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Existence and Well-Posedness of Tripled Fixed Points with Application to a System of Differential Equations

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Abstract: The purpose of this manuscript is to demonstrate the existence and uniqueness of triple fixed-point results for Geraghty-type contractions in ordinary metric spaces with binary relations. Moreover, the well-posedness of the tripled fixed point problem is investigated. Consequently, α-dominated mapping on such space is discussed. Ultimately, to promote our paper, some illustrative examples are derived, and the existence of the solution to a system of differential equations is obtained as an application.

Keywords: tripled fixed point; metric space; well-posedness; differential equation; integral equation

MSC: 47H10; 34B15; 54H25

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1. Introduction

Fixed-point techniques in complete metric spaces (CMSs) became popular in 1922 after Banach presented their principle [1]. This technique has particular resonance in many important disciplines, such as topology, dynamical systems, differential and integral equations, economics, game theory, biological sciences, computer science and chemistry [2,3]. As of the importance of this approach, it became the main controller in the study of the existence and uniqueness of the solution to many differential and integral equations [4–6].

Bhaskar and Lakshmikantham [7] introduced the concept of mixed monotone maps and coupled fixed points. Consequently, many authors established coupled fixed point results for contractive mappings under suitable conditions in partially ordered metric spaces (POMSs) with some important applications. For more details, see [8,9].

In 2011, Berinde and Borcut [10] initiated the notion of triple fixed points (TFPs) and established some TFP results for contractive mappings in POMSs. Afterward, many investigators established TFP theorems for contraction mappings in various spaces. For more contributions in this regard, see [11–15].

In 1973, Geraghty [16] extended the Banach contraction principle [1] by replacing a contraction coefficient with a function satisfying certain conditions. Later, the results of [16] were extended by Hamini and Emami [5] in POMSs. In 2013, the idea of α-Geraghty contraction type mappings and some nice fixed point consequences were established in a CMS by Cho et al. [17]. As many scholars are interested in this regard, it is sufficient to mention [18,19].

In 1989, Blassi and Myjak [20] defined another approach to fixed points, the so-called well-posedness of a fixed point problem. The concept of well-posedness of a fixed point problem for a single valued mapping has evoked much interest for several authors, see [21,22].

In this manuscript, the existence and uniqueness of a TFP are established for Geraghty-type contraction mappings under appropriate conditions. Furthermore, an example is given to support our results. Moreover, the well-posedness of a TFP problem and $\alpha-$dominated mappings are obtained. As an application, the existence solution to a system of differential equations is derived.

2. Preliminaries

In this section, we present some notations and basic definitions that are useful in the sequel. Assume that Ω and Ψ are two non-empty sets and $\Re$ is a relation from Ω to Ψ, i.e., $\Re \subseteq \Omega \times \Psi$. Here, the pair $(\eta, \varkappa) \in \Re$ or $\eta \Re \varkappa$ means η is $\Re$ related to $\varkappa$, the domain of $\Re$ defined by the set $D = \{\eta \in \Omega : (\eta, \varkappa) \in \Re$ for some $\varkappa \in \Psi\}$, the range of $\Re$ defined by the set $G = \{\varkappa \in \Psi : (\eta, \varkappa) \in \Re$ for some $\eta \in \Omega\}$ and the inverse of $\Re$ is $\Re^{-1}$, which is defined by $\{(\varkappa, \eta) : (\eta, \varkappa) \in \Re\}$.

A relation $\Re$ from Ω to Ω is said to be a relation on Ω. Suppose that $\Re$ is a relation on Ω. The relation $\Re$ is called directed if for given $\eta, \varkappa \in \Omega$ there is $\varrho \in \Omega$ so that $(\eta, \varrho) \in \Re$ and $(\varkappa, \varrho) \in \Re$. If the relation $\Re$ is reflexive, anti-symmetric and transitive, then it is called a partial order relation on Ω.

Definition 1. *Assume that $(\Omega, \eth)$ is a CMS with a binary relation $\Re$ on it. We say that Ω has $\Re-$regular property if for each sequence $\{\eta_\varpi\} \in \Omega$ converging to $\eta \in \Omega$ with $(\eta_\varpi, \eta_{\varpi+1}) \in \Re$, we have $(\eta_\varpi, \eta) \in \Re$, for each ϖ or if $(\eta_{\varpi+1}, \eta_\varpi) \in \Re$, then $(\eta, \eta_\varpi) \in \Re$, for each ϖ.*

Definition 2 ([10])**.** *Let Ω be a non-empty set. A trio $(\eta, \varkappa, \varrho) \in \Omega^3$ is said to be a TFP of the mapping $r : \Omega^3 \to \Omega$ if $\eta = r(\eta, \varkappa, \varrho)$, $\varkappa = r(\varkappa, \varrho, \eta)$ and $\varrho = r(\varrho, \eta, \varkappa)$.*

Example 1. *Assume that $\Omega = [0, \infty)$ and $r : \Omega^3 \to \Omega$ is a mapping described as*

$$r(\eta, \varkappa, \varrho) = \frac{\eta + \varkappa + \varrho}{3}, \text{ for all } \eta, \varkappa, \varrho \in \Omega.$$

Then, there is a unique TFP of r, whenever $\eta = \varkappa = \varrho$.

Now, in order to achieve the goal of this paper, we present the following auxiliary functions.

Assume that Θ is a class of all functions $\zeta : [0, \infty) \to [0, 1)$ so that, for each sequence $\{\ell_\varpi\} \in [0, \infty)$, $\lim_{\varpi \to +\infty} \zeta(\ell_\varpi) = 1$ implies $\lim_{\varpi \to +\infty} \ell_\varpi = 0$.

Let Θ^* be a class of all functions $\Im : [0, \infty) \to [0, 1)$ so that, for any sequence $\{\ell_\varpi\} \in [0, \infty)$, $\limsup_{\varpi \to +\infty} \Im(\ell_\varpi) = 1$ implies $\lim_{\varpi \to +\infty} \ell_\varpi = 0$.

Clearly, the class Θ^* is more extended than Θ, that is Θ^* contains Θ.

The example below illustrates this containment.

Example 2. *Let $\Im : [0, \infty) \to [0, 1)$ be a function defined by*

$$\Im(\ell_\varpi) = \left\{ (1 + (-1)^\varpi + \ell_\varpi)^{\frac{1}{\varpi}} : \ell_\varpi = \left(\frac{1}{2^\varpi}\right), \text{ for all } \varpi \in \mathbb{N} \right\}.$$

It is clear that $\ell_\varpi \in [0, \infty)$ and $\lim_{\varpi \to +\infty} \Im(\ell_\varpi)$ does not exist (the value of the limit is not unique). So $\Im \notin \Theta$. The sequence $\{\Im(\ell_\varpi)\}_{\varpi \in \mathbb{N}}$ is bounded and has two subsequences $\{\frac{1}{2}\}$ and $\{2 + \frac{1}{2^\varpi}\}^{\frac{1}{\varpi}}$. Thus, the limits are $\frac{1}{2}$ and 1. Therefore, $\liminf_{\varpi \to +\infty} \Im(\ell_\varpi) = \frac{1}{2}$ and $\limsup_{\varpi \to +\infty} \Im(\ell_\varpi) = 1$, whenever $\lim_{\varpi \to +\infty} \ell_\varpi = 0$. Hence, $\Im \in \Theta^*$.

3. Main Results

In this part, we obtain some TFP results under certain conditions. Furthermore, illustrative examples are given to support the theoretical results. Now, we begin with the following definition:

Definition 3. *Suppose that $\Omega \neq \emptyset$ with a binary relation $\Re$ on it. A mapping $r : \Omega^3 \to \Omega$ is called an $\Re-$dominated mapping if for all $(\eta, \varkappa, \varrho) \in \Omega^3$,*

$$(\eta, r(\eta, \varkappa, \varrho)) \in \Re,\ (r(\varkappa, \varrho, \eta), \varkappa) \in \Re \text{ and } (\varrho, r(\varrho, \eta, \varkappa)) \in \Re.$$

Example 3. *Suppose that $\Omega = [0, 1]$ and the mapping $r : \Omega^3 \to \Omega$ is defined by*

$$r(\eta, \varkappa, \varrho) = \frac{\eta + \varkappa + \varrho}{9 + \eta + \varkappa + \varrho}, \text{for all } \eta, \varkappa, \varrho \in \Omega.$$

Assume that a binary relation $\Re$ on Ω is described by

$$\begin{aligned} \Re \;=\; & \left\{((\eta, \varkappa, \varrho), (\mu, \nu, \vartheta)) : \left[\left(0 \leq \eta \leq 1;\ 0 \leq \mu \leq \frac{1}{3}\right) \text{ or } \left(0 \leq \eta \leq \frac{1}{3};\ 0 \leq \mu \leq 1\right)\right],\right. \\ & \left[\left(0 \leq \varkappa \leq 1;\ 0 \leq \nu \leq \frac{1}{3}\right) \text{ or } \left(0 \leq \varkappa \leq \frac{1}{3};\ 0 \leq \nu \leq 1\right)\right], \\ & \left.\left[\left(0 \leq \varrho \leq 1;\ 0 \leq \vartheta \leq \frac{1}{3}\right) \text{ or } \left(0 \leq \varrho \leq \frac{1}{3};\ 0 \leq \vartheta \leq 1\right)\right]\right\}. \end{aligned}$$

Thus,

$$r(\eta, \varkappa, \varrho) = r(\varkappa, \varrho, \eta) = r(\varrho, \eta, \varkappa) \in \left[0, \frac{1}{3}\right], \text{ for all } \eta, \varkappa, \varrho \in \Omega.$$

It follows that

$$(\eta, r(\eta, \varkappa, \varrho)) \in \Re,\ (r(\varkappa, \varrho, \eta), \varkappa) \in \Re \text{ and } (\varrho, r(\varrho, \eta, \varkappa)) \in \Re, \text{ for all } (\eta, \varkappa, \varrho) \in \Omega^3.$$

Hence, r is an $\Re-$dominated mapping.

Problem (W) : Suppose that (Ω, ∂) is a metric space. We consider the problem of finding a TFP $(\eta, \varkappa, \varrho) \in \Omega^3$ of the mapping $r : \Omega^3 \to \Omega$, so that

$$\eta = r(\eta, \varkappa, \varrho),\ \varkappa = r(\varkappa, \varrho, \eta) \text{ and } \varrho = r(\varrho, \eta, \varkappa). \tag{1}$$

Definition 4. *The problem (W) is said to be well-posed if the following conditions hold:*

(W_1) *the point $(\eta^*, \varkappa^*, \varrho^*)$ is a TFP of r;*
(W_2) *$\eta_\varpi \to \eta^*$, $\varkappa_\varpi \to \varkappa^*$ and $\varrho_\varpi \to \varrho^*$ as $\varpi \to \infty$, whenever $\{(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)\}$ is any sequence in Ω^3 such that $\limsup_{\varpi \to +\infty}[\partial(\eta_\varpi, \eta^*) + \partial(\varkappa_\varpi, \varkappa^*) + \partial(\varrho_\varpi, \varrho^*)]$ is finite and*

$$\lim_{\varpi \to +\infty} \partial(\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) = \lim_{\varpi \to +\infty} \partial(\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)) = \lim_{\varpi \to +\infty} \partial(\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)) = 0.$$

Now, we establish our first main result as follows:

Theorem 1. *Assume that (Ω, ∂) is a CMS with a transitive relation $\Re$ on it so that Ω has $\Re-$regular property. Let $r : \Omega^3 \to \Omega$ be the $\Re-$dominated mapping and there is $\Im \in \Theta^*$. If for $(\eta, \varkappa, \varrho), (\mu, \nu, \vartheta), (a, b, c) \in \Omega^3$ so that $((\eta, \mu, a), (\varkappa, \nu, b), (\varrho, \vartheta, c) \in \Omega^3)$ or $((\mu, a, \eta), (\nu, b, \varkappa), (\vartheta, c, \varrho) \in \Omega^3)$ or $((a, \eta, \mu), (b, \varkappa, \nu), (c, \varrho, \vartheta) \in \Omega^3)$, and the inequality below holds*

$$\partial(r(\eta, \varkappa, \varrho), r(\mu, \nu, \vartheta)) \leq \Im(\aleph(\eta, \varkappa, \varrho, \mu, \nu, \vartheta))\aleph(\eta, \varkappa, \varrho, \mu, \nu, \vartheta), \tag{2}$$

where

$$\aleph(\eta, \varkappa, \varrho, \mu, \nu, \vartheta) = \left(\frac{\partial(\eta, \mu) + \partial(\varkappa, \nu) + \partial(\varrho, \vartheta)}{3}\right).$$

Then, r has a TFP.

Proof. Let $(\eta_0, \varkappa_0, \varrho_0) \in \Omega^3$ be an arbitrary point. Define three sequences $\{\eta_\varpi\}$, $\{\varkappa_\varpi\}$ and $\{\varrho_\varpi\}$ in Ω by

$$\begin{aligned} \eta_{\varpi+1} &= r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi), \\ \varkappa_{\varpi+1} &= r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi) \\ \text{and } \varrho_{\varpi+1} &= r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi), \text{ for all } \varpi \geq 0. \end{aligned} \tag{3}$$

As r is $\Re-$dominated, we find

$$\begin{aligned} (\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) &= (\eta_\varpi, \eta_{\varpi+1}) \in \Re, \\ (\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)) &= (\varkappa_\varpi, \varkappa_{\varpi+1}) \in \Re \\ \text{and } (\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)) &= (\varrho_\varpi, \varrho_{\varpi+1}) \in \Re, \text{ for all } \varpi \geq 0. \end{aligned} \tag{4}$$

Let

$$\delta_\varpi = \eth(\eta_\varpi, \eta_{\varpi+1}) + \eth(\varkappa_\varpi, \varkappa_{\varpi+1}) + \eth(\varrho_\varpi, \varrho_{\varpi+1}), \text{ for all } \varpi \geq 0. \tag{5}$$

From (2)–(5), we obtain

$$\begin{aligned} \eth(\eta_{\varpi+1}, \eta_{\varpi+2}) &= \eth(r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi), r(\eta_{\varpi+1}, \varkappa_{\varpi+1}, \varrho_{\varpi+1})) \\ &\leq \Im(\aleph(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi, \eta_{\varpi+1}, \varkappa_{\varpi+1}, \varrho_{\varpi+1})) \\ &\quad \times \aleph(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi, \eta_{\varpi+1}, \varkappa_{\varpi+1}, \varrho_{\varpi+1}), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \aleph(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi, \eta_{\varpi+1}, \varkappa_{\varpi+1}, \varrho_{\varpi+1}) &= \Im\left(\frac{\eth(\eta_\varpi, \eta_{\varpi+1}) + \eth(\varkappa_\varpi, \varkappa_{\varpi+1}) + \eth(\varrho_\varpi, \varrho_{\varpi+1})}{3}\right) \\ &= \Im\left(\frac{\delta_\varpi}{3}\right). \end{aligned} \tag{7}$$

Applying (6) in (7), we have

$$\eth(\eta_{\varpi+1}, \eta_{\varpi+2}) \leq \Im\left(\frac{\delta_\varpi}{3}\right)\frac{\delta_\varpi}{3}. \tag{8}$$

Analogously, we can write

$$\eth(\varkappa_{\varpi+1}, \varkappa_{\varpi+2}) \leq \Im\left(\frac{\delta_\varpi}{3}\right)\frac{\delta_\varpi}{3} \tag{9}$$

and

$$\eth(\varrho_{\varpi+1}, \varrho_{\varpi+2}) \leq \Im\left(\frac{\delta_\varpi}{3}\right)\frac{\delta_\varpi}{3}. \tag{10}$$

By adding (8)–(10), we have

$$\begin{aligned} \delta_{\varpi+1} &= \eth(\eta_{\varpi+1}, \eta_{\varpi+2}) + \eth(\varkappa_{\varpi+1}, \varkappa_{\varpi+2}) + \eth(\varrho_{\varpi+1}, \varrho_{\varpi+2}) \\ &\leq \Im\left(\frac{\delta_\varpi}{3}\right)\delta_\varpi. \end{aligned} \tag{11}$$

Assume that $0 \leq \delta_\varpi < \delta_{\varpi+1}$. Thus, by (11), one obtains

$$\delta_{\varpi+1} \leq \Im\left(\frac{\delta_\varpi}{3}\right)\delta_\varpi \leq \delta_\varpi < \delta_{\varpi+1},$$

a contradiction. Thus, $\delta_{\varpi+1} \leq \delta_{\varpi}$, for each $\varpi \geq 0$, which means that a sequence of positive real numbers $\{\delta_{\varpi}\}$ is decreasing. Hence, there is $\delta \geq 0$ so that $\lim_{\varpi \to +\infty} \delta_{\varpi} = \delta$. Based on (11), we obtain

$$\delta_{\varpi+1} \leq \Im\left(\frac{\delta_{\varpi}}{3}\right)\delta_{\varpi}, \text{ for all } \varpi \geq 0. \tag{12}$$

If possible, suppose that $\delta > 0$. Taking the limit supremum on both sides of (12), we have

$$\delta \leq \limsup_{\varpi \to +\infty} \Im\left(\frac{\delta_{\varpi}}{3}\right)\delta_{\varpi},$$

which leads to

$$1 \leq \limsup_{\varpi \to +\infty} \Im\left(\frac{\delta_{\varpi}}{3}\right) \leq 1,$$

which implies that

$$\limsup_{\varpi \to +\infty} \Im\left(\frac{\delta_{\varpi}}{3}\right) = 1.$$

Using the property of $\Im$, we have

$$\lim_{\varpi \to +\infty} \frac{\delta_{\varpi}}{3} = \frac{\delta}{3} = 0.$$

Hence, $\delta = 0$, which is contrary to our assumption. Based on the foregoing, we can write $\delta = 0$ and

$$\begin{aligned}
& \lim_{\varpi \to +\infty} [\eth(\eta_{\varpi}, \eta_{\varpi+1}) + \eth(\varkappa_{\varpi}, \varkappa_{\varpi+1}) + \eth(\varrho_{\varpi}, \varrho_{\varpi+1})] \\
= & \lim_{\varpi \to +\infty} \eth(\eta_{\varpi}, \eta_{\varpi+1}) = \lim_{\varpi \to +\infty} \eth(\varkappa_{\varpi}, \varkappa_{\varpi+1}) = \lim_{\varpi \to +\infty} \eth(\varrho_{\varpi}, \varrho_{\varpi+1}) \\
= & 0.
\end{aligned} \tag{13}$$

Now, we shall show that $\{\eta_{\varpi}\}$, $\{\varkappa_{\varpi}\}$ and $\{\varrho_{\varpi}\}$ are Cauchy sequences. Assume on the contrary that either $\{\eta_{\varpi}\}$ or $\{\varkappa_{\varpi}\}$ or $\{\varrho_{\varpi}\}$ is not a Cauchy sequence. Then, either

$$\begin{aligned}
\lim_{\wp,\, \varpi \to +\infty} \eth(\eta_{\wp}, \eta_{\varpi}) &\neq 0 \text{ or} \\
\lim_{\wp,\, \varpi \to +\infty} \eth(\varkappa_{\wp}, \varkappa_{\varpi}) &\neq 0 \text{ or} \\
\lim_{\wp,\, \varpi \to +\infty} \eth(\varrho_{\wp}, \varrho_{\varpi}) &\neq 0.
\end{aligned}$$

Therefore,

$$\lim_{\wp,\, \varpi \to +\infty} [\eth(\eta_{\wp}, \eta_{\varpi}) + \eth(\varkappa_{\wp}, \varkappa_{\varpi}) + \eth(\varrho_{\wp}, \varrho_{\varpi})] \neq 0,$$

which implies that, for each $\varepsilon > 0$, we can find subsequences $\left\{\varpi_{(\beta)}\right\}$ and $\left\{\wp_{(\beta)}\right\}$ of positive integer with $\varpi_{(\beta)} > \wp_{(\beta)} > \beta$ so that, for every $\beta > 0$,

$$\eth\left(\eta_{\wp_{(\beta)}}, \eta_{\varpi_{(\beta)}}\right) + \eth\left(\varkappa_{\wp_{(\beta)}}, \varkappa_{\varpi_{(\beta)}}\right) + \eth\left(\varrho_{\wp_{(\beta)}}, \varrho_{\varpi_{(\beta)}}\right) \geq \varepsilon \tag{14}$$

and

$$\eth\left(\eta_{\wp_{(\beta)}}, \eta_{\varpi_{(\beta)}-1}\right) + \eth\left(\varkappa_{\wp_{(\beta)}}, \varkappa_{\varpi_{(\beta)}-1}\right) + \eth\left(\varrho_{\wp_{(\beta)}}, \varrho_{\varpi_{(\beta)}-1}\right) < \varepsilon. \tag{15}$$

Consider

$$\begin{aligned}\varepsilon &\leq \eth\left(\eta_{\varpi_{(\beta)}},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\wp_{(\beta)}}\right)\\ &\leq \left[\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\varpi_{(\beta)}-1}\right)\right]\\ &\quad+\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}}\right)\right]\\ &< \eth\left(\eta_{\varpi_{(\beta)}},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\varpi_{(\beta)}-1}\right)+\varepsilon.\end{aligned}$$

As $\beta\to+\infty$ in the above inequality and by (13), we find

$$\lim_{\beta\to+\infty}\left[\eth\left(\eta_{\wp_{(\beta)}},\eta_{\varpi_{(\beta)}}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\varpi_{(\beta)}}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\varpi_{(\beta)}}\right)\right]=\varepsilon. \tag{16}$$

In addition,

$$\begin{aligned}&\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\\ \leq&\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}}\right)\right]\\ &+\left[\eth\left(\eta_{\wp_{(\beta)}},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\wp_{(\beta)}-1}\right)\right]\\ <&\ \eth\left(\eta_{\wp_{(\beta)}},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\wp_{(\beta)}-1}\right)+\varepsilon.\end{aligned} \tag{17}$$

Again,

$$\begin{aligned}&\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\wp_{(\beta)}}\right)\\ \leq&\left[\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\varpi_{(\beta)}-1}\right)\right]\\ &+\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\right]\\ &+\left[\eth\left(\eta_{\wp_{(\beta)}},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\wp_{(\beta)}-1}\right)\right],\end{aligned}$$

which leads to

$$\begin{aligned}&\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\wp_{(\beta)}}\right)\\ &-\left[\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\varpi_{(\beta)}-1}\right)\right]\\ &-\left[\eth\left(\eta_{\wp_{(\beta)}},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\wp_{(\beta)}-1}\right)\right]\\ \leq&\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\right]\\ <&\ \eth\left(\eta_{\wp_{(\beta)}},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\wp_{(\beta)}-1}\right)+\varepsilon.\end{aligned} \tag{18}$$

Taking the limit as $\beta\to+\infty$ in (17) and (18) and by (13) and (16), we obtain

$$\lim_{\beta\to+\infty}\left[\eth\left(\eta_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right]=\varepsilon. \tag{19}$$

Hence,

$$\begin{aligned}
&\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\wp_{(\beta)}}\right)\\
\leq\;&\left[\eth\left(\eta_{\varpi_{(\beta)}},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}},\varrho_{\varpi_{(\beta)}-1}\right)\right]\\
&+\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\right]\\
\leq\;&\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\right]\\
&+\left[\eth\left(\eta_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}}\right)\right].
\end{aligned}$$

Once again, letting $\beta\to+\infty$ in the above inequality and using (13), (16) and (19), we conclude that

$$\lim_{\beta\to+\infty}\left[\eth\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}}\right)+\eth\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}}\right)+\eth\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}}\right)\right]=\varepsilon. \tag{20}$$

From (4) and the transitivity hypothesis of $\Re$, one finds

$$\left(\eta_{\varpi_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right)\in\Re,\;\left(\varkappa_{\varpi_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right)\in\Re\text{ and }\left(\varrho_{\varpi_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right)\in\Re.$$

Applying (2), we find

$$\begin{aligned}
\eth\left(\eta_{\wp_{(\beta)}},\eta_{\varpi_{(\beta)}}\right)&=\eth\left(r\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1}\right),r\left(\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\\
&\leq\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\\
&\quad\times\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right),
\end{aligned}\tag{21}$$

where

$$\begin{aligned}
&\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\\
=\;&\frac{\eth\left(\eta_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1}\right)+\eth\left(\varkappa_{\wp_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1}\right)+\eth\left(\varrho_{\wp_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)}{3}.
\end{aligned}\tag{22}$$

Similarly, one can write

$$\begin{aligned}
\eth\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\varpi_{(\beta)}}\right)&=\eth\left(r\left(\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1}\right),r\left(\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1}\right)\right)\\
&\leq\Im\left(\aleph\left(\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1}\right)\right)\\
&\quad\times\aleph\left(\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1}\right)\\
&=\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\\
&\quad\times\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)
\end{aligned}\tag{23}$$

and

$$\begin{aligned}
\eth\left(\varrho_{\wp_{(\beta)}},\varrho_{\varpi_{(\beta)}}\right)&=\eth\left(r\left(\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1}\right),r\left(\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1}\right)\right)\\
&\leq\Im\left(\aleph\left(\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1}\right)\right)\\
&\quad\times\aleph\left(\varrho_{\wp_{(\beta)}-1},\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1}\right)\\
&=\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\\
&\quad\times\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right).
\end{aligned}\tag{24}$$

Combining (21), (23) and (24), we can write

$$\begin{aligned}
&\partial\left(\eta_{\wp_{(\beta)}},\eta_{\varpi_{(\beta)}}\right)+\partial\left(\varkappa_{\wp_{(\beta)}},\varkappa_{\varpi_{(\beta)}}\right)+\partial\left(\varrho_{\wp_{(\beta)}},\varrho_{\varpi_{(\beta)}}\right)\\
\leq\ &3\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\\
&\times\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right).
\end{aligned}\tag{25}$$

When $\beta\to+\infty$ in (22) and by (13), (19) and (20), we have

$$\lim_{\beta\to+\infty}\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)=\frac{\varepsilon}{3}.\tag{26}$$

Taking the limit supremum in (25) and applying (14) and (26), we find

$$\begin{aligned}
\varepsilon\ &\leq\ 3\limsup_{\beta\to+\infty}\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\times\frac{\varepsilon}{3}\\
&=\ \varepsilon\limsup_{\beta\to+\infty}\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right).
\end{aligned}$$

Based on the property of $\Im$, we can write

$$1\leq\limsup_{\beta\to+\infty}\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)\leq 1,$$

which leads to

$$\limsup_{\beta\to+\infty}\Im\left(\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)\right)=1.$$

Again, the property of $\Im$ implies that

$$\lim_{\beta\to+\infty}\aleph\left(\eta_{\wp_{(\beta)}-1},\varkappa_{\wp_{(\beta)}-1},\varrho_{\wp_{(\beta)}-1},\eta_{\varpi_{(\beta)}-1},\varkappa_{\varpi_{(\beta)}-1},\varrho_{\varpi_{(\beta)}-1}\right)=\frac{\varepsilon}{3}=0,$$

that is, $\varepsilon=0$. This contradicts our assumption. Therefore, $\{\eta_{\varpi}\}$, $\{\varkappa_{\varpi}\}$ and $\{\varrho_{\varpi}\}$ are Cauchy sequences in Ω. As Ω is complete, there are η, $\varkappa$, $\varrho\in\Omega$ so that

$$\eta_{\varpi}\to\eta,\ \varkappa_{\varpi}\to\varkappa\text{ and }\varrho_{\varpi}\to\varrho,\text{ as }\varpi\to+\infty.$$

Hence, we can write

$$\lim_{\varpi\to+\infty}\partial(\eta_{\varpi},\eta)=\lim_{\varpi\to+\infty}\partial(\eta_{\varpi},\eta_{\wp})=\lim_{\varpi\to+\infty}\partial(\eta,\eta)=0.\tag{27}$$

Similarly,

$$\lim_{\varpi\to+\infty}\partial(\varkappa_{\varpi},\varkappa)=\lim_{\varpi\to+\infty}\partial(\varkappa_{\varpi},\varkappa_{\wp})=\lim_{\varpi\to+\infty}\partial(\varkappa,\varkappa)=0,\tag{28}$$

and

$$\lim_{\varpi\to+\infty}\partial(\varrho_{\varpi},\varrho)=\lim_{\varpi\to+\infty}\partial(\varrho_{\varpi},\varrho_{\wp})=\lim_{\varpi\to+\infty}\partial(\varrho,\varrho)=0.\tag{29}$$

It follows from (27)–(29) and (2) that

$$\begin{aligned}
&\partial(\eta_{\varpi+1},r(\eta,\varkappa,\varrho))\\
\leq\ &\Im\left(\frac{\partial(\eta_{\varpi},\eta)+\partial(\varkappa_{\varpi},\varkappa)+\partial(\varrho_{\varpi},\varrho)}{3}\right)\\
&\times\left(\frac{\partial(\eta_{\varpi},\eta)+\partial(\varkappa_{\varpi},\varkappa)+\partial(\varrho_{\varpi},\varrho)}{3}\right).
\end{aligned}\tag{30}$$

Taking the limit as $\varpi \to +\infty$, in (30), we have

$$\eth(\eta, r(\eta, \varkappa, \varrho)) \leq 0.$$

Analogously, we obtain

$$\eth(\varkappa, r(\varkappa, \varrho, \eta)) \leq 0 \text{ and } \eth(\varrho, r(\varrho, \eta, \varkappa)) \leq 0,$$

since

$$\eth(\eta, r(\eta, \varkappa, \varrho)) \nless 0,\ \eth(\varkappa, r(\varkappa, \varrho, \eta)) \nless 0 \text{ and } \eth(\varrho, r(\varrho, \eta, \varkappa)) \nless 0.$$

Then, we find

$$\eth(\eta, r(\eta, \varkappa, \varrho)) = 0,\ \eth(\varkappa, r(\varkappa, \varrho, \eta)) = 0 \text{ and } \eth(\varkappa, r(\varkappa, \varrho, \eta)) = 0,$$

which implies that

$$\eta = r(\eta, \varkappa, \varrho),\ \varkappa = r(\varkappa, \varrho, \eta) \text{ and } \varrho = r(\varrho, \eta, \varkappa).$$

Therefore, a trio $(\eta, \varkappa, \varrho)$ is a TFP of r. This finishes the proof. □

The following result is released, if we take $\Re$ as a partially ordered relation:

Corollary 1. *Assume that $(\Omega, \eth)$ is a CMS with a partial order $\preceq$ on it so that $\eth$ has regular property (means if $\{\eta_\varpi\}$ is a monotone convergent sequence with limit η, then $\eta_\varpi \preceq \eta$ or $\eta \preceq \eta_\varpi$ according to the sequence is increasing or decreasing). Let $r : \Omega^3 \to \Omega$ be a dominated map (means for each $(\eta, \varkappa, \varrho) \in \Omega^3$, $\eta \preceq r(\eta, \varkappa, \varrho)$, $\varkappa \preceq r(\varkappa, \varrho, \eta)$ and $\varkappa \preceq r(\varkappa, \varrho, \eta)$) and there is $\Im \in \Theta^*$ so that the condition (2) of Theorem 1 is fulfilled for all $(\eta, \varkappa, \varrho), (\mu, \nu, \vartheta) \in \Omega^3$ with ($\eta \preceq \mu$, $\varkappa \succeq \nu$ and $\varrho \preceq \vartheta$) or ($\eta \succeq \mu$, $\varkappa \preceq \nu$ and $\varrho \succeq \vartheta$). Then, r has a TFP.*

If we take $\Re$ is the universal relation, that is, $\Re = \Omega^3$ in *Theorem 1*, then we obtain the result below:

Corollary 2. *Suppose that $r : \Omega^3 \to \Omega$ is a mapping defined on a CMS $(\Omega, \eth)$. Suppose also there is $\Im \in \Theta^*$ so that (2) of Theorem 1 is verified for all $(\eta, \varkappa, \varrho), (\mu, \nu, \vartheta) \in \Omega^3$. Then, r has a TFP.*

In order to obtain the uniqueness of a TFP of r, we present the following theorem:

Theorem 2. *In addition to the stipulations of Theorem 1, assume that both $\Re$ and $\Re^{-1}$ are directed. Then, r has a unique TFP.*

Proof. Based on *Theorem 1, the set of TFPs of* r is non-empty. Suppose that $(\eta, \varkappa, \varrho)$ and $(\eta^*, \varkappa^*, \varrho^*)$ are two TFPs of r, i.e.,

$$\begin{aligned} \eta &= r(\eta, \varkappa, \varrho);\ \varkappa = r(\varkappa, \varrho, \eta);\ \varrho = r(\varrho, \eta, \varkappa) \\ \text{and } \eta^* &= r(\eta^*, \varkappa^*, \varrho^*);\ \varkappa^* = r(\varkappa^*, \varrho^*, \eta^*);\ \varrho^* = r(\varrho^*, \eta^*, \varkappa^*). \end{aligned}$$

Our goal is to prove $\eta = \eta^*$, $\varkappa = \varkappa^*$ and $\varrho = \varrho^*$. Using the directed property of $\Re$ and $\Re^{-1}$, there are $\mu \in \Omega$, $\nu \in \Omega$ and $\vartheta \in \Omega$ so that $(\eta, \mu) \in \Re; (\eta^*, \mu) \in \Re$, $(\varkappa, \nu) \in \Re^{-1}; (\varkappa^*, \nu) \in \Re^{-1}$ and $(\varrho, \vartheta) \in \Re; (\varrho^*, \vartheta) \in \Re$, this implies that $(\eta, \mu) \in \Re; (\eta^*, \mu) \in \Re$, $(\nu, \varkappa) \in \Re; (\nu, \varkappa^*) \in \Re$ and $(\varrho, \vartheta) \in \Re; (\varrho^*, \vartheta) \in \Re$. Take $\mu_0 = \mu$, $\nu_0 = \nu$ and $\vartheta_0 = \vartheta$. Therefore, $(\eta, \mu_0) \in \Re$, $(\nu_0, \varkappa) \in \Re$ and $(\varrho, \vartheta_0) \in \Re$. Assume that

$$\mu_1 = r(\mu_0, \nu_0, \vartheta_0),\ \nu_1 = r(\nu_0, \vartheta_0, \mu_0) \text{ and } \vartheta_1 = r(\vartheta_0, \mu_0, \nu_0).$$

In the same way as the proof of Theorem 1, we built three sequences $\{\mu_\varpi\}$, $\{\nu_\varpi\}$ and $\{\vartheta_\varpi\}$ as follows:

$$\mu_{\varpi+1} = r(\mu_\varpi, \nu_\varpi, \vartheta_\varpi),\ \nu_{\varpi+1} = r(\nu_\varpi, \vartheta_\varpi, \mu_\varpi) \text{ and } \vartheta_{\varpi+1} = r(\vartheta_\varpi, \mu_\varpi, \nu_\varpi), \tag{31}$$

for all $\varpi \geq 0$. As r is $\Re-$dominated, we have

$$(\mu_\varpi, \mu_{\varpi+1}) \in \Re, (\nu_{\varpi+1}, \nu_\varpi) \in \Re \text{ and } (\vartheta_\varpi, \vartheta_{\varpi+1}) \in \Re, \text{ for all } \varpi \geq 0. \tag{32}$$

Again, following the same mechanism used in Theorem 1, the sequences $\{\mu_\varpi\}$, $\{\nu_\varpi\}$ and $\{\vartheta_\varpi\}$ are Cauchy sequences in Ω and there are $\widehat{\mu}, \widehat{\nu}, \widehat{\vartheta} \in \Omega$ so that

$$\lim_{\varpi\to+\infty} \mu_\varpi = \widehat{\mu},\ \lim_{\varpi\to+\infty} \nu_\varpi = \widehat{\nu} \text{ and } \lim_{\varpi\to+\infty} \vartheta_\varpi = \widehat{\vartheta}. \tag{33}$$

Now, we show that $\eta = \widehat{\mu}$, $\varkappa = \widehat{\nu}$ and $\varrho = \widehat{\vartheta}$, which implies that

$$\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right) = 0.$$

Suppose, on the contrary, that $\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right) \neq 0$. We claim that

$$(\eta, \mu_\varpi) \in \Re, (\nu_\varpi, \varkappa) \in \Re \text{ and } (\varrho, \vartheta_\varpi) \in \Re, \text{ for all } \varpi \geq 0. \tag{34}$$

Since $((\eta, \mu_0), (\mu_0, \mu_1) \in \Re)$, $((\nu_1, \nu_0), (\nu_0, \varkappa) \in \Re)$ and $((\varrho, \vartheta_0), (\vartheta_0, \vartheta_1) \in \Re)$, by the transitivity property of $\Re$, we find $(\eta, \mu_1) \in \Re, (\nu_1, \varkappa) \in \Re$ and $(\varrho, \vartheta_1) \in \Re$. Hence, our assumption holds for $\varpi = 1$. Suppose that (34) is true for some $\wp > 1$, which implies that $(\eta, \mu_\wp) \in \Re$, $(\nu_\wp, \varkappa) \in \Re$ and $(\varrho, \vartheta_\wp) \in \Re$. By (32), $\left(\mu_\wp, \mu_{\wp+1}\right) \in \Re$, $\left(\nu_{\wp+1}, \nu_\wp\right) \in \Re$ and $\left(\vartheta_\wp, \vartheta_{\wp+1}\right) \in \Re$. The transitivity property of $\Re$ implies that $\left(\eta, \mu_{\wp+1}\right) \in \Re$, $\left(\nu_{\wp+1}, \varkappa\right) \in \Re$ and $\left(\varrho, \vartheta_{\wp+1}\right) \in \Re$. Hence, our claim is proved. Using (2) and (34), we find

$$\begin{aligned} \eth(\eta, \mu_{\varpi+1}) &= \eth(r(\eta, \varkappa, \varrho), r(\mu_\varpi, \nu_\varpi, \vartheta_\varpi)) \\ &\leq \Im(\aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi)) \\ &\quad \times \aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi), \end{aligned} \tag{35}$$

where

$$\aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi) = \Im\left(\frac{\eth(\eta, \mu_\varpi) + \eth(\varkappa, \nu_\varpi) + \eth(\varrho, \vartheta_\varpi)}{3}\right). \tag{36}$$

Analogously,

$$\begin{aligned} \eth(\varkappa, \nu_{\varpi+1}) &= \eth(r(\varkappa, \varrho, \eta), r(\nu_\varpi, \vartheta_\varpi, \mu_\varpi)) \\ &\leq \Im(\aleph(\varkappa, \varrho, \eta, \nu_\varpi, \vartheta_\varpi, \mu_\varpi)) \\ &\quad \times \aleph(\varkappa, \varrho, \eta, \nu_\varpi, \vartheta_\varpi, \mu_\varpi) \\ &= \Im(\aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi)) \\ &\quad \times \aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi), \end{aligned} \tag{37}$$

and

$$\begin{aligned} \eth(\varrho, \vartheta_{\varpi+1}) &= \eth(r(\varrho, \eta, \varkappa), r(\vartheta_\varpi, \mu_\varpi, \nu_\varpi)) \\ &\leq \Im(\aleph(\varrho, \eta, \varkappa, \vartheta_\varpi, \mu_\varpi, \nu_\varpi)) \\ &\quad \times \aleph(\varrho, \eta, \varkappa, \vartheta_\varpi, \mu_\varpi, \nu_\varpi) \\ &= \Im(\aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi)) \\ &\quad \times \aleph(\eta, \varkappa, \varrho, \mu_\varpi, \nu_\varpi, \vartheta_\varpi). \end{aligned} \tag{38}$$

Adding (35), (37) and (38), we find

$$\begin{aligned} & \eth(\eta, \mu_{\varpi+1}) + \eth(\varkappa, \nu_{\varpi+1}) + \eth(\varrho, \vartheta_{\varpi+1}) \\ \leq \ & 3\Im(\aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi})) \\ & \times \aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi}), \end{aligned} \tag{39}$$

Passing limit in (36) as $\varpi \to +\infty$ and applying (33), we obtain

$$\lim_{\varpi \to +\infty} \aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi}) = \frac{\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right)}{3}. \tag{40}$$

Taking the limit supremum in (39) as $\varpi \to +\infty$, using (33) and (40), one can write

$$\begin{aligned} & \eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right) \\ \leq \ & \left[\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right)\right] \\ & \times \limsup_{\varpi \to +\infty} \Im(\aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi})), \end{aligned} \tag{41}$$

that is

$$1 \leq \limsup_{\varpi \to +\infty} \Im(\aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi})) \leq 1, \tag{42}$$

which implies that $\limsup_{\varpi \to +\infty} \Im(\aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi})) = 1$. From the property of $\Im$, we can write

$$\lim_{\varpi \to +\infty} \aleph(\eta, \varkappa, \varrho, \mu_{\varpi}, \nu_{\varpi}, \vartheta_{\varpi}) = \frac{\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right)}{3} = 0,$$

which contradicts with our assumption that $\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right) \neq 0$. Hence,

$$\eth(\eta, \widehat{\mu}) + \eth(\varkappa, \widehat{\nu}) + \eth\left(\varrho, \widehat{\vartheta}\right) = 0 \text{ implies } \eth(\eta, \widehat{\mu}) = \eth(\varkappa, \widehat{\nu}) = \eth\left(\varrho, \widehat{\vartheta}\right) = 0,$$

that is

$$\eta = \widehat{\mu}, \ \varkappa = \widehat{\nu} \text{ and } \varrho = \widehat{\vartheta}. \tag{43}$$

With the same manner, we can show that

$$\eta^* = \widehat{\mu}, \ \varkappa^* = \widehat{\nu} \text{ and } \varrho^* = \widehat{\vartheta}. \tag{44}$$

By (43) and (44), we find $\eta = \eta^*$, $\varkappa = \varkappa^*$ and $\varrho = \varrho^*$. Therefore, the TFP is unique. □

We support our study by the example below.

Example 4. *Let $\Omega = [0,1]$ and $\eth(\eta, \varkappa) = |\eta - \varkappa|$ be a usual metric. Define a function $\Im : [0, \infty) \to [0, 1)$ by $\Im(\ell) = \frac{\ln(1+\ell)}{\ell}$, if $\ell > 0$ and $\Im(\ell) = 0$, if $\ell = 0$. Define the mapping $r : \Omega^3 \to \Omega$ by*

$$r(\eta, \varkappa, \varrho) = \ln\left(1 + \frac{\eta + \varkappa + \varrho}{3}\right), \textit{ for all } (\eta, \varkappa, \varrho) \in \Omega^3,$$

and a binary relation $\Re$ on Ω as follows:

$$\begin{aligned}\Re \;=\; & \{((\eta,\varkappa,\varrho),\ (\mu,\nu,\vartheta)):\\ & [(0\leq\eta\leq 1;\ 0\leq\mu\leq\ln 2)\ or\ (0\leq\eta\leq\ln 2;\ 0\leq\mu\leq 1)],\\ & [(0\leq\varkappa\leq 1;\ 0\leq\nu\leq\ln 2)\ or\ (0\leq\varkappa\leq\ln 2;\ 0\leq\nu\leq 1)],\\ & [(0\leq\varrho\leq 1;\ 0\leq\vartheta\leq\ln 2)\ or\ (0\leq\varrho\leq\ln 2;\ 0\leq\vartheta\leq 1)]\}.\end{aligned}$$

It is easy to see that Ω is regular with respect to $\Re$ and the mapping is an $\Re-$dominated. Suppose that $(\eta,\varkappa,\varrho),(\mu,\nu,\vartheta),(a,b,c)\in\Omega^3$ so that $((\eta,\mu,a),\ (\varkappa,\nu,b),\ (\varrho,\vartheta,c)\in\Omega^3)$ or $((\mu,a,\eta),\ (\nu,b,\varkappa),\ (\vartheta,c,\varrho)\in\Omega^3)$ or $((a,\eta,\mu),\ (b,\varkappa,\nu),\ (c,\varrho,\vartheta)\in\Omega^3)$. Therefore, $(\eta\in[0,1]$ or $\eta\in[0,\ln 2])$; $(\mu\in[0,1]$ or $\mu\in[0,\ln 2])$; $(\varkappa\in[0,1]$ or $\varkappa\in[0,\ln 2])$; $(\nu\in[0,1]$ or $\nu\in[0,\ln 2])$; $(\varrho\in[0,1]$ or $\varrho\in[0,\ln 2])$ and $(\vartheta\in[0,1]$ or $\vartheta\in[0,\ln 2])$. Hence, we have

$$\begin{aligned}
& \eth(r(\eta,\varkappa,\varrho),r(\mu,\nu,\vartheta))\\
= \;& \eth\left(\ln\left(1+\frac{\eta+\varkappa+\varrho}{3}\right),\ln\left(1+\frac{\mu+\nu+\vartheta}{3}\right)\right)\\
= \;& \left|\ln\left(1+\frac{\eta+\varkappa+\varrho}{3}\right)-\ln\left(1+\frac{\mu+\nu+\vartheta}{3}\right)\right|\\
= \;& \left|\ln\left(\frac{\left(1+\frac{\eta+\varkappa+\varrho}{3}\right)}{\left(1+\frac{\mu+\nu+\vartheta}{3}\right)}\right)\right| = \left|\ln\left(1+\frac{\frac{\eta+\varkappa+\varrho}{3}-\frac{\mu+\nu+\vartheta}{3}}{1+\frac{\mu+\nu+\vartheta}{3}}\right)\right|\\
\leq \;& \left|\ln\left(1+\frac{\left|\frac{\eta+\varkappa+\varrho}{3}-\frac{\mu+\nu+\vartheta}{3}\right|}{1+\frac{\mu+\nu+\vartheta}{3}}\right)\right| \leq \left|\ln\left(1+\left|\frac{\eta+\varkappa+\varrho}{3}-\frac{\mu+\nu+\vartheta}{3}\right|\right)\right|\\
\leq \;& \left|\ln\left(1+\frac{|\mu-\eta|+|\nu-\varkappa|+|\vartheta-\varrho|}{3}\right)\right| = \ln\left(1+\frac{|\mu-\eta|+|\nu-\varkappa|+|\vartheta-\varrho|}{3}\right)\\
\leq \;& \ln(1+\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta)) = \frac{\ln(1+\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta))}{\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta)}\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta)\\
= \;& \Im(\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta))\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta).
\end{aligned}$$

Therefore, all requirements of Theorem 1 are satisfied and (0,0,0) is a TFP of r.

4. Well-Posedness

We begin this part with the following assumption:

(Q) If $r(\eta^*,\varkappa^*,\varrho^*)$ is any solution of the problem (W)—that is, by (1) and $\{(\eta_\varpi,\varkappa_\varpi,\varrho_\varpi)\}$ is any sequence in Ω^3 for which

$$\begin{aligned}\lim_{\varpi\to+\infty}\eth(\eta_\varpi,r(\eta_\varpi,\varkappa_\varpi,\varrho_\varpi)) \;&=\; \lim_{\varpi\to+\infty}\eth(\varkappa_\varpi,r(\varkappa_\varpi,\varrho_\varpi,\eta_\varpi))\\ &=\; \lim_{\varpi\to+\infty}\eth(\varrho_\varpi,r(\varrho_\varpi,\eta_\varpi,\varkappa_\varpi))=0,\end{aligned}$$

then $(\eta^*,\eta_\varpi)\in\Re$, $(\varkappa_\varpi,\varkappa^*)\in\Re$ and $(\varrho^*,\varrho_\varpi)\in\Re$, for all ϖ.

Theorem 3. *In addition to the assumption of Theorem 2, the TFP problem (W) is well-posed, provided that the hypothesis (Q) is satisfied.*

Proof. Theorem 2 says that the point $(\eta^*,\varkappa^*,\varrho^*)$ is a TFP of r. This means the point $(\eta^*,\varkappa^*,\varrho^*)$ is a solution of (1), that is $\eta^*=r(\eta^*,\varkappa^*,\varrho^*)$, $\varkappa^*=r(\varkappa^*,\varrho^*,\eta^*)$ and $\varrho^*=r(\varrho^*,\eta^*,\varkappa^*)$. Let $\{(\eta_\varpi,\varkappa_\varpi,\varrho_\varpi)\}$ be any sequence in Ω^3 such that

$$\limsup_{\varpi\to+\infty}[\eth(\eta_\varpi,\eta^*)+\eth(\varkappa_\varpi,\varkappa^*)+\eth(\varrho_\varpi,\varrho^*)]$$

is finite, and

$$\lim_{\varpi\to+\infty} \eth(\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) = \lim_{\varpi\to+\infty} \eth(\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)) = \lim_{\varpi\to+\infty} \eth(\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)) = 0.$$

Then, there is $\aleph > 0$ so that

$$\limsup_{\varpi\to+\infty} [\eth(\eta_\varpi, \eta^*) + \eth(\varkappa_\varpi, \varkappa^*) + \eth(\varrho_\varpi, \varrho^*)] = \aleph,$$

and also by the hypothesis $(\mathcal{Q})$, $(\eta^*, \eta_\varpi) \in \Re$, $(\varkappa_\varpi, \varkappa^*) \in \Re$ and $(\varrho^*, \varrho_\varpi) \in \Re$, for all ϖ. By (2), we find

$$\begin{aligned} \eth(\eta_\varpi, \eta^*) &= \eth(\eta_\varpi, r(\eta^*, \varkappa^*, \varrho^*)) \\ &\leq \eth(\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) + \eth(r(\eta^*, \varkappa^*, \varrho^*), r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) \\ &\leq \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi))\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) \\ &\quad + \eth(\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)), \end{aligned} \tag{45}$$

where

$$\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) = \Im\left(\frac{\eth(\eta^*, \eta_\varpi) + \eth(\varkappa^*, \varkappa_\varpi) + \eth(\varrho^*, \varrho_\varpi)}{3}\right). \tag{46}$$

Similarly, we can obtain

$$\begin{aligned} \eth(\varkappa_\varpi, \varkappa^*) &\leq \Im(\aleph(\varkappa^*, \varrho^*, \eta^*, \varkappa_\varpi, \varrho_\varpi, \eta_\varpi))\aleph(\varkappa^*, \varrho^*, \eta^*, \varkappa_\varpi, \varrho_\varpi, \eta_\varpi) \\ &\quad + \eth(\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)) \\ &\leq \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi))\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) \\ &\quad + \eth(\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)), \end{aligned} \tag{47}$$

and

$$\begin{aligned} \eth(\varrho_\varpi, \varrho^*) &\leq \Im(\aleph(\varrho^*, \eta^*, \varkappa^*, \varrho_\varpi, \eta_\varpi, \varkappa_\varpi))\aleph(\varrho^*, \eta^*, \varkappa^*, \varrho_\varpi, \eta_\varpi, \varkappa_\varpi) \\ &\quad + \eth(\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)) \\ &\leq \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi))\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) \\ &\quad + \eth(\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)). \end{aligned} \tag{48}$$

Adding (45), (47) and (48), we have

$$\begin{aligned} &\eth(\eta_\varpi, \eta^*) + \eth(\varkappa_\varpi, \varkappa^*) + \eth(\varrho_\varpi, \varrho^*) \\ \leq\ & 3\Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi))\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) \\ &+ \eth(\eta_\varpi, r(\eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) + \eth(\varkappa_\varpi, r(\varkappa_\varpi, \varrho_\varpi, \eta_\varpi)) \\ &+ \eth(\varrho_\varpi, r(\varrho_\varpi, \eta_\varpi, \varkappa_\varpi)). \end{aligned} \tag{49}$$

Taking the limit supremum as $\varpi \to +\infty$ in (46), we have

$$\limsup_{\varpi\to+\infty} \aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) = \frac{\aleph}{3}. \tag{50}$$

Assume that

$$\limsup_{\varpi\to+\infty} [\eth(\eta_\varpi, \eta^*) + \eth(\varkappa_\varpi, \varkappa^*) + \eth(\varrho_\varpi, \varrho^*)] = \aleph \neq 0.$$

Hence, $\aleph > 0$. Taking the limit supremum as $\varpi \to +\infty$ in (49) and using (50), we can write

$$\aleph \leq \aleph \limsup_{\varpi\to+\infty} \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi)),$$

which implies that

$$1 \leq \limsup_{\varpi \to +\infty} \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) \leq 1.$$

Therefore, $\limsup_{\varpi \to +\infty} \Im(\aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi)) = 1$. Using the property of $\Im$, we obtain that

$$\limsup_{\varpi \to +\infty} \aleph(\eta^*, \varkappa^*, \varrho^*, \eta_\varpi, \varkappa_\varpi, \varrho_\varpi) = 0,$$

that is

$$\lim_{\varpi \to +\infty} [\partial(\eta_\varpi, \eta^*) + \partial(\varkappa_\varpi, \varkappa^*) + \partial(\varrho_\varpi, \varrho^*)] = 0,$$

which is a contradiction. Hence, we find

$$\limsup_{\varpi \to +\infty} [\partial(\eta^*, \eta_\varpi) + \partial(\varkappa^*, \varkappa_\varpi) + \partial(\varrho^*, \varrho_\varpi)] = 0.$$

Then, we have

$$\begin{aligned} 0 &\leq \liminf_{\varpi \to +\infty} [\partial(\eta^*, \eta_\varpi) + \partial(\varkappa^*, \varkappa_\varpi) + \partial(\varrho^*, \varrho_\varpi)] \\ &\leq \limsup_{\varpi \to +\infty} [\partial(\eta^*, \eta_\varpi) + \partial(\varkappa^*, \varkappa_\varpi) + \partial(\varrho^*, \varrho_\varpi)] = 0, \end{aligned}$$

which implies that

$$\lim_{\varpi \to +\infty} [\partial(\eta^*, \eta_\varpi) + \partial(\varkappa^*, \varkappa_\varpi) + \partial(\varrho^*, \varrho_\varpi)] = 0.$$

It follows that

$$\lim_{\varpi \to +\infty} \partial(\eta_\varpi, \eta^*) = \lim_{\varpi \to +\infty} \partial(\varkappa_\varpi, \varkappa^*) = \lim_{\varpi \to +\infty} \partial(\varrho_\varpi, \varrho^*) = 0,$$

which leads to $\eta_\varpi \to \eta^*$, $\varkappa_\varpi \to \varkappa^*$ and $\varrho_\varpi \to \varrho^*$ as $\varpi \to +\infty$. Hence, the TFP problem (W) is well-posed. □

5. Some Results for α-Dominated Mappings

In this section, we introduce α-dominated mappings and discuss the extension of mappings equipped with admissibility conditions in the TFP theory.

Definition 5. *Let Ω be a non-empty set and $\alpha : \Omega^2 \to \mathbb{R}$ be a given mapping. A mapping $r : \Omega^3 \to \Omega$ is called an $\alpha-$dominated mapping if for all $(\eta, \varkappa, \varrho) \in \Omega^3$, we have*

$$\alpha(\eta, r(\eta, \varkappa, \varrho)) \geq 1, \ \alpha(\varkappa, r(\varkappa, \varrho, \eta)) \geq 1 \text{ and } \alpha(\varrho, r(\varrho, \eta, \varkappa)) \geq 1.$$

Definition 6. *Let Ω be a non-empty set and $\alpha : \Omega^2 \to \mathbb{R}$ be a given mapping. We say that α has triangular property if for each $\eta, \varkappa, \varrho \in \Omega$,*

$$\alpha(\eta, \varkappa) \geq 1, \ \alpha(\varkappa, \varrho) \geq 1 \text{ implies } \alpha(\eta, \varrho) \geq 1.$$

Definition 7. *Let (Ω, ∂) be a metric space and $\alpha : \Omega^2 \to \mathbb{R}$ be a mapping. We say that Ω has $\alpha-$regular property if for each convergent sequence $\{\eta_\varpi\}$ with limit $\eta \in \Omega$, $\alpha(\eta_\varpi, \eta_{\varpi+1}) \geq 1$, for all ϖ implies $\alpha(\eta_\varpi, \eta) \geq 1$, for all ϖ.*

Theorem 4. *Suppose that (Ω, ∂) is a CMS and $\alpha : \Omega^2 \to \mathbb{R}$ is a mapping so that Ω fulfills $\alpha-$regular property and α has triangular property. Let $r : \Omega^3 \to \Omega$ be an $\alpha-$dominated mapping and there is $\Im \in \Theta^*$ so that (2) of Theorem 1 is fulfilled for all $(\eta, \varkappa, \varrho)$, $(\mu, \nu, \vartheta) \in \Omega^3$ with $\alpha(\eta, \mu) \geq 1$, $\alpha(\varkappa, \nu) \geq 1$ and $\alpha(\varrho, \vartheta) \geq 1$. Then, r has a TFP.*

Proof. Define a binary relation $\Re$ on Ω by

$$(\eta, \varkappa) \in \Re \text{ iff } \alpha(\eta, \varkappa) \geq 1 \text{ or } \alpha(\varkappa, \eta) \geq 1.$$

Then

(i) $\alpha(\eta, \mu) \geq 1$, $\alpha(\varkappa, \nu) \geq 1$ and $\alpha(\varrho, \vartheta) \geq 1$, leads to $(\eta, \mu) \in \Re$, $(\nu, \varkappa) \in \Re$ and $(\varrho, \vartheta) \in \Re$;

(ii) $\alpha(\eta, r(\eta, \varkappa, \varrho)) \geq 1$, $\alpha(\varkappa, r(\varkappa, \varrho, \eta)) \geq 1$ and $\alpha(\varrho, r(\varrho, \eta, \varkappa)) \geq 1$, leads to $(\eta, r(\eta, \varkappa, \varrho)) \in \Re$, $(r(\varkappa, \varrho, \eta), \varkappa) \in \Re$ and $(\varrho, r(\varrho, \eta, \varkappa)) \in \Re$, *for all* $(\eta, \varkappa, \varrho) \in \Omega^3$;

(iii) $\alpha(\eta_\varpi, \eta_{\varpi+1}) \geq 1$ and $\alpha(\eta_\varpi, \eta) \geq 1$, leads to $(\eta_\varpi, \eta_{\varpi+1}) \in \Re$ and $(\eta_\varpi, \eta) \in \Re$, whenever $\{\eta_\varpi\}$ is a convergent sequence with $\eta_\varpi \to \eta$ and $\alpha(\eta_\varpi, \eta_{\varpi+1}) \geq 1$.

Therefore, all assumptions boil down to the hypotheses of Theorem 1. Hence, according to Theorem 1, the map r has a TFP in Ω^3. □

6. Solving a System of Differential Equations

In this section, we apply Theorems 1 and 2 to discuss the existence and uniqueness solution for the following differential equation:

$$\begin{cases} \eta''(\ell) = \psi(\ell, \eta(\ell), \varkappa(\ell), \varrho(\ell)), \\ \varkappa''(\ell) = \psi(\ell, \varkappa(\ell), \varrho(\ell), \eta(\ell)), \\ \varrho''(\ell) = \psi(\ell, \varrho(\ell), \eta(\ell), \varkappa(\ell)), \\ \eta(0) = \eta'(1) = \varkappa(0) = \varkappa'(1) = \varrho(0) = \varrho'(1), \end{cases} \tag{51}$$

for each $\ell \in [0, 1]$. Problem (51) is equivalent to the following integral system:

$$\begin{cases} \eta(\ell) = \int_0^1 \mho(\ell, \zeta)\psi(\zeta, \eta(\zeta), \varkappa(\zeta), \varrho(\zeta))d\zeta, \\ \varkappa(\ell) = \int_0^1 \mho(\zeta, \ell)\psi(\zeta, \varkappa(\zeta), \varrho(\zeta), \eta(\zeta))d\zeta, \\ \varrho(\ell) = \int_0^1 \mho(\ell, \zeta)\psi(\zeta, \varrho(\zeta), \eta(\zeta), \varkappa(\zeta))d\zeta, \end{cases} \tag{52}$$

for all $\ell, \zeta \in [0, 1]$, where $\mho$ is the Green's function defined by

$$\mho(\ell, \zeta) = \begin{cases} \ell, & \ell \leq \zeta, \\ \zeta, & \ell > \zeta. \end{cases}$$

Suppose that $\Omega = C([0, 1], \mathbb{R})$ is the space of all real valued continuous functions defined on $[0, 1]$. Define a metric $\eth$ by

$$\eth(\eta, \varkappa) = \max_{\ell \in [0,1]} |\eta(\ell) - \varkappa(\ell)|, \text{ for all } \ell \in [0, 1].$$

Clearly, $(\Omega, \eth)$ is a CMS. Let Ω be equipped with the universal relation U, that is, $(\eta, \varkappa) \in U$, for all $\eta, \varkappa \in \Omega$. Define a mapping $r : \Omega^3 \to \Omega$ by

$$r(\eta, \varkappa, \varrho)(\ell) = \int_0^1 \mho(\ell, \zeta)\psi(\ell, \zeta, \eta(\zeta), \varkappa(\zeta), \varrho(\zeta))d\zeta, \text{ for all } \ell, \zeta \in [0, 1]. \tag{53}$$

Solving system (51) is equivalent to finding a unique solution to the mapping (53). Now, system (51) will be considered under the following postulates:

(H_1) The function $\psi : [0, 1] \times [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous;

(H_2) $|\mho(\ell, \zeta)| \leq \rho$, where $\rho > 0$ is a fixed number;

(H_3) For all $(\eta,\varkappa,\varrho),(\mu,\nu,\vartheta)\in\Omega^3$, we have

$$|\psi(\ell,\zeta,\eta,\varkappa,\varrho)-\psi(\ell,\zeta,\mu,\nu,\vartheta)|\leq\Lambda(\ell,\zeta,\eta,\varkappa,\varrho,\mu,\nu,\vartheta),\text{ for all }\ell,\zeta\in[0,1],$$

where

$$\Lambda(\ell,\zeta,\eta,\varkappa,\varrho,\mu,\nu,\vartheta)=\frac{1}{\rho}\ln\left(1+\frac{|\eta-\mu|+|\varkappa-\nu|+|\varrho-\vartheta|}{3}\right).$$

Our main theorem in this part is as follows:

Theorem 5. *Under assumptions* (H_1)–(H_3)*, system (51) has a unique solution in* Ω.

Proof. Since U is the universal relation on Ω, from the definition of r, we have

$$(\eta,r(\eta,\varkappa,\varrho))\in\Re,\ (r(\varkappa,\varrho,\eta),\varkappa)\in\Re\text{ and }(\varrho,r(\varrho,\eta,\varkappa))\in\Re.$$

for all $\eta,\varkappa,\varrho\in\Omega$. This means that r is $U-$dominated mapping. Furthermore, every universal relation is a binary relation, so Ω has $U-$regular property.

Now, from our hypotheses (H_1) and (H_2), for each $(\eta,\varkappa,\varrho),(\mu,\nu,\vartheta)\in\Omega^3$, we have

$$\begin{aligned}
&|r(\eta,\varkappa,\varrho)(\ell)-r(\mu,\nu,\vartheta)(\ell)|\\
=&\left|\int_0^1\mho(\ell,\zeta)[\psi(\ell,\zeta,\eta(\zeta),\varkappa(\zeta),\varrho(\zeta))-\psi(\ell,\zeta,\mu(\zeta),\nu(\zeta),\vartheta(\zeta))]d\zeta\right|\\
\leq&\int_0^1|\mho(\ell,\zeta)||[\psi(\ell,\zeta,\eta(\zeta),\varkappa(\zeta),\varrho(\zeta))-\psi(\ell,\zeta,\mu(\zeta),\nu(\zeta),\vartheta(\zeta))]|d\zeta\\
\leq&\rho\int_0^1|[\psi(\ell,\zeta,\eta(\zeta),\varkappa(\zeta),\varrho(\zeta))-\psi(\ell,\zeta,\mu(\zeta),\nu(\zeta),\vartheta(\zeta))]|d\zeta.
\end{aligned}$$

Applying the condition (H_3), we find

$$\begin{aligned}
|r(\eta,\varkappa,\varrho)(\ell)-r(\mu,\nu,\vartheta)(\ell)|&\leq\rho\int_0^1\Lambda(\ell,\zeta,\eta,\varkappa,\varrho,\mu,\nu,\vartheta)d\zeta\\
&=\rho\int_0^1\frac{1}{\rho}\ln\left(1+\frac{|\eta-\mu|+|\varkappa-\nu|+|\varrho-\vartheta|}{3}\right)d\zeta\\
&=\int_0^1\ln\left(1+\frac{|\eta-\mu|+|\varkappa-\nu|+|\varrho-\vartheta|}{3}\right)d\zeta\\
&\leq\int_0^1\ln\left(1+\frac{\eth(\eta,\mu)+\eth(\varkappa,\nu)+\eth(\varrho,\vartheta)}{3}\right)d\zeta\\
&=\ln\left(1+\frac{\eth(\eta,\mu)+\eth(\varkappa,\nu)+\eth(\varrho,\vartheta)}{3}\right)\int_0^1d\zeta,
\end{aligned}$$

It follows that

$$\begin{aligned} |r(\eta,\varkappa,\varrho)(\ell) - r(\mu,\nu,\vartheta)(\ell)| &\leq \ln\left(1 + \frac{\eth(\eta,\mu) + \eth(\varkappa,\nu) + \eth(\varrho,\vartheta)}{3}\right) \\ &\leq \ln(1 + \aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta)) \\ &= \frac{\ln(1 + \aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta))}{\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta)} \aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta) \\ &= \Im(\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta))\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta), \end{aligned}$$

where

$$\Im(\ell) = \frac{\ln(1+\ell)}{\ell},\ \ell > 0 \text{ and } \Im(\ell) = 0 \text{ if } \ell = 0,$$

and

$$\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta) = \left(\frac{\eth(\eta,\mu) + \eth(\varkappa,\nu) + \eth(\varrho,\vartheta)}{3}\right).$$

Thus,

$$\eth(r(\eta,\varkappa,\varrho) - r(\mu,\nu,\vartheta)) \leq \Im(\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta))\aleph(\eta,\varkappa,\varrho,\mu,\nu,\vartheta).$$

Therefore, all hypotheses of Theorems 1 and 2 are fulfilled. Hence, the problem (51) has a unique solution on Ω. □

7. Conclusions

Fixed-point techniques are considered the backbone of mathematical analysis because of their many applications. Hence, this method has attracted many authors who are interested in this direction. Amongst the interesting applications is the study of algorithms and what they mean by convergence and divergence in the field of optimization, game theory, ordinary and fractional differential equations, differential and integral equations, and many other applications.

In our manuscript, we investigate the existence and uniqueness of TFPs for Geraghty-type contraction maps under appropriate assumptions. Furthermore, the main results are supported by an example. In addition, well-posed and α-dominated mappings for the TFP problem are presented. Finally, the existence solution to a system of differential equations is derived. As future work, motivated by the work of [23,24], the main results of this article can be generalized to $n-$tuple fixed point theorems.

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