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Application of Einstein Function on Bi-Univalent Functions Defined on the Unit Disc

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Abstract: Motivated by q-calculus, we define a new family of Σ, which is the family of bi-univalent analytic functions in the open unit disc U that is related to the Einstein function $E(z)$. We establish estimates for the first two Taylor–Maclaurin coefficients $|a_2|$, $|a_3|$, and the Fekete–Szegö inequality $\left|a_3 - \mu a_2^2\right|$ for the functions that belong to these families.

Keywords: analytic function; Einstein function; bernoulli numbers; bi-univalent function; quantum calculus; subordination

MSC: 30C45; 30C55; 11B68

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1. Introduction and Basic Concepts

Let $\mathcal{A}$ denote the family of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. In addition, an important class of functions will be called $\mathcal{P}$. $\mathcal{P}$ is the family of analytic univalent functions ϕ with positive real part mapping U onto domains symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ such that $\phi'(0) > 0$. In 1994, Ma and Minda [1] introduced the following subset of functions:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec \phi(z), \phi \in \mathcal{P}, z \in U \right\},$$

where the symbol "$\prec$" refers to the subordination given in Definition 1 below. Ma and Minda [1] investigated certain useful problems, including distortion, growth and covering theorems.

Now, taking some particular functions instead of ϕ in $\mathcal{S}^*(\phi)$, we achieve many subfamilies of the collection $\mathcal{A}$ which have different geometric interpretations, as for example:

(i) If $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \le B < A \le 1$, then $\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions; see [2]. Some interesting problems such as convolution properties, coefficient inequalities, sufficient conditions, subordinate results and integral preserving were discussed recently in [3–7] for some of the generalized families associated with circular domains;

(ii) The class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokól and Stankiewicz [8], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\left|w^2 - 1\right| < 1$;

(iii) When we take $\phi(z) = e^z$, then we have $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ [9];
(iv) The family $\mathcal{S}^*_R := \mathcal{S}^*\left(1 + \frac{z}{k}\frac{k+z}{k-z}\right), k = \sqrt{2}+1$, the rational function is studied in [10];
(v) For $\mathcal{S}^*_{sin} := \mathcal{S}^*(1 + \sin z)$, the class $\mathcal{S}^*_{sin}$ is introduced in [11];
(vi) By setting $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, the family $\mathcal{S}^*(\phi)$ reduces to S_{car} introduced by Sharma and his coauthors [12], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the cardioid given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$, for more subclasses see [13–17].

In mathematics, Einstein function is a name occasionally used for one of the functions (see [18,19]): $E_1(z) := \dfrac{z}{e^z - 1}$, $E_2(z) := \dfrac{z^2 e^z}{(e^z - 1)^2}$, $E_3(z) := \log(1 - e^{-z})$, $E_4(z) := \dfrac{z}{e^z - 1} - \log(1 - e^{-z})$.

It is easily noticed that both E_1 and E_2 have these nice properties (see Figure 1); the image domain of $E_{1,2}$ ($E_{1,2}$ are convex functions with $\mathrm{Re}(E_{1,2}(z)) > 0 \ \forall\, z \in U$) is symmetric along the real axis and starlike about $E_{1,2}(0) = 1$. Unfortunately, $E'_{1,2}(0) \not> 0$, thus, we shall define the new functions $E(z) := E_1(z) + z$ and $\mathbb{E}(z) := E_2(z) + \frac{1}{2}z$. Now, we can say that $E, \mathbb{E} \in \mathcal{P}$ (see Figure 2).

The series representations are given as follows:

$$E(z) = 1 + z + \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n, \tag{2}$$

and

$$\mathbb{E}(z) = 1 + \frac{1}{2}z + \sum_{n=1}^{\infty} \frac{(1-n)B_n}{n!} z^n, \tag{3}$$

where B_n is the nth Bernoulli number; it is known that the Bernoulli numbers B_n can be defined by the contour integral (see [20])

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \tag{4}$$

where the contour encloses the origin, has radius less than $2\pi i$, and is traversed in a counterclockwise direction; the first few members are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots$$

and

$$B_{2n+1} = 0 \ \ \forall n \in \mathbb{N}.$$

Here, in this paper, we will deal with the first function E, the function $\mathbb{E}$ is left as open problem.

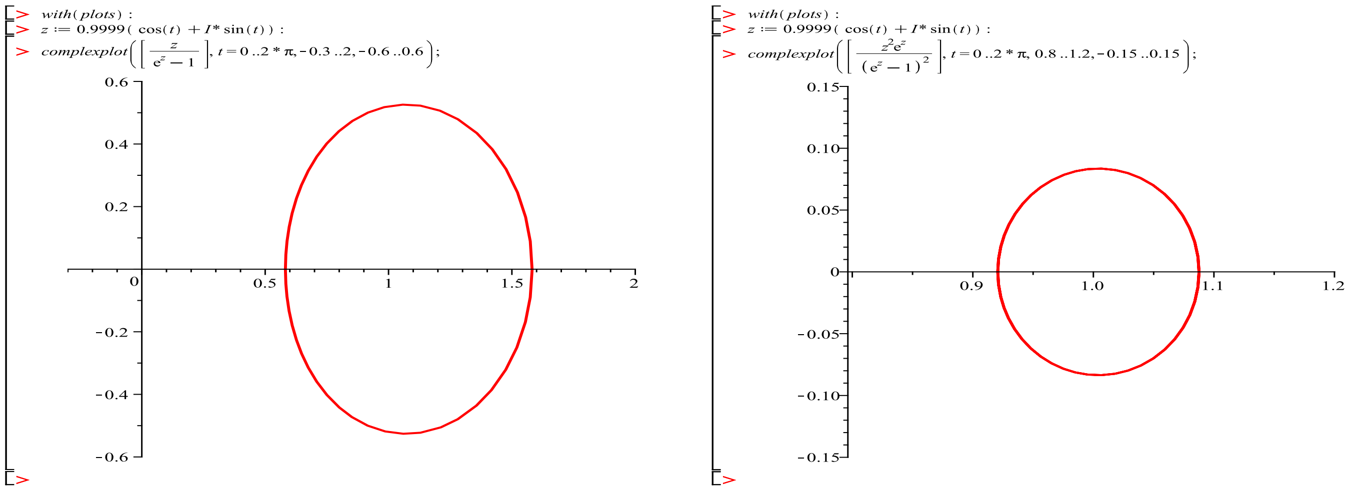
Let $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of all functions of the form (1) which are univalent in U.

It is well known, by using the Koebe one-quarter theorem [21], that every univalent function $f \in \mathcal{S}$ containing a disc of radius $\frac{1}{4}$ has an inverse function f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \ \ (z \in \mathcal{U}),$$

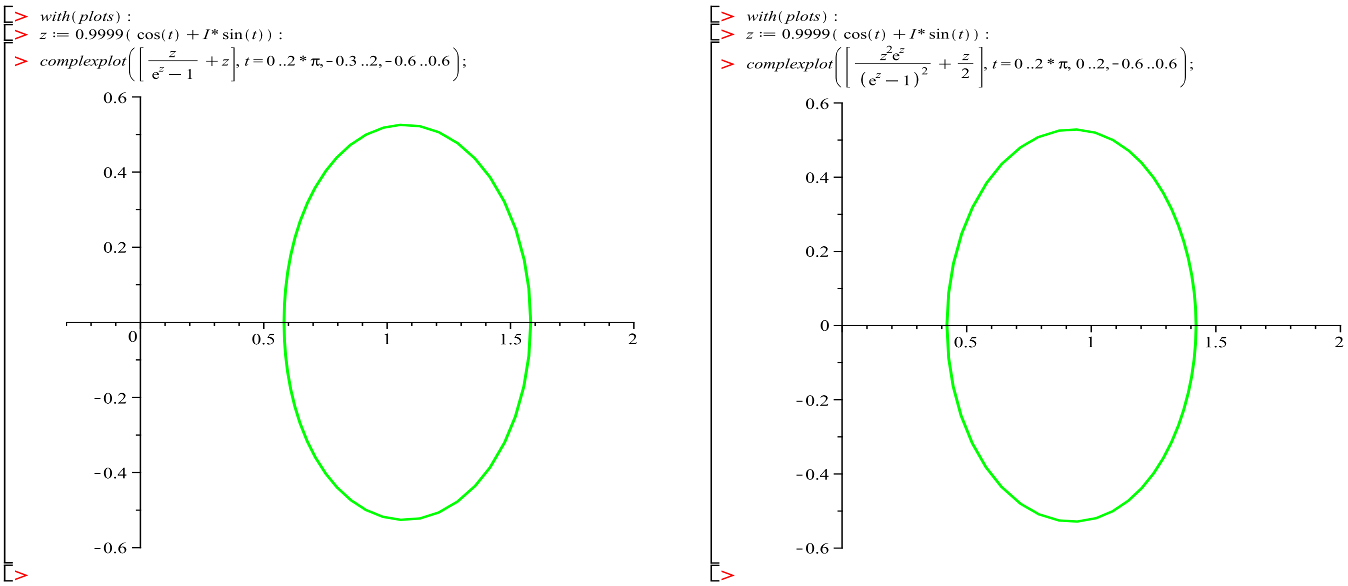
and

$$f(f^{-1}(\omega)) = \omega \quad \left(\omega \in \Delta = \left\{\omega \in \mathbb{C} : |\omega| < \frac{1}{4}\right\}\right).$$



(**a**) $E_1(U)$ (**b**) $E_2(U)$

Figure 1. The images of unit disc U of the Einstein functions E_1 and E_2.



(**a**) $E(U)$ (**b**) $\mathbb{E}(U)$

Figure 2. The images of unit disc U of the modified Einstein functions E and $\mathbb{E}$.

A function $f \in \mathcal{S}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the subfamily of $\mathcal{S}$, consisting of all bi-univalent functions defined on the unit disc U. Since $f \in \Sigma$ has the Maclaurin series expansion given by (1), a simple calculation shows that its inverse $g = f^{-1}$ has the series expansion

$$g(\omega) = f^{-1}(\omega) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \ldots. \tag{5}$$

Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -log(1-z) \qquad \text{and} \qquad \frac{1}{2}log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$, such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2},$$

are also not members of Σ.

Now, we recall some notations about the q-difference operator which is used in investigating our main families. In view of Annaby and Mansour [22], the q-difference operator is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{z(q-1)} & , \quad z \neq 0\,; \\ f'(0) & , \quad z = 0\,; \end{cases}$$

and

$$\partial_q^0 f(z) = f(z), \quad \partial_q^1 f(z) = \partial_q f(z) \quad \text{and} \quad \partial_q^m f(z) = \partial_q(\partial_q^{m-1} f(z)) \quad (m \in \mathbb{N}).$$

Thus, for the function $f \in \mathcal{A}$ defined by (1), we have

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \neq 0), \tag{6}$$

where

$$[\nu]_q = \frac{q^\nu - 1}{q-1} = \sum_{j=0}^{\nu-1} q^j, \quad \nu \in \mathbb{N}.$$

We note that $\lim_{q\to 1^-} [n]_q = n$ and $\lim_{q\to 1^-} \partial_q f(z) = f'(z)$.

Definition 1 ([23,24])**.** *An analytic function f is said to be subordinate to another analytic function g, written as $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function ω, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then we have the following equivalence:*

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The aim of this article is to introduce new subfamilies of analytic bi-univalent functions subordinate to the Einstein function $E(z)$. Furthermore, we deduce some estimations to $|a_2|$, $|a_3|$ and also the Fekete–Szegö inequalities for the functions that belong to these subfamilies.

Definition 2. *Consider $0 \le \delta \le 1$, $0 \le \lambda \le 1$ and $q \in (0,1)$. The function $f \in \Sigma$ is said to be in $\mathcal{M}_\Sigma(\delta, \lambda; E)$ if it satisfies*

$$(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda} \partial_q f(z) + \delta \frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)} \prec E(z), \tag{7}$$

and

$$(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda} \partial_q g(\omega) + \delta \frac{\partial_q(\omega\partial_q g(\omega))}{\partial_q g(\omega)} \prec E(\omega), \tag{8}$$

where $g = f^{-1}$ is given by (5) and $z, \omega \in U$.

Definition 3. *Consider $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $q \in (0,1)$. The function $f \in \mathcal{A}$ is said to be in $\mathcal{N}_\Sigma(\alpha, \beta; E)$ if it satisfies*

$$(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) + \beta z \partial_q^2 f(z) \prec E(z), \tag{9}$$

and

$$(1-\alpha)\frac{g(\omega)}{\omega}+\alpha\partial_q g(\omega)+\beta\omega\partial_q^2 g(\omega)\prec E(\omega), \tag{10}$$

where $g=f^{-1}$ is given by (5) and $z,\omega\in U$.

Lemma 1 ([25,26])**.** *Let $\alpha,\beta\in\mathbb{R}$ and $p_1,p_2\in\mathbb{C}$. If $|p_1|,|p_2|<\zeta$, then*

$$\left|(\alpha+\beta)p_1+(\alpha-\beta)p_2\right|\le\begin{cases}2|\alpha|\zeta, & |\alpha|\ge|\beta|,\\ 2|\beta|\zeta, & |\alpha|\le|\beta|.\end{cases}$$

Lemma 2 ([21])**.** *Suppose that $\chi(z)$ is analytic in the unit open disc U with $\chi(0)=0$, $|\chi(z)|<1$, and that*

$$\chi(z)=\rho_1 z+\sum_{n=2}^{\infty}\rho_n z^n\quad\text{for all}\quad z\in U. \tag{11}$$

Then,

$$|\rho_1|\le 1,\qquad\text{and}\qquad |\rho_n|\le 1-|\rho_1|^2\qquad(n\in\mathbb{N}\backslash\{1\}). \tag{12}$$

2. Main Results

Unless otherwise mentioned, we assume in the reminder of this article that $0\le\delta\le1$, $0\le\lambda\le1$, $0\le\alpha\le1$, $0\le\beta\le1$, $q\in(0,1)$, and also $z,\omega\in U$.

Theorem 1. *Let $f\in\mathcal{M}_\Sigma(\delta,\lambda;E)$, then*

$$|a_2|\le\frac{1}{\sqrt{\left|K_2+K_4-\frac{2}{3}K_1^2\right|+4K_1^2}}, \tag{13}$$

$$|a_3|\le\frac{|K_2|+|K_4|}{2K_3|K_2+K_4|}, \tag{14}$$

where

$$\left.\begin{aligned}K_1&=(1-\delta)([2]_q+\lambda-1)+\delta[2]_q([2]_q-1),\\ K_2&=(1-\delta)(\lambda-1)([2]_q+\tfrac{\lambda}{2}-1)-\delta[2]_q([2]_q-1),\\ K_3&=(1-\delta)([3]_q+\lambda-1)+\delta[3]_q([3]_q-1),\\ K_4&=(1-\delta)(\lambda-1)([2]_q+\tfrac{\lambda}{2}+1)-\delta[2]_q^2([2]_q-1)+2(1-\delta)[3]_q+2\delta[3]_q([3]_q-1).\end{aligned}\right\} \tag{15}$$

Proof. Let f and g be in $\mathcal{M}_\Sigma(\delta,\lambda;E)$, then, it satisfies the conditions (7) and (8). However, according to subordination principle Definition 1 and Lemma 2, there exist two Schwarz functions $u(z)$ and $v(\omega)$ of the form

$$u(z)=\sum_{n=1}^{\infty}c_n z^n,\quad\text{and}\quad v(\omega)=\sum_{n=1}^{\infty}d_n\omega^n,$$

such that

$$(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda}\partial_q f(z)+\delta\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}=E(u(z)), \tag{16}$$

and

$$(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda}\partial_q g(\omega)+\delta\frac{\partial_q(\omega\partial_q g(\omega))}{\partial_q g(\omega)}=E(v(\omega)). \tag{17}$$

After some simple calculations, we deduce

$$\begin{aligned} E(u(z)) &= \frac{u(z)e^{u}(z)}{e^{u(z)}-1} \\ &= 1+\frac{u(z)}{2}+\frac{(u(z))^2}{12}-\frac{(u(z))^4}{720}+\cdots \\ &= 1+\frac{c_1}{2}z+\frac{1}{2}\left(c_2+\frac{c_1^2}{6}\right)z^2+\cdots, \end{aligned} \tag{18}$$

$$\begin{aligned} E(v(\omega)) &= \frac{v(\omega)e^{v}(\omega)}{e^{v(\omega)}-1} \\ &= 1+\frac{v(\omega)}{2}+\frac{(v(\omega))^2}{12}-\frac{(v(\omega))^4}{720}+\cdots \\ &= 1+\frac{d_1}{2}\omega+\frac{1}{2}\left(d_2+\frac{d_1^2}{6}\right)\omega^2+\cdots. \end{aligned} \tag{19}$$

Moreover,

$$(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda}\partial_q f(z)+\delta\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}=1+K_1a_2z+\left(K_3a_3+K_2a_2^2\right)z^2+\cdots \tag{20}$$

$$(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda}\partial_q g(\omega)+\delta\frac{\partial_q(\omega\partial_q g(\omega))}{\partial_q g(\omega)}=1-K_1a_2\omega+\left(K_4a_2^2-K_3a_3\right)\omega^2+\cdots, \tag{21}$$

where $K_j : j=1,2,3,4$ are stated in (15).

By substituting from (20), (21), (18) and (19) into (16) and (17), and by comparing the coefficients on both sides, we obtain

$$K_1a_2=\frac{c_1}{2}, \tag{22}$$

$$K_3a_3+K_2a_2^2=\frac{1}{2}\left(c_2+\frac{c_1^2}{6}\right), \tag{23}$$

$$-K_1a_2=\frac{d_1}{2}, \tag{24}$$

$$-K_3a_3+K_4a_2^2=\frac{1}{2}\left(d_2+\frac{d_1^2}{6}\right). \tag{25}$$

As a direct result of Equations (22) and (23), we get

$$c_1=-d_1, \tag{26}$$

and also,

$$c_1^2+d_1^2=8K_1^2a_2^2. \tag{27}$$

By adding (23) to (25) and then using (27), we obtain

$$\left(K_2+K_4-\frac{2}{3}K_1^2\right)a_2^2=\frac{1}{2}(c_2+d_2). \tag{28}$$

Equations (26) and (28) together with using Lemma 2 imply that

$$\left|K_2+K_4-\frac{2}{3}K_1^2\right||a_2|^2\le 1-|c_1|^2. \tag{29}$$

However, from Equation (22), we can deduce

$$|c_1|^2 = 4K_1^2|a_2|^2. \tag{30}$$

By using (30) into (29), we obtain

$$|a_2| \leq \frac{1}{\sqrt{\left|K_2 + K_4 - \frac{2}{3}K_1^2\right| + 4K_1^2}}. \tag{31}$$

Further, from (23) and (25) and also using (26), we get

$$K_3(K_2 + K_4)a_3 = \frac{1}{2}\left(c_2K_4 - K_2d_2 + \frac{c_1^2}{6}(K_4 - K_2)\right). \tag{32}$$

Thus, by virtue of Lemma 2, we find

$$K_3|K_2 + K_4||a_3| \leq \frac{1}{2}\left(|K_2| + |K_4| + |c_1|^2\left(\frac{|K_4 - K_2|}{6} - |K_2| - |K_4|\right)\right). \tag{33}$$

On the other hand, from the properties of the modulus, the term $\frac{|K_4-K_2|}{6} - |K_2| - |K_4| \leq 0$. Then, we conclude

$$|a_3| \leq \frac{|K_2| + |K_4|}{2K_3|K_2 + K_4|}. \tag{34}$$

Thus, the proof is completed. □

Theorem 2. *Let $f \in \mathcal{N}_\Sigma(\alpha, \beta; E)$, then*

$$|a_2| \leq \frac{1}{\sqrt{2|\Upsilon(\alpha,\beta;q)| + 4\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}}, \tag{35}$$

$$|a_3| \leq \begin{cases} \dfrac{1}{2\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)}, & \dfrac{2\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}{1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q} \geq 1, \\ \dfrac{\Upsilon(\alpha,\beta;q)+1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q}{2\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)\left(\Upsilon(\alpha,\beta;q)+2\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2\right)}, & \dfrac{2\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}{1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q} \leq 1, \end{cases} \tag{36}$$

where

$$\Upsilon(\alpha,\beta;q) = 1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q - \frac{1}{3}\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2. \tag{37}$$

Proof. Suppose f and g are in $\mathcal{N}_\Sigma(\alpha, \beta; E)$, then they satisfy the conditions (7) and (8). According to subordination principle Definition 1 and Lemma 2, there exist two Schwarz functions $u(z)$ and $v(\omega)$ of the form

$$u(z) = \sum_{n=1}^{\infty} c_n z^n, \quad \text{and} \quad v(\omega) = \sum_{n=1}^{\infty} d_n \omega^n,$$

such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) + \beta z\partial_q^2 f(z) = E(u(z)), \tag{38}$$

and

$$(1-\alpha)\frac{g(\omega)}{\omega} + \alpha\partial_q g(\omega) + \beta\omega\partial_q^2 g(\omega) = E(v(\omega)). \tag{39}$$

With some simple calculations, we get

$$(1-\alpha)\frac{f(z)}{z}+\alpha\partial_q f(z)+\beta z\partial_q^2 f(z)=1+\sum_{n=2}^{\infty}\Big(1+\alpha\Big([n]_q-1\Big)+\beta[n-1]_q[n]_q\Big)a_n z^n \tag{40}$$

and

$$(1-\alpha)\frac{g(\omega)}{\omega}+\alpha\partial_q g(\omega)+\beta\omega\partial_q^2 g(\omega)=1-\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)a_2\omega+\Big(1+\alpha\Big([3]_q-1\Big)+\beta[2]_q[3]_q\Big)a_2\omega^2+\cdots. \tag{41}$$

By substituting from (18), (19), (40) and (41) into (38) and (39) as well as by comparing the coefficients on both sides, we conclude

$$\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)a_2=\frac{c_1}{2}, \tag{42}$$

$$\Big(1+\alpha\Big([3]_q-1\Big)+\beta[2]_q[3]_q\Big)a_3=\frac{1}{2}\left(c_2+\frac{c_1^2}{6}\right), \tag{43}$$

$$-\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)a_2=\frac{d_1}{2}, \tag{44}$$

and

$$\Big(1+\alpha\Big([3]_q-1\Big)+\beta[2]_q[3]_q\Big)(2a_2^2-a_3)=\frac{1}{2}\left(d_2+\frac{d_1^2}{6}\right). \tag{45}$$

From (42) and (44), we obtain

$$c_1=-d_1, \tag{46}$$

and also,

$$c_1^2+d_1^2=8\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)^2a_2^2. \tag{47}$$

By adding (43) to (45) with using (47), we get

$$2\Upsilon(\alpha,\beta;q)a_2^2=\frac{1}{2}(c_2+d_2). \tag{48}$$

In view of Lemma 2, Equation (48) together with (46) imply that

$$2\Big|\Upsilon(\alpha,\beta;q)\Big||a_2|^2\leq 1-|c_1|^2. \tag{49}$$

On the other hand, from Equation (42), we can write

$$|c_1|^2=4\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)^2|a_2|^2. \tag{50}$$

By using (50) in (49), we get

$$|a_2|\leq\frac{1}{\sqrt{2|\Upsilon(\alpha,\beta;q)|+4\Big(1+\alpha\Big([2]_q-1\Big)+\beta[2]_q\Big)^2}}, \tag{51}$$

where $\Upsilon(\alpha,\beta;q)$ is defined in (37).

Further, by subtracting (45) from (43) and using (46), we have

$$a_3=a_2^2+\frac{c_2-d_2}{4\Big(1+\alpha\Big([3]_q-1\Big)+\beta[2]_q[3]_q\Big)}. \tag{52}$$

In view of Lemma 2, Equation (52) together with (50) imply that

$$|a_3| \leq \left(1 - \frac{2\left(1+\alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}{1+\alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q}\right)|a_2|^2 + \frac{1}{2\left(1+\alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}. \tag{53}$$

By the virtue of (51), we can get the desired result. Thus, we completed the proof. □

Theorem 3. *Suppose $f \in \mathcal{M}_\Sigma(\delta, \lambda; E)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2K_3}\begin{cases} \left|\Phi(\mu,\delta,\lambda;q)\right|, & \left|\Phi(\mu,\delta,\lambda;q)\right| \geq 1, \\ 1, & \left|\Phi(\mu,\delta,\lambda;q)\right| \leq 1, \end{cases} \tag{54}$$

where

$$\Phi(\mu,\delta,\lambda;q) = \frac{K_4 - K_2 - 2\mu K_3}{K_2 + K_4 - \frac{2}{3}K_1^2}, \tag{55}$$

and K_1, K_2, K_3, and K_4 are given by (15).

Proof. To investigate the desired result, subtract (25) from (23) by using (26), we get

$$a_3 = \frac{K_4 - k - 2}{2K_3}a_2^2 + \frac{(c_2 - d_2)}{4K_3}. \tag{56}$$

Thus,

$$a_3 - \mu a_2^2 = \left(\frac{K_4 - K_2 - 2\mu K_3}{2K_3}\right)a_2^2 + \frac{(c_2 - d_2)}{4K_3}. \tag{57}$$

As a result of subsequent computations performed by using (28), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{4K_3}\left|(\Phi(\mu,\delta,\lambda;q) + 1)c_2 + (\Phi(\mu,\delta,\lambda;q) - 1)d_2\right|, \tag{58}$$

where $\Phi(\mu,\delta,\lambda;q)$ is given by (55).

However, in view of Kanas et al. [27] and (12), we can obtain

$$|c_2| \leq 1 - |c_1|^2 \leq 1 \text{ and also } |d_2| \leq 1 - |d_1|^2 \leq 1. \tag{59}$$

Now, applying Lemma 1 to (58), we can obtain the desired result directly. Thus, we completed the proof. □

Theorem 4. *Let us consider $f \in \mathcal{N}_\Sigma(\alpha, \beta; E)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \left|\frac{1-\mu}{2\Upsilon(\alpha,\beta;q)}\right|, & \left|\frac{1-\mu}{\Upsilon(\alpha,\beta;q)}\right| \geq \frac{1}{1+\alpha([3]_q-1)+\beta[2]_q[3]_q}, \\ \frac{1}{2\left(1+\alpha([3]_q-1)+\beta[2]_q[3]_q\right)}, & \left|\frac{1-\mu}{\Upsilon(\alpha,\beta;q)}\right| \leq \frac{1}{1+\alpha([3]_q-1)+\beta[2]_q[3]_q}, \end{cases} \tag{60}$$

where $\Upsilon(\alpha, \beta; q)$ is defined by (37).

Proof. In order to investigate the desired result (60), subtract (45) from (43) taking in consideration (46), we conclude

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + \frac{c_2 - d_2}{4\left(1+\alpha([3]_q - 1) + \beta[2]_q[3]_q\right)}. \tag{61}$$

By virtue of (48), we can get that

$$a_3 - \mu a_2^2 = c_2\left(\frac{1-\mu}{4\Upsilon(\alpha,\beta;q)} + \frac{1}{4\left(1+\alpha([3]_q-1)+\beta[2]_q[3]_q\right)}\right) + d_2\left(\frac{1-\mu}{4\Upsilon(\alpha,\beta;q)} - \frac{1}{4\left(1+\alpha([3]_q-1)+\beta[2]_q[3]_q\right)}\right). \quad (62)$$

By applying Lemma 1 to (62) and using (59), we obtain the required result which completes the proof. □

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