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Editorial

Special Issue Editorial "Symmetry in the Mathematical Inequalities"

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Introduction

The theory of inequalities represents a long-standing topic in many mathematical areas and remains an attractive research domain with many applications. Inequalities Theory represents an old topic of many mathematical areas which still remains an attractive research domain with many applications. The study of convex functions occupied and still occupies a central role in Inequalities Theory because the convex functions develop a series of inequalities.

The research results presented here are concerned with the improvement in classical inequalities resulting from convex functions and highlighting their applications.

Related to probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, known as *Jensen's inequality*, underlies many important inequalities.

Another important result related to convex function is the *Hermite–Hadamard inequality*, due to Hermite and Hadamard, which asserts that for every continuous convex function $f : [a, b] \to \mathbb{R}$ the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Related to the Hermite–Hadamard inequality, many mathematicians have worked with great interest to generalise, refine, and extend it for different classes of functions such as quasi-convex functions, log-convex, r-convex functions, etc., and apply it for special means (logarithmic mean, Stolarsky mean, etc.).

Ujević in [1] obtained sharp inequalities for Simpson and Ostrowski types. Liu et al., in [2], used the MT-convexity-class-derived Ostrowski fractional inequalities. Kaijser et al., in [3], established Hardy-type inequalities via convexity. Rashid et al., in [4], using generalized k-fractional integrals, found Grüss inequalities.

There are many types of inequalities for functionals and inequalities for invertible positive operators. Among these, we found the Jensen functional under superquadraticity conditions and the Jensen functional related to a strongly convex function. Some applications of these functionals are related characterizations of generalized entropies. Generalized entropies have been studied by many researchers. Rényi and Tsallis entropies are well known as one-parameter generalizations of Shannon's entropy, being intensively studied not only in the field of classical statistical physics but also in the field of quantum physics.

Due to the nature of convexity theory, there exists a strong relationship between convexity and symmetry. When working on either of the concepts, it can be applied to the other one as well. Integral inequalities concerned with convexity have a lot of applications in various fields of mathematics in which symmetry has a great part to play.

Recently, fractional calculus has been the center of attraction for researchers in mathematical sciences because of its basic definitions, properties, and applications in tackling real-life problems.

Quantum information theory, an interdisciplinary field that includes computer science, information theory, philosophy, cryptography, and symmetry, has various applications for quantum calculus. The inequalities have a strong association with convex and symmetric convex functions. From the past to the present, various works have been dedicated to Simpson's inequality for differentiable convex functions. Simpson-type inequalities for twice-differentiable functions have been the subject of some research.

This Special Issue brings together original research papers in all areas of mathematics that are concerned with inequalities or their role. The research results presented in this Special Issue are related to the improvement of classical inequalities and highlight their applications, promoting an exchange of ideas between mathematicians from many parts of the world dedicated to the theory of inequalities.

For example, the study of convex functions has occupied a central role in the theory of inequalities because such functions develop a series of inequalities. There is a strong correlation between the concepts of convexity and symmetry. In number theory, a number of inequalities characterize arithmetic functions. Other important types of inequalities are those related to invertible positive operators that have applications in operator equations, network theory, and quantum information theory (inequalities for generalized entropies).

The following manuscripts were selected for publication. The articles were prepared by scientists working in leading universities and research centers in Albania, China, Ecuador, Greece, India, Iraq, Italy, Japan, Pakistan, Romania, Serbia, Saudi Arabia, South Africa, Taiwan, Thailand, Tunisia, Turkey, and the USA.

S. K. Sahoo et al., in the paper "New Ostrowski-Type Fractional Integral Inequalities via Generalized Exponential-Type Convex Functions and Applications" [5], presented some fractional integral inequalities of the Ostrowski type for a new class of convex mapping. Specifically, n-polynomials that are exponentially s-convex via a fractional operator are established. Additionally, they showed a new Hermite–Hadamard fractional integral inequality. Some special cases of the results are discussed as well. Finally, in applications, some new limits for special means of positive real numbers and midpoint formulae are given. These new outcomes yield a few generalizations of the earlier outcomes already published in the literature.

X. Huang et al., in the paper "A More Accurate Half-Discrete Hilbert-Type Inequality Involving One upper Limit Function and One Partial Sum" [6], constructed proper weight coefficients by virtue of the symmetry principle and used them to establish a more accurate half-discrete, Hilbert-type inequality involving one upper limit function and one partial sum. The authors proved the new inequality with the help of the Euler–Maclaurin summation formula and Abel's partial summation formula. Finally, it is illustrated how the obtained results can generate some new half-discrete Hilbert-type inequalities.

T. Sitthiwirattham et al., in the paper "On Some New Trapezoidal Type Inequalities for Twice (p, q) Differentiable Convex Functions in Post-Quantum Calculus" [7], studied a (p, q)-integral identity involving the second (p, q)-derivative and then used this result to prove some new trapezoidal-type inequalities for twice (p, q)-differentiable convex functions. It is also shown that the newly established results are the refinements of some existing results in the field of integral inequalities.

T. Sitthiwirattham et al., in the paper "On Some New Fractional Ostrowski- and Trapezoid-Type Inequalities for Functions of Bounded Variations with Two Variables" [8], proved three identities for functions of bounded variations. Then, by using these equalities, several trapezoid- and Ostrowski-type inequalities were obtained via generalized fractional integrals for the functions of bounded variations with two variables. Moreover, presented some results were presented for Riemann–Liouville fractional integrals by the special choice of the main results. Finally, the connections between their results and those in earlier works were investigated.

I. B. Sial et al., in the paper "On Some New Inequalities of Hermite–Hadamard Midpoint and Trapezoid Type for Preinvex Functions in p,q-Calculus" [9], established some new Hermite–Hadamard-type inequalities for preinvex functions and left–right estimates of newly established inequalities for (p, q)-differentiable pre-invex functions in the context of (p, q)-calculus. The authors also show that the results established in this paper are generalizations of comparable results in the literature of integral inequalities. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Y. Li and P. Zeng, in the paper "Continuous Dependence on the Heat Source of 2D Large-Scale Primitive Equations in Oceanic Dynamics" [10], studied the initial boundary value problem for the two-dimensional primitive equations of large-scale oceanic dynamics. These models are often used to predict the weather and climate change. Using the differential inequality technique, rigorous a priori bounds of the solutions and the continuous dependence on the heat source are established. The application of symmetry in mathematical inequalities are shown in practice.

S. Simić and B. Bin-Mohsin, in the paper "Global Bounds for the Generalized Jensen Functional with Applications" [11], found sharp global bounds for the generalized Jensen functional $J_n(g, h; p, x)$. In particular, exact bounds are determined for the generalized power mean in terms from the class of the Stolarsky means. As a consequence, they obtained the best possible global converses of quotients and differences of the generalized arithmetic, geometric, and harmonic means.

M. A. Ali et al., in the paper "Some New Simpson's–Formula–Type Inequalities for Twice-Differentiable Convex Functions via Generalized Fractional Operators" [12], established a new generalized fractional integral identity involving twice-differentiable functions. The authors used this result to prove some new Simpson's-formula-type inequalities for twice-differentiable convex functions. Furthermore, a few special cases of newly established inequalities were examined, and several new and old Simpson's-formula-type inequalities were obtained. These types of analytic inequalities, as well as the methodologies for solving them, have applications in a wide range of fields where symmetry is crucial.

S. Luo et al., in the paper "Phragmén-Lindelöf Alternative Results for a Class of Thermoelastic Plate" [13], studied the spatial properties of solutions for a class of thermoelastic plates with biharmonic operators. The energy method was used. A differential inequality in which the energy expression was controlled by a second-order differential inequality was deduced. The Phragmén–Lindelöf alternative results of the solutions were obtained by solving the inequality. These results show that the Saint-Venant principle is also valid for the hyperbolic–hyperbolic coupling equations. Their results can been seen as a version of symmetry in inequality for studying the Phragmén–Lindelöf alternative results.

T. Zhang et al., in the paper "Schur-Convexity for Elementary Symmetric Composite Functions and Their Inverse Problems and Applications" [14], investigated the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity for the elementary symmetric composite function and its dual form. The inverse problems are also considered. New inequalities on special means are established by using the theory of majorization.

S. Furuichi and N. Minculete, in the paper "Bounds for the Differences between Arithmetic and Geometric Means and Their Applications to Inequalities" [15], provide some bounds for the differences between the weighted arithmetic and geometric means, using known inequalities. It improved the results given by Furuichi–Ghaemi–Gharakhanlu and Sababheh–Choi. The authors also found some bounds on entropies, applying the results in a different approach. Finally, explored certain convex or concave functions are explored, which are symmetric functions on the axis $t = 1/2$.

M. J. Vivas-Cortez et al., in the paper "On Some New Simpson's Formula Type Inequalities for Convex Functions in Post–Quantum Calculus" [16], proved a new (p, q)-integral identity involving a (p, q)-derivative and (p, q)-integral. The newly established identity is then used to show some new Simpson's formula type inequalities for (p, q)-differentiable convex functions. Finally, the newly discovered results are shown to be refinements of comparable results in the literature. Analytic inequalities of this type, as well as the techniques used to solve them, have applications in a variety of fields where symmetry is important.

X. Chen et al., in the paper "Spatial Decay Bounds for the Brinkman Fluid Equations in Double-Diffusive Convection" [17], studied the Brinkman equations pipe flow, which includes the salinity and the temperature. Assuming that the fluid satisfies nonlinear boundary conditions at the finite end of the cylinder, using the symmetry of differential inequalities and the energy analysis methods, the exponential decay estimates for homogeneous Brinkman equations are established.

J. Reunsumrit et al., in the paper "On Generalization of Different Integral Inequalities for Harmonically Convex Functions" [18], proved a parameterized integral identity involving differentiable functions. Then, for differentiable harmonically convex functions, they use this result to establish some new inequalities of a midpoint type, trapezoidal type, and Simpson type. Analytic inequalities of this type, as well as the approaches for solving them, have applications in a variety of domains where symmetry is important. Finally, several particular cases of recently discovered results are discussed, as well as applications to the special means of real numbers.

A. Kashuri et al., in the paper "New Generalized Class of Convex Functions and Some Related Integral Inequalities" [19], studied the new generic class of functions called the (n, m)-generalized convex and studied its basic algebraic properties. The Hermite–Hadamard inequality for the (n, m)-generalized convex function, for the products of two functions and of this type, were proven. Moreover, this class of functions was applied to several known identities; midpoint-type inequalities of Ostrowski and Simpson were derived. Our results are extensions of many previous contributions related to integral inequalities via different convexities.

This volume will be of interest to mathematicians specializing in inequalities theory and beyond. Many of the results presented here can be very useful in demonstrating new results.

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References

1. Ujević, N. Sharp inequalities of Simpson type and Ostrowski type. *Comput. Math. Appl.* **2004**, *48*, 145–151. [CrossRef]
2. Liu, W.; Wen, W.; Park, J. Ostrowski type fractional integral inequalities for MT-convex functions. *Math. Notes* **2015**, *16*, 249–256. [CrossRef]
3. Kaijser, S.; Nikolova, L.; Persson, L.E.; Wedestig, A. Hardy type inequalities via convexity. *Math. Inequal. Appl.* **2005**, *8*, 403–417. [CrossRef]
4. Rashid, S.; Jarad, F.; Noor, M.A.; Noor, K.I.; Baleanu, D.; Liu, J.B. On Grüss inequalities within generalized k–fractional integrals. *Adv. Differ. Equ.* **2020**, *2020*, 203. [CrossRef]
5. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Nasir, J.; Aydi, H.; Mukheimer, A. New Ostrowski–Type Fractional Integral Inequalities via Generalized Exponential–Type Convex Functions and Applications. *Symmetry* **2021**, *13*, 1429. [CrossRef]
6. Huang, X.; Wu, S.; Yang, B. A More Accurate Half–Discrete Hilbert–Type Inequality Involving One upper Limit Function and One Partial Sum. *Symmetry* **2021**, *13*, 1548. [CrossRef]
7. Sitthiwirattham, T.; Murtaza, G.; Ali, M.A.; Ntouyas, S.K.; Adeel, M.; Soontharanon, J. On Some New Trapezoidal Type Inequalities for Twice (p, q) Differentiable Convex Functions in Post–Quantum Calculus. *Symmetry* **2021**, *13*, 1605. [CrossRef]
8. Sitthiwirattham, T.; Budak, H.; Kara, H.; Ali, M.A.; Reunsumrit, J. On Some New Fractional Ostrowski– and Trapezoid–Type Inequalities for Functions of Bounded Variations with Two Variables. *Symmetry* **2021**, *13*, 1724. [CrossRef]
9. Sial, I.B.; Ali, M.A.; Murtaza, G.; Ntouyas, S.K.; Soontharanon, J.; Sitthiwirattham, T. On Some New Inequalities of Hermite–Hadamard Midpoint and Trapezoid Type for Preinvex Functions in p, q–Calculus. *Symmetry* **2021**, *13*, 1864. [CrossRef]
10. Li, Y.; Zeng, P. Continuous Dependence on the Heat Source of 2D Large–Scale Primitive Equations in Oceanic Dynamics. *Symmetry* **2021**, *13*, 1961. [CrossRef]
11. Simić, S.; Bin-Mohsin, B. Global Bounds for the Generalized Jensen Functional with Applications. *Symmetry* **2021**, *13*, 2105. [CrossRef]

12. Ali, M.A.; Kara, H.; Tariboon, J.; Asawasamrit, S.; Budak, H.; Hezenci, F. Some New Simpson's–Formula–Type Inequalities for Twice–Differentiable Convex Functions via Generalized Fractional Operators. *Symmetry* **2021**, *13*, 2249. [CrossRef]
13. Luo, S.; Shi, J.; Ouyang, B. Phragmén-Lindelöf Alternative Results for a Class of Thermoelastic Plate. *Symmetry* **2021**, *13*, 2256. [CrossRef]
14. Zhang, T.; Chen, A.; Shi, H.; Saheya, B.; Xi, B. Schur–Convexity for Elementary Symmetric Composite Functions and Their Inverse Problems and Applications. *Symmetry* **2021**, *13*, 2351. [CrossRef]
15. Furuichi, S.; Minculete, N. Bounds for the Differences between Arithmetic and Geometric Means and Their Applications to Inequalities. *Symmetry* **2021**, *13*, 2398. [CrossRef]
16. Vivas-Cortez, M.J.; Ali, M.A.; Qaisar, S.; Sial, I.B.; Jansem, S.; Mateen, A. On Some New Simpson's Formula Type Inequalities for Convex Functions in Post–Quantum Calculus. *Symmetry* **2021**, *13*, 2419. [CrossRef]
17. Chen, X.; Li, Y.; Li, D. Spatial Decay Bounds for the Brinkman Fluid Equations in Double–Diffusive Convection. *Symmetry* **2022**, *14*, 98. [CrossRef]
18. Reunsumrit, J.; Vivas-Cortez, M.J.; Ali, M.A.; Sitthiwirattham, T. On Generalization of Different Integral Inequalities for Harmonically Convex Functions. *Symmetry* **2022**, *14*, 302. [CrossRef]
19. Kashuri, A.; Agarwal, R.P.; Mohammed, P.O.; Nonlaopon, K.; Abualnaja, K.M.; Hamed, Y.S. New Generalized Class of Convex Functions and Some Related Integral Inequalities. *Symmetry* **2022**, *14*, 722. [CrossRef]