follows immediately from Theorem 1 and Proposition 1, if we replace a_k, b_k and $\phi(z)$ in Equation (24) by their values in terms of $f(z)$ as given by the system (23). □

Proposition 2. *For $z \in A_q^*$ and $n \in \mathbb{N}$, the Fourier series for q-Lidstone polynomials $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ are given by*

$$\widetilde{A}_n(z) = (-1)^n \sum_{k=1}^{\infty} \frac{2}{w_k^{2n+1} S_q'(w_k)} S_q(w_k z), \tag{47}$$

$$\widetilde{B}_n(z) = (-1)^n \sum_{k=1}^{\infty} \frac{2}{w_k^{2n+1} S_q'(w_k) C_q(q^{1/2} w_k)} S_q(w_k z), \tag{48}$$

where $\{w_k : k \in \mathbb{N}$ with $w_1 < w_2 < w_3 < \dots\}$ is the set of positive zeroes of $S_q(z)$.

Proof. By using Equation (40) and Theorem 3 we get (47). Similarly, Equation (48) follows immediately from (41), (36) and (37). □

We end this section by determining the asymptotic behavior of $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ for large n.

Proposition 3. *Let $z \in A_q^*$. Then, there exist some constants K_q and L_q such that*

$$\left| (-1)^n \widetilde{A}_n(z) - \frac{2\, S_q(w_1 z)}{w_1^{2n+1}\, S_q'(w_1)} \right| < \frac{K_q}{w_1^{2n}}, \tag{49}$$

$$\left| (-1)^n B_n(z) - \frac{2\, S_q(w_1 z)}{w_1^{2n+1}\, S_q'(w_1) C_q(\sqrt{q} w_1)} \right| < \frac{L_q}{w_1^{2n}}, \tag{50}$$

where w_1 is the smallest positive zero of $S_q(z)$.

Proof. From Equation (47), we get

$$\left| (-1)^n \widetilde{A}_n(z) - \frac{2}{w_1^{2n+1}\, S_q'(w_1)} S_q(w_1 z) \right| = \left| \sum_{k=2}^{\infty} \frac{2}{w_k^{2n+1} S_q'(w_k)} S_q(w_k z) \right|.$$

Since the function $S_q(.)$ is bounded on A_q^*, there exists a constant $M > 0$ such that

$$\begin{aligned} &\left| \sum_{k=2}^{\infty} \frac{2}{w_k^{2n+1} S_q'(w_k)} S_q(w_k z) \right| \\ &< \frac{M}{w_2^{2n+1} S_q'(w_2)} \Big[1 + \left(\frac{w_2}{w_3}\right)^{2n+1} \frac{S_q'(w_2)}{S_q'(w_3)} + \left(\frac{w_2}{w_4}\right)^{2n+1} \frac{S_q'(w_2)}{S_q'(w_4)} + \dots\dots \Big]. \end{aligned}$$

Note that $w_1 < w_2 < \dots$, this implies the series in brackets tends to unity when $n \to \infty$. Set $K_q = \frac{M}{w_1 S_q'(w_2)}$, we get (49). Inequality (50) can be proved in the same manner by using Equation (48). □

6. Conclusions and Future Work

In this paper, we have introduced some definitions of the q-Lidstone polynomials which are q-Bernoulli polynomials generated by the third Jackson q-Bessel function, based on the Green's function of the q-difference equation

$$\begin{cases} \frac{\delta_q^{2n} f(z)}{\delta_q z^{2n}} = \phi(z), \\ \\ \frac{\delta_q^{2k} f(0)}{\delta_q z^{2k}} = a_k, \ \frac{\delta_q^{2k} f(1)}{\delta_q z^{2k}} = b_k \ (k = 0, 1, \dots, n-1). \end{cases} \tag{51}$$

New results are obtained; particularly the q-Fourier series expansions of these functions.

Another study to give a characterization of those functions on the plane given by absolutely convergent of q-Lidstone series expansion (1), using the results in Section 5, is in progress.

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