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The Unique Solution for Sequential Fractional Differential Equations with Integral Multi-Point and Anti-Periodic Type Boundary Conditions [†]

Zhaocai Hao * and Beibei Chen

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China; beibeichen@qfnu.edu.cn

* Correspondence: zchao@qfnu.edu.cn

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Abstract: In this paper, we obtain the existence of the unique solution of anti-periodic type (anti-symmetry) integral multi-point boundary conditions for sequential fractional differential equations. We apply the Banach contraction mapping principle to get the desired results. Our results specialize and extend some existing results.

Keywords: sequential fractional differential equations; anti-periodic; multi-point; integral; uniqueness

MSC: 39A12; 26A33



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1. Introduction

Fractional calculus has played a key role in improving the mathematical modeling of several phenomena occurring in engineering and scientific disciplines, such as blood flow problems, control theory, aerodynamics, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, polymer rheology, regular variation in thermodynamics, etc. For more details and explanation, see, for instance [1,2]. The literature on these topics are very enriched and contains a variety of important results ranging from the existence theory to the methods of solution for such problems [3–27]. For example, [22,25] studied fractional differential equations (We call it FDEs for short) with Riemann–Liouville fractional derivatives. Influenced by the importance of differential inclusions, researchers in [3,4,9] turned to the study of fractional differential inclusions. The authors in [10] studied anti-periodic boundary value problems (We call it BVPs for short) of FDEs with nonlinear term depending on lower order fractional derivatives. FDEs with semipositone BVPs were studied in [24]. Authors in [5] investigated FDEs with Hadamard fractional derivatives. Some results on Caputo–Hadamard FDEs can be found in [16,21]. Refs. [6–8,13,16,17,19–21,26] analyzed FDEs system.

Recently, Alsaedi et al. [18] obtained the existence of solutions of the following FDEs with the nonlinear anti-periodic type multi-point boundary value conditions:

$$\begin{cases} ({}^C D^q + k {}^C D^{q-1})u(t) = f(t, u(t)), & 2 < q \leq 3, \quad t \in (0, T), \\ \alpha_1 u(0) + \sum_{i=1}^m a_i u(\eta_i) + \gamma_1 u(T) = \beta_1, \\ \alpha_2 u'(0) + \sum_{i=1}^m b_i u'(\eta_i) + \gamma_2 u'(T) = \beta_2, \\ \alpha_3 u''(0) + \sum_{i=1}^m c_i u''(\eta_i) + \gamma_3 u''(T) = \beta_3, \end{cases} \quad (1)$$

where ${}^C D^q$ means the Caputo fractional derivative of order $q, \alpha_j, \beta_j, \gamma_j \in R (j = 1, 2, 3), a_i, b_i, c_i \in R (i = 1, 2, 3, \dots, m), k \in R^+, T$ is a positive constant and f is a continuous function. In paper [15], authors investigated the existence of solutions for the following fractional BVPs

$$\begin{cases} ({}^C D^\alpha + k {}^C D^{\alpha-1})u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad t \in (0, T), \\ \alpha_1 u(0) + \rho_1 u(T) = \lambda_2, \\ \alpha_2 u'(0) + \rho_2 u'(T) = \mu_2. \end{cases} \quad (2)$$

and

$$\begin{cases} ({}^C D^\alpha + k {}^C D^{\alpha-1})u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad t \in (0, T), \\ \alpha_1 u(0) + \rho_1 u(T) = \lambda_1 \int_0^\eta u(t)dt + \lambda_2, \\ \alpha_2 u'(0) + \rho_2 u'(T) = \mu_1 \int_\xi^T u(t)dt + \mu_2, \end{cases} \quad (3)$$

where ${}^C D^\alpha$ denotes the Liouville–Caputo fractional derivative of order $\alpha, \alpha_j, \beta_j, \rho_j, \lambda_j, \mu_j \in R (j = 1, 2), 0 < \eta < \xi < T, k > 0, T > 0$ and f is continuous. Agarwal et al. [14] studied the following FDEs with anti-periodic BVPs

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), & 3 < \alpha \leq 4, \quad t \in [0, T], T > 0, \\ y(0) = -y(T), \quad y'(0) = -y'(T), \quad y''(0) = -y''(T), \quad y'''(0) = -y'''(T), \end{cases} \quad (4)$$

where ${}^C D^q$ denotes the Caputo fractional derivative, f is a given continuous function. In [27], the author investigated the following fractional BVPs with integral and anti-periodic boundary conditions:

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t)), & 1 < \alpha < 2, \quad t \in [0, 1], \\ u(1) = \mu \int_0^1 u(s)ds, \quad u'(0) + u'(1) = 0, \end{cases} \quad (5)$$

where ${}^C D^\alpha$ denotes the Caputo fractional derivative of α, f is a given continuous function. Bashir et al. [12] studied the following Caputo sequential FDEs system

$$\begin{cases} ({}^C D^\alpha + k_1 {}^C D^{\alpha-1})x(t) = f(t, x(t), y(t)), & 1 < \alpha \leq 2, t \in [0, T], \\ ({}^C D^\beta + k_2 {}^C D^{\beta-1})y(t) = g(t, x(t), y(t)), & 1 < \beta \leq 2, t \in [0, T], \end{cases} \quad (6)$$

with boundary conditions

$$\begin{cases} x(0) = a_1 y(T), \quad x'(0) = a_2 y'(T), \\ y(0) = b_1 x(T), \quad y'(0) = b_2 x'(T). \end{cases} \quad (7)$$

In [12,14,15,18,27], the authors obtained the uniqueness of solutions of their problems. Motivated by the works mentioned above, especially by [12,14,15,18,27], in this paper, we consider the following sequential FDEs with more general boundary conditions

$$\begin{cases} ({}^C D^\alpha + k {}^C D^{\alpha-1})u(t) = f(t, u(t), {}^C D^\beta u(t)), & 2 < \alpha \leq 3, 0 < \beta < 1, t \in (0, T), \\ \alpha_1 u(0) + \sum_{i=1}^m a_i u(\eta_i) + \gamma_1 u(T) + \lambda_1 \int_0^\eta u(t)dt = \beta_1, \\ \alpha_2 u'(0) + \sum_{i=1}^m b_i u'(\eta_i) + \gamma_2 u'(T) + \lambda_2 \int_{\xi_1}^{\xi_2} u'(t)dt = \beta_2, \\ \alpha_3 u''(0) + \sum_{i=1}^m c_i u''(\eta_i) + \gamma_3 u''(T) + \lambda_3 \int_{\xi}^T u''(t)dt = \beta_3, \end{cases} \quad (8)$$

where ${}^C D^\alpha$ and ${}^C D^\beta$ denote the Caputo fractional derivative, $\alpha_j, \beta_j, \gamma_j, \lambda_j \in \mathbb{R} (j = 1, 2, 3)$, $a_i, b_i, c_i \in \mathbb{R} (i = 1, 2, 3, \dots, m)$, $k, \xi, \xi_1, \xi_2, \eta \in \mathbb{R}^+$, T is a positive constant and f is Lipschitz continuous. Boundary value conditions imposed on u at 0 and T have some symmetry characteristics. By using the contraction mapping principle, we also obtain the uniqueness of the solution of problem (8). Our results include and extend some existing results (see the Section 5 for detail).

The remainder of our paper is organized as follows: In Section 2, we introduce some preliminaries. The uniqueness of the solution of the problem (8) is given in Section 3. In Section 4, we present an examples to illustrate our results. The last section is the conclusion.

2. Preliminaries

In this section, we introduce some preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 1 ([2,23]). The fractional integral of order $\alpha > 0$ of a function $x : [0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad (9)$$

provided that the right side is pointwise defined on $[0, \infty)$ where $\Gamma(\cdot)$ is the gamma functional.

Definition 2 ([2,23]). The Caputo fractional derivative of order $\alpha > 0$ of a function $x : [0, \infty) \rightarrow \mathbb{R}$ is defined as

$${}^C D_{0+}^\alpha x(t) = I_{0+}^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds, \quad (10)$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $[0, \infty)$.

The following lemma can be obtained by using the method similar to Lemma 13 in [11].

Lemma 1. Let $g(t)$ be a continuous function, $2 < \alpha < 3$, the general solution of following linear equation

$$({}^C D^\alpha + k {}^C D^{\alpha-1})u(t) = g(t), \quad t \in [0, T], \quad (11)$$

is given by

$$u(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds, \quad (12)$$

where $T > 0$, $A_i (i = 0, 1, 2)$ are constants.

Lemma 2. Suppose $g \in C[0, T]$, $u \in C^3[0, T]$, $T > 0$, then the problem

$$\begin{cases} ({}^C D^\alpha + k {}^C D^{\alpha-1})u(t) = g(t), & 2 < \alpha \leq 3, \quad t \in (0, T), \\ \alpha_1 u(0) + \sum_{i=1}^m a_i u(\eta_i) + \gamma_1 u(T) + \lambda_1 \int_0^\eta u(t) dt = \beta_1, \\ \alpha_2 u'(0) + \sum_{i=1}^m b_i u'(\eta_i) + \gamma_2 u'(T) + \lambda_2 \int_{\xi_1}^{\xi_2} u'(t) dt = \beta_2, \\ \alpha_3 u''(0) + \sum_{i=1}^m c_i u''(\eta_i) + \gamma_3 u''(T) + \lambda_3 \int_{\xi}^T u''(t) dt = \beta_3, \end{cases} \quad (13)$$

is equivalent to the fractional integral equation

$$\begin{aligned}
u(t) = & v_1(t) + \sum_{i=1}^m \omega_i(t) \int_0^{\eta_i} e^{-k(\eta_i-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau ds \\
& + v_2(t) \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau ds \\
& + \sum_{i=1}^m \varphi_i(t) \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau + v_3(t) \int_0^T \frac{(T-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau \\
& + \sum_{i=1}^m \psi_i(t) \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} g(\tau) d\tau + v_4(t) \int_0^T \frac{(T-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} g(\tau) d\tau \\
& + v_5(t) \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau ds dt + v_6(t) \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau dt \\
& + v_7(t) \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} g(\tau) d\tau dt + v_8(t) \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau ds dt \\
& + v_9(t) \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau dt - \frac{\lambda_1}{\delta_3} \int_0^{\eta} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau ds dt \\
& + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau \right) ds,
\end{aligned} \tag{14}$$

where

$$\omega_i(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) c_i k^2 - k b_i \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right) - \frac{1}{\delta_3} a_i, i = 1, 2, 3, \dots, m, \tag{15}$$

$$\varphi_i(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) c_i k - b_i \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right), i = 1, 2, 3, \dots, m, \tag{16}$$

$$\psi_i(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) c_i, i = 1, 2, 3, \dots, m, \tag{17}$$

$$v_1(t) = \frac{\beta_3 e^{-kt}}{\delta_1} + \frac{\beta_1}{\delta_3} - \frac{\beta_3 \delta''_1}{\delta_3 \delta_1} - \frac{\beta_2 \delta'_2}{\delta_3 \delta_2} + \frac{\beta_3 \delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} + \frac{\beta_2}{\delta_2} t - \frac{\beta_3 \delta'_1}{\delta_1 \delta_2} t, \tag{18}$$

$$v_2(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \gamma_3 k^2 - \gamma_2 k \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right) - \frac{\gamma_1}{\delta_3}, \tag{19}$$

$$v_3(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \gamma_3 k + \gamma_2 \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right), \tag{20}$$

$$v_4(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \gamma_3, \quad v_5(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \lambda_3 k^2, \tag{21}$$

$$v_6(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \lambda_3 k, \quad v_7(t) = \left(-\frac{e^{-kt}}{\delta_1} + \frac{\delta'_1 t}{\delta_1 \delta_2} + \frac{\delta''_1}{\delta_1 \delta_3} - \frac{\delta'_1 \delta'_2}{\delta_1 \delta_2 \delta_3} \right) \lambda_3, \tag{22}$$

$$v_8(t) = -\lambda_2 k \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right), \quad v_9(t) = \lambda_2 \left(-\frac{t}{\delta_2} + \frac{\delta'_2}{\delta_2 \delta_3} \right), \tag{23}$$

$$\delta_1 = \alpha_3 k^2 + \sum_{i=1}^m c_i k^2 e^{-k\eta_i} + \gamma_3 k^2 e^{-kT} + k(e^{-k\xi} - e^{-kT}), \quad \delta_2 = \alpha_2 + \sum_{i=1}^m b_i + \gamma_2 + \lambda_2(\xi_2 - \xi_1), \tag{24}$$

$$\delta_3 = \alpha_1 + \sum_{i=1}^m a_i + \gamma_1 + \lambda_1 \eta, \quad \delta'_1 = -k\alpha_2 - \sum_{i=1}^m b_i k e^{-k\eta_i} - \gamma_2 k e^{-kT} + \lambda_2(e^{-\xi_2} - e^{-\xi_1}), \tag{25}$$

$$\delta'_2 = \sum_{i=1}^m a_i \eta_i + \gamma_1 T + \frac{1}{2} \lambda_1 \eta^2, \quad \delta'_1 = \alpha_1 + \sum_{i=1}^m a_i e^{-k\eta_i} + \gamma_1 e^{-kT} + \frac{\lambda_1}{k} (1 - e^{-k\eta}). \quad (26)$$

Proof. By Lemma 1, we say that the solution of (8) can be expressed in terms of an integral equation as

$$u(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds, \quad (27)$$

where A_0 , A_1 and A_2 are arbitrary constants and $I^{\alpha-1} g(s) = \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} g(\tau) d\tau$. By differentiating (14) with respect to t , we obtain

$$u'(t) = -kA_0 e^{-kt} + A_2 - k \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds + I^{\alpha-1} g(t), \quad (28)$$

$$u''(t) = k^2 A_0 e^{-kt} + k^2 \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(t) + I^{\alpha-2} g(t). \quad (29)$$

By using the boundary conditions given by (13) in (27)–(29), we obtain

$$\begin{aligned} \beta_1 = & \delta'_1 A_0 + \delta_3 A_1 + \delta'_2 A_2 + \sum_{i=1}^m a_i \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds \\ & + \gamma_1 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds + \lambda_1 \int_0^\eta \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt, \end{aligned} \quad (30)$$

$$\begin{aligned} \beta_2 = & - \sum_{i=1}^m b_i \left[k \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds + I^{\alpha-1} g(\eta_i) \right] \\ & - \gamma_2 \left[k \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - I^{\alpha-1} g(T) \right] \\ & - \lambda_2 \left[k \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - \int_{\xi_1}^{\xi_2} I^{\alpha-1} g(t) dt \right] \delta'_1 + A_0 + \delta_2 A_2, \end{aligned} \quad (31)$$

$$\begin{aligned} \beta_3 = & \delta_1 A_0 + \sum_{i=1}^m c_i \left[k^2 \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(\eta_i) + I^{\alpha-2} g(\eta_i) \right] \\ & + \gamma_3 \left[k^2 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(T) + I^{\alpha-2} g(T) \right] \\ & + \lambda_3 \left[k^2 \int_{\xi}^T \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - k \int_{\xi}^T I^{\alpha-1} g(t) dt + \int_{\xi}^T I^{\alpha-2} g(t) dt \right]. \end{aligned} \quad (32)$$

By solving the system (30)–(32) for A_0 , A_1 , A_2 , we obtain

$$\begin{aligned} A_0 = & \frac{1}{\delta'_1} \left\{ \beta_3 - \sum_{i=1}^m c_i \left[k^2 \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(\eta_i) + I^{\alpha-2} g(\eta_i) \right] \right. \\ & - \gamma_3 \left[k^2 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(T) + I^{\alpha-2} g(T) \right] \\ & \left. - \lambda_3 \left[k^2 \int_{\xi}^T \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - k \int_{\xi}^T I^{\alpha-1} g(t) dt + \int_{\xi}^T I^{\alpha-2} g(t) dt \right] \right\}. \end{aligned} \quad (33)$$

$$\begin{aligned}
A_1 = & \frac{1}{\delta_3} \left[\beta_1 - \sum_{i=1}^m a_i \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds - \gamma_1 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - \lambda_1 \int_0^\eta \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt \right] \\
& + \frac{\delta'_1 \delta'_2 - \delta''_1 \delta_2}{\delta_2 \delta_3} \frac{1}{\delta_1} \left\{ \beta_3 - \sum_{i=1}^m c_i [k^2 \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds - I^{\alpha-1} h(\eta_i) + I^{\alpha-2} g(\eta_i)] \right. \\
& - \gamma_3 [k^2 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(T) + I^{\alpha-2} g(T)] \\
& - \lambda_3 [k^2 \int_{\xi}^T \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - k \int_{\xi}^T I^{\alpha-1} g(t) dt + \int_{\xi}^T I^{\alpha-2} g(t) dt] \Big\} \\
& - \frac{\delta'_2}{\delta_2 \delta_3} \left\{ \beta_2 + \sum_{i=1}^m b_i [k \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds + I^{\alpha-1} g(\eta_i)] + \gamma_2 [k \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - I^{\alpha-1} g(T)] \right. \\
& \left. + \lambda_2 [k \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - \int_{\xi_1}^{\xi_2} I^{\alpha-1} g(t) dt] \right\}.
\end{aligned} \tag{34}$$

$$\begin{aligned}
A_2 = & \frac{1}{\delta_2} \beta_2 - \frac{\delta'_1}{\delta_1 \delta_2} \left\{ \beta_3 - \sum_{i=1}^m c_i [k^2 \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds - I^{\alpha-1} g(\eta_i) + I^{\alpha-2} g(\eta_i)] \right. \\
& - \gamma_3 [k^2 \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - k I^{\alpha-1} g(T) + I^{\alpha-2} g(T)] \\
& - \lambda_3 [k^2 \int_{\xi}^T \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - k \int_{\xi}^T I^{\alpha-1} g(t) dt + \int_{\xi}^T I^{\alpha-2} g(t) dt] \Big\} \\
& + \frac{1}{\delta_2} \left\{ \sum_{i=1}^m b_i [k \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{\alpha-1} g(s) ds + I^{\alpha-1} g(\eta_i)] + \gamma_2 [k \int_0^T e^{-k(T-s)} I^{\alpha-1} g(s) ds - I^{\alpha-1} g(T)] \right. \\
& \left. + \lambda_2 [k \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} I^{\alpha-1} g(s) ds dt - \int_{\xi_1}^{\xi_2} I^{\alpha-1} g(t) dt] \right\}.
\end{aligned} \tag{35}$$

By substituting the values of A_0, A_1 and A_2 in (27), we obtain the desired solution (14). The converse of the lemma follows by direct computation. This completes the proof. $\square$

3. Uniqueness Results

Define the space

$$P = \left\{ x : x \in C[0, T], {}^C D^\beta x \in C[0, T] \right\}, \tag{36}$$

with the norm

$$\|x\| = \sup_{t \in [0, T]} |x(t)|, \|x\|_1 = \|x\| + \|{}^C D^\beta x\|. \tag{37}$$

Then, it is well known that $(P, \|\cdot\|_1)$ is a Banach space. By Lemma 2, (8) can be converted into a fixed-point problem

$$u = Au, \tag{38}$$

where $A : P \rightarrow P$ is presented by

$$\begin{aligned}
& (Au)(t) \\
&= v_1(t) + \sum_{i=1}^m \omega_i(t) \int_0^{\eta_i} e^{-k(\eta_i-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds \\
&+ v_2(t) \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds + \sum_{i=1}^m \varphi_i(t) \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau \\
&+ v_3(t) \int_0^T \frac{(T-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau + \sum_{i=1}^m \psi_i(t) \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau \\
&+ v_4(t) \int_0^T \frac{(T-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau + v_5(t) \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds dt \\
&+ v_6(t) \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau dt + v_7(t) \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau dt \\
&+ v_8(t) \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds dt + v_9(t) \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau dt \\
&- \frac{\lambda_1}{\delta_3} \int_0^{\eta} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds dt + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau \right) ds,
\end{aligned} \tag{39}$$

where

$$F(\cdot) := f(\cdot, u(\cdot), {}^C D^{\beta} u(\cdot)), u \in P. \tag{40}$$

Clearly, (8) has solutions if and only if the associated fixed-point problem $u = Au$ has fixed points. For computational convenience, we set

$$\begin{aligned}
Q_1 = \sup_{t \in [0, T]} & \left\{ \frac{t^{\alpha-1}(1-e^{-kt})}{k\Gamma(\alpha)} + \frac{|\sum_{i=1}^m \omega_i(t) \eta_i^{\alpha-1}(1-e^{-k\eta_i})|}{k\Gamma(\alpha)} + \frac{|v_2(t)| T^{\alpha-1}(1-e^{-kT})}{k\Gamma(\alpha)} \right. \\
&+ \frac{|\sum_{i=1}^m \varphi_i(t) \eta_i^{\alpha-2}|}{\Gamma(\alpha)} + \frac{|v_3(t)| T^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\sum_{i=1}^m \psi_i(t) \eta_i^{\alpha-2}|}{\Gamma(\alpha-1)} + \frac{|v_4(t)| T^{\alpha-2}}{\Gamma(\alpha-1)} \\
&+ \frac{|v_5(t)| (T-\xi + e^{-k(T-\xi)})}{k^2\Gamma(\alpha)} + \frac{|v_6(t)| (T^{\alpha}-\xi^{\alpha})}{\Gamma(\alpha+1)} + \frac{|v_7(t)| (T^{\alpha-1}-\xi^{\alpha-1})}{\Gamma(\alpha)} \\
&\left. + \frac{|v_8(t)| (\xi_2-\xi_1 + e^{-k(\xi_2-\xi_1)})}{k^2\Gamma(\alpha)} + \frac{|v_9(t)| (\xi_2^{\alpha}-\xi_1^{\alpha})}{\Gamma(\alpha+1)} + \frac{\lambda_1 |\eta + e^{-k\eta}|}{\delta_3 k^2\Gamma(\alpha)} \right\}.
\end{aligned} \tag{41}$$

$$\begin{aligned}
Q_2 = \sup_{t \in [0, T]} & \left\{ \frac{t^{\alpha-1}(1-e^{-kt})}{k\Gamma(\alpha)} + \frac{|\sum_{i=1}^m \omega'_i(t) \eta_i^{\alpha-1}(1-e^{-k\eta_i})|}{k\Gamma(\alpha)} + \frac{|v'_2(t)| T^{\alpha-1}(1-e^{-kT})}{k\Gamma(\alpha)} \right. \\
&+ \frac{|\sum_{i=1}^m \varphi'_i(t) \eta_i^{\alpha-2}|}{\Gamma(\alpha)} + \frac{|v'_3(t)| T^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\sum_{i=1}^m \psi'_i(t) \eta_i^{\alpha-2}|}{\Gamma(\alpha-1)} + \frac{|v'_4(t)| T^{\alpha-2}}{\Gamma(\alpha-1)} \\
&+ \frac{|v'_5(t)| (T-\xi + e^{-k(T-\xi)})}{k^2\Gamma(\alpha)} + \frac{|v'_6(t)| (T^{\alpha}-\xi^{\alpha})}{\Gamma(\alpha+1)} + \frac{|v'_7(t)| (T^{\alpha-1}-\xi^{\alpha-1})}{\Gamma(\alpha)} \\
&\left. + \frac{|v'_8(t)| (\xi_2-\xi_1 + e^{-k(\xi_2-\xi_1)})}{k^2\Gamma(\alpha)} + \frac{|v'_9(t)| (\xi_2^{\alpha}-\xi_1^{\alpha})}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha)} \right\}.
\end{aligned} \tag{42}$$

$$Q = \max_{t \in [0, T]} \{Q_1, Q_2\}. \tag{43}$$

Now, we prove the uniqueness of solutions of problem (8).

Theorem 1. Suppose that $f : [0, T] \times R \times R \rightarrow R$ is a Lipschitz continuous function with respect to the second and third variable, that is, there exists a positive constant l such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq l(|x_1 - x_2| + |y_1 - y_2|), \forall t \in [0, T], \forall x_1, x_2, y_1, y_2 \in R, T > 0. \tag{44}$$

Then, the boundary value problem (8) has an unique solution on $[0, T]$ if

$$l < \frac{\Gamma(2-\beta)}{Q[\Gamma(2-\beta) + T^{1-\beta}]}, \quad (45)$$

where Q is given by (43).

Proof. Due to the continuity of f , we know there exists a constant M satisfied

$$\sup_{t \in [0, T]} |f(t, 0, 0)| = M. \quad (46)$$

Let $P_\sigma = \{u \in P : \|u\|_1 \leq \sigma\}$, where

$$\sigma \geq \frac{(QM + \|v_1\|_1)}{1 - lQ} + \frac{(QM + \|v_1\|_1)}{\Gamma(2-\beta)T^{\beta-1} - lQ}, \quad (47)$$

and Q, v_1 are given by (18) and (43), respectively. We divide the following long proof into three steps. $\square$

Step 1: We shall show that $A(P_\sigma) \subset P_\sigma$, where the operator A is defined by (39). For any $u \in P_\sigma$, $t \in [0, T]$, we have

$$\begin{aligned} & |f(t, u(t), {}^C D^\beta u(t))| \\ & \leq |f(t, u(t), {}^C D^\beta u(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ & \leq l(|u(t) - 0| + |{}^C D^\beta u(t) - 0|) + M \\ & \leq l(\|u\| + \|{}^C D^\beta u\|) + M \leq l\|u\|_1 + M \leq l\sigma + M. \end{aligned} \quad (48)$$

Then, for any $u \in P_\sigma$, we have from (39) and (48) that

$$\begin{aligned} & \|A(u)\| \\ & \leq (l\sigma + M) \sup_{t \in [0, T]} \left\{ |v_1(t)| + \sum_{i=1}^m |\omega_i(t)| \int_0^{\eta_i} e^{-k(\eta_i-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds \right. \\ & + |v_2(t)| \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds + \sum_{i=1}^m |\varphi_i(t)| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ & + |v_3(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^m |\psi_i(t)| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-3}}{\Gamma(\alpha-2)} ds + |v_4(t)| \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} ds \\ & + |v_5(t)| \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt + |v_6(t)| \int_{\xi}^T \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds dt \\ & + |v_7(t)| \int_{\xi}^T \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} ds dt + |v_8(t)| \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt \\ & + |v_9(t)| \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds dt + \frac{\lambda_1}{\delta_3} \int_0^\eta \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt \\ & \left. + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) ds \right\} \\ & \leq (l\sigma + M)Q_1 + \|v_1\|_1 \\ & \leq (l\sigma + M)Q + \|v_1\|_1. \end{aligned} \quad (49)$$

The inequality (49) tells us that

$$\begin{aligned}
& |(Au)'(t)| \\
& \leq \sup_{t \in [0, T]} \left\{ |v'_1(t)| + \sum_{i=1}^m |\omega'_i(t)| \int_0^{\eta_i} e^{-k(\eta_i-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds \right. \\
& + |v'_2(t)| \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds + \sum_{i=1}^m |\varphi'_i(t)| \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau \\
& + |v'_3(t)| \int_0^T \frac{(T-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau + \sum_{i=1}^m |\psi'_i(t)| \int_0^{\eta_i} \frac{(\eta_i-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau \\
& + |v'_4(t)| \int_0^T \frac{(T-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau + |v'_5(t)| \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds dt \\
& + |v'_6(t)| \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau dt + |v'_7(t)| \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} F(\tau) d\tau dt \\
& + |v'_8(t)| \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau ds dt + |v'_9(t)| \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau dt \\
& + t \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau + k \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} F(\tau) d\tau \right) ds \Big\} \\
& \leq (l\sigma + M)Q_2 + \|v_1\|_1 \leq (l\sigma + M)Q + \|v_1\|_1,
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
|{}^C D^\beta(Au)(t)| & \leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} |(Au)'(s)| ds \\
& \leq \frac{(l\sigma + M)Q + \|v_1\|_1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} ds \\
& = \frac{t^{1-\beta}}{\Gamma(2-\beta)} [(l\sigma + M)Q + \|v_1\|_1] \\
& \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)} [(l\sigma + M)Q + \|v_1\|_1].
\end{aligned} \tag{51}$$

Thus (47), (49) and (51) deduce that

$$\begin{aligned}
\|Au\|_1 & = \|Au\| + \|{}^C D^\beta(Au)\| \\
& \leq [(l\sigma + M)Q + \|v_1\|_1] + \frac{T^{1-\beta}}{\Gamma(2-\beta)} [(l\sigma + M)Q + \|v_1\|_1] \\
& \leq \sigma.
\end{aligned} \tag{52}$$

Therefore, $A(P_\sigma) \subset P_\sigma$.

Step 2. We prove that the operator A given by (59) is completely continuous. Firstly, the continuity of f implies that the operator A is continuous. Secondly, we prove that the operator A is compact. Let Ω be an arbitrary bounded set in X . Then, there exists $\tilde{\sigma}$ such that $\|u\|_1 \leq \tilde{\sigma}$. Then, by the similar method to prove (48)–(52), we get

$$\|Au\|_1 \leq [(l\tilde{\sigma} + M)Q + \|v_1\|_1] + \frac{T^{1-\beta}}{\Gamma(2-\beta)} [(l\tilde{\sigma} + M)Q + \|v_1\|_1] := \text{constant}, \forall u \in \Omega. \tag{53}$$

This tells us that $A\Omega$ is uniformly bounded. Furthermore, we prove that the operator A is equicontinuous. Similar to (50), we know

$$|(Au)'(t)| \leq (l\tilde{\sigma} + M)Q + \|v_1\|_1, \forall t \in [0, T], u \in \Omega. \tag{54}$$

Consequently, for any $u \in \Omega$ and $t_1, t_2 \in [0, T], t_1 < t_2$, we have

$$|(Au)(t_2) - (Au)(t_1)| \leq |(Au)'(\theta)| |t_2 - t_1| \leq [(l\tilde{\sigma} + M)Q + \|v_1\|_1] |t_2 - t_1|, \tag{55}$$

where $\theta \in (t_1, t_2)$. The inequality (55) implies that A is equicontinuous. By the Arzela-Ascoli theorem, A is a compact operator.

From step 1, step 2 and Schauder fixed point theorem, we know that the operator A has a fixed point in P_σ . This means the existence of solutions of problem (8). To prove the uniqueness of solutions of problem (8), we need the next step.

Step 3. We show that the operator A is a contraction mapping. For any $u, v \in P_\sigma$, denote

$$G(\cdot) := |f(\cdot, u(\cdot), {}^C D^\beta u(\cdot)) - f(\cdot, v(\cdot), {}^C D^\beta v(\cdot))|. \quad (56)$$

Then,

$$\begin{aligned} & \|Au - Av\| \\ & \leq \sup_{t \in [0, T]} \left\{ \sum_{i=1}^m |\omega_i(t)| \int_0^{\eta_i} e^{-k(\eta_i - \tau)} \int_0^s G(\tau) d\tau ds + |v_2(t)| \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau ds \right. \\ & + \sum_{i=1}^m |\varphi_i(t)| \int_0^{\eta_i} \frac{(\eta_i - \tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau + |v_3(t)| \int_0^T \frac{(T-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau + \sum_{i=1}^m |\psi_i(t)| \int_0^{\eta_i} \frac{(\eta_i - \tau)^{\alpha-3}}{\Gamma(\alpha-2)} G(\tau) d\tau \\ & + |v_4(t)| \int_0^T \frac{(T-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} G(\tau) d\tau + |v_5(t)| \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau ds dt \\ & + |v_6(t)| \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau dt + |v_7(t)| \int_{\xi}^T \int_0^t \frac{(t-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} G(\tau) d\tau dt \\ & + |v_8(t)| \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau ds dt + |v_9(t)| \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau dt \\ & + \frac{\lambda_1}{\delta_3} \int_0^\eta \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau ds dt + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} G(\tau) d\tau \right) ds \Big\} \\ & \leq l \|u - v\|_1 \sup_{t \in [0, T]} \left\{ |v_1(t)| + \sum_{i=1}^m |\omega_i(t)| \int_0^{\eta_i} e^{-k(\eta_i - s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds \right. \\ & + |v_2(t)| \int_0^T e^{-k(T-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds + \sum_{i=1}^m |\varphi_i(t)| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ & + |v_3(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^m |\psi_i(t)| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-3}}{\Gamma(\alpha-2)} ds + |v_4(t)| \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} ds \\ & + |v_5(t)| \int_{\xi}^T \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt + |v_6(t)| \int_{\xi}^T \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds dt \\ & + |v_7(t)| \int_{\xi}^T \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} ds dt + |v_8(t)| \int_{\xi_1}^{\xi_2} \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt \\ & + |v_9(t)| \int_{\xi_1}^{\xi_2} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds dt + \frac{\lambda_1}{\delta_3} \int_0^\eta \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau ds dt \\ & + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) ds \Big\} \\ & \leq l Q_1 \|u - v\|_1 \\ & \leq l Q \|u - v\|_1. \end{aligned} \quad (57)$$

Similarly, we get

$$|(Au)'(t) - (Av)'(t)| \leq l Q_2 \|u - v\|_1 \leq l Q \|u - v\|_1. \quad (58)$$

Consequently we know

$$\begin{aligned}\|{}^C D^\beta(Au) - {}^C D^\beta(Av)\| &= \sup_{t \in [0, T]} \frac{1}{\Gamma(1-\beta)} \left| \int_0^t (t-s)^{-\beta} |(Au)'(s) - (Av)'(s)| ds \right| \\ &\leq \frac{lQ\|u-v\|_1}{\Gamma(2-\beta)T^{\beta-1}}.\end{aligned}\quad (59)$$

Then, we know from (57) and (59)

$$\|Au - Av\|_1 = \|Au - Av\| + \|{}^C D^\beta(Au) - {}^C D^\beta(Av)\| \leq lQ \left[1 + \frac{1}{\Gamma(2-\beta)T^{\beta-1}} \right] \|u - v\|_1 < \|u - v\|_1, \quad (60)$$

which and (45) imply that A is a contraction mapping. Thus, from the Banach contraction mapping principle, we deduce that the operator A has an unique fixed point, that is, (8) has an unique solution on $[0, T]$. This ends the proof.

4. An Example

Motivated by examples in [12,14,15,18,27] we give an example to illustrate Theorem 1. Consider the following problem:

$$\begin{cases} ({}^C D^{\frac{5}{2}} + 2{}^C D^{\frac{3}{2}})u(t) = f(t, u(t), {}^C D^{\frac{1}{2}}u(t)), & t \in (0, 2), \\ u(0) + u(\frac{1}{4}) + \frac{1}{2}u(\frac{3}{4}) - u(\frac{5}{4}) + u(\frac{7}{4}) - u(2) + \int_0^1 u(t)dt = 1, \\ u'(0) - u'(\frac{1}{4}) + u'(\frac{3}{4}) - u'(\frac{5}{4}) + u'(\frac{7}{4}) - \frac{1}{2}u'(2) + 2 \int_{\frac{1}{4}}^{\frac{3}{4}} u'(t)dt = 2, \\ \frac{1}{2}u''(0) + \frac{1}{4}u''(\frac{1}{4}) - u''(\frac{3}{4}) + u''(\frac{5}{4}) + u''(\frac{7}{4}) + u''(2) + \int_{\frac{3}{2}}^2 u''(t)dt = 1, \end{cases} \quad (61)$$

where

$$f(t, u, v) = \frac{1}{200} (\tan^{-1} u + \frac{5-t}{\sqrt{t+25}} \sin v) + (1 + \frac{e^{-t}}{\sqrt{t+25}}) \cos t,$$

$T = 2$, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $k = 2$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \frac{1}{2}$, $m = 4$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$, $\eta_3 = \frac{5}{4}$, $\eta_4 = \frac{7}{4}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = -1$, $a_4 = 1$, $b_1 = b_3 = -1$, $b_2 = b_2 = 1$, $c_1 = \frac{1}{4}$, $c_2 = -1$, $c_3 = c_4 = 1$, $\gamma_1 = -1$, $\gamma_2 = -\frac{1}{2}$, $\gamma_3 = 1$, $\lambda_1 = \lambda_3 = 1$, $\lambda_2 = 2$, $\eta = 1$, $\xi = \frac{3}{2}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{3}{4}$.

We find that $Q_1 \approx 49.286$, $Q_2 \approx 9.711$, $Q = 49.286$, where Q_1 , Q_2 , Q are given by (41), (42) and (43), respectively. Let $l = \frac{1}{200}$, then we have

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{200} (|u_1 - u_2| + |v_1 - v_2|), \forall t \in [0, 2], u_1, v_1, u_2, v_2 \in \mathbb{R},$$

and that

$$l = \frac{1}{200} < \frac{\Gamma(2-\beta)}{Q[\Gamma(2-\beta) + T^{1-\beta}]} \approx 0.0078 < 1.$$

Thus, we have verified (44) and (45), and the uniqueness of solutions of (61) follows the Theorem 1.

5. Conclusions

It is necessary to point out that the results presented in this paper specialize to some known theorems with an appropriate choice of the parameters involved in the problems at hand. By fixing the values of the parameters involved in the problem (8), we come to some special cases. For instance, the function f in [14,15,18,27] has no the third argument ${}^C D^\beta u$. Moreover, problem (1) is a special case if

$$\lambda_j = 0, j = 1, 2, 3,$$

of problem (8). Problem (2) is a special case if

$$\alpha_2 = \lambda_2 = \beta_2 = \gamma_j = 0, j = 1, 2, 3, a_i = b_i = c_i = 0, i = 1, 2, 3, \dots, m,$$

of problem (8). Problem (3) is a special case if

$$\alpha_2 = \gamma_2 = \lambda_2 = \beta_2 = 0, a_i = b_i = c_i = 0, i = 1, 2, 3, \dots, m,$$

of problem (8). If we deal with (8) without the term ${}^C D^{\alpha-1}u$ and with one more boundary value condition

$$\alpha_4 u'''(0) + \sum_{i=1}^m d_i u'''(\eta_i) + \gamma_4 u'''(T) + \lambda_4 \int_{\xi}^T u'''(t) dt = \beta_4,$$

then problem (4) becomes a special case if

$$\lambda_j = \beta_j = a_i = b_i = c_i = d_i = 0, \alpha_j = \gamma_j, i = 1, 2, 3, \dots, m, j = 1, 2, 3, 4,$$

of problem (8); in case we take

$$\alpha_1 = \alpha_3 = \gamma_3 = \lambda_2 = \lambda_3 = a_i = b_i = c_i = \beta_j = 0, i = 1, 2, 3, \dots, m, j = 1, 2, 3,$$

$$\alpha_2 = \gamma_2, \gamma_1 \lambda_1 \neq 0, T = 1, 1 < \alpha < 2,$$

in (8) without ${}^C D^{\alpha-1}u$, we obtain the problem (5). Consequently, our main result Theorem 1 includes and extends Theorem 3.4 in [14], Theorem 4.1 in [15], Theorem 3.1 in [18], Theorem 3.2 in [27].

At the same time we know that there are still more works to do in the future. For example, how to apply fixed point due to Kranselskii, Leray-Schauder alternative criterion and Leray-Schauder degree theory to obtain the existence of solution of problem (8) and other FDEs with more general boundary value conditions. Can we add some impulsive boundary conditions to problem (8)? Furthermore, there are more problems need to solve, such as fractional differential inclusions problems, semipositone problems, FDEs systems and so on.

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