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On a Nonlocal Coupled System of Hilfer Generalized Proportional Fractional Differential Equations

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Abstract: This paper studies the existence and uniqueness of solutions for a coupled system of Hilfer-type generalized proportional fractional differential equations supplemented with nonlocal asymmetric multipoint boundary conditions. We consider both the scalar and the Banach space case. We apply standard fixed-point theorems to derive the desired results. In the scalar case, we apply Banach's fixed-point theorem, the Leray–Schauder alternative, and Krasnosel'skiĭ's fixed-point theorem. The Banach space case is based on Mönch's fixed-point theorem and the technique of the measure of noncompactness. Examples illustrating the main results are presented. Symmetric distance between itself and its derivative can be investigated by replacing the proportional number equal to one half.

Keywords: Hilfer proportional fractional derivative; Hilfer proportional fractional integral; system; fixed-point theorems; existence results; measure of noncompactness

Citation: Samadi, A.; Ntouyas, S.K.; Tariboon, J. On a Nonlocal Coupled System of Hilfer Generalized Proportional Fractional Differential Equations. *Symmetry* **2022**, *14*, 738. https://doi.org/10.3390/sym14040738

Academic Editors: Manuel Duarte Ortigueira and Jose M. Carvalho

Received: 16 March 2022
Accepted: 2 April 2022
Published: 4 April 2022

1. Introduction

Fractional calculus appears as a developed area of mathematical analysis to consider many real-world problems [1–8]. Fractional calculus is composed of massive traits to construct wide applications in multiple scientific studies. Considered equations in fractional calculus are often unable to study complex systems, and a diversity of new fractional operators were introduced to improve the field of fractional calculus [9–14]. In the literature, one can find several kinds of fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, Hilfer, and Katugampola. The Riemann–Liouville and Caputo fractional derivatives were extended to the Hilfer fractional derivative [15], and many applications of Hilfer fractional derivative were then obtained in many fields of mathematics and physics. (see [16–18]). In [19], the study of the Hilfer generalized proportional was introduced by the authors. In [20], the study of boundary value problems of the Hilfer generalized proportional fractional derivative of order in $(1,2]$, supplemented with nonlocal multipoint boundary conditions, given by

$$\begin{cases} \left(D_{a^+}^{\alpha,\eta,\sigma} + kD_{a^+}^{\alpha-1,\eta,\sigma}\right)x(t) = f(t,x(t)), \ \ t \in [a,b], \\ x(a) = 0, \qquad x(b) = \sum_{j=1}^{m} \theta_j x(\xi_j) \end{cases} \tag{1}$$

was initiated, where $D_{a^+}^{\alpha,\eta,\sigma}$ is the fractional derivative of a Hilfer generalized proportional type of order $1 < \alpha < 2$, Hilfer parameter $0 \le \eta \le 1$, $\sigma \in (0,1]$, $k \in \mathbb{R}$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a \ge 0$, $\theta_j \in \mathbb{R}$, $\xi_j \in (a,b)$ for $j = 1,2,3,\ldots,m$. Existence and uniqueness results were considered first in the scalar case by applying some classical fixed-point theorems. The notion of measure of measure of noncompactness and Mönch's fixed-point theorem were then applied to obtain an existence result for Problem (1), where $f : [a,b] \times E \to E$ is a given function, and $(E, \|\cdot\|_\infty)$ is a real Banach space.

In recent years, very few papers have been published dealing with Hilfer generalized proportional fractional derivative of order in $(1,2]$. Motivated by the above paper, we enrich this new research area. Thus, in this paper, a coupled system of Hilfer-type generalized proportional fractional differential equations with nonlocal multipoint boundary conditions of form

$$\begin{cases} \left(D_{c^+}^{\delta_1,\eta,\sigma} + kD_{c^+}^{\delta_1-1,\eta,\sigma}\right) r(z) = h_1(z, r(z), s(z)), \ \ z \in [c,d], \\ \left(D_{c^+}^{\delta_2,\eta,\sigma} + k_1 D_{c^+}^{\delta_2-1,\eta,\sigma}\right) s(z) = h_2(z, r(z), s(z)), \\ r(c) = 0, \qquad r(d) = \sum_{j=1}^{m} \theta_j s(\xi_j), \\ s(c) = 0, \qquad s(d) = \sum_{i=1}^{n} \varepsilon_i r(\lambda_i), \end{cases} \tag{2}$$

is investigated, in which $D_{a^+}^{\delta_1,\eta,\sigma}$ and $D_{a^+}^{\delta_2,\eta,\sigma}$ are the fractional derivatives of Hilfer generalized proportional type of order $1 < \delta_1, \delta_2 < 2$, Hilfer parameter $0 \le \eta \le 1$, $\sigma \in (0,1]$, $k, k_1 \in \mathbb{R}$, $h_1, h_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $a \ge 0$, $\theta_j, \varepsilon_i \in \mathbb{R}$, $\xi_j, \lambda_i \in (a,b)$ for $i = 1,2,3,\ldots,n$ and $j = 1,2,\ldots,m$.

We prove existence and uniqueness results in the scalar case by applying classical Banach and Krasnosel'skiĭ fixed-point theorems, and the Leray–Schauder alternative. Next, by using the measure of noncompactness and Mönch's fixed point theorem, we establish an existence result for Problem (2) when $f : [c,d] \times E \to E$ is a given function, and $(E, \|\cdot\|_\infty)$ is a real Banach space.

The remainder of this work is organized as follows. Section 2 outlines some basic notations, definitions, and basic results of fractional calculus. An auxiliary lemma concerning a linear variant of Problem (1) is also proved. This lemma help in converting nonlinear Problem (1) into a fixed-point problem. Our main results are presented in Section 3 for the scalar case, and Section 4 for the Banach space case. Section 5 is devoted to constructing illustrative numerical examples. The work in this paper is new and enriches the literature on coupled systems of Hilfer-type generalized proportional fractional differential equations. The used methods are standard, but their configuration in the present problem is new.

2. Preliminaries

Here, some notations, definitions, and lemmas from fractional calculus are recalled.

Let $C([c,d], E)$ be the Banach space of all continuous functions $u : [c,d] \to E$ endowed by

$$\|u\|_\infty = \sup\{\|u(z)\|, z \in [c,d]\}.$$

In the case when $E = \mathbb{R}$, we use notation

$$\|u\| = \sup\{|u(z)|, z \in [c,d]\}.$$

Definition 1. *Let $h \in L^1([c,d], \mathbb{R})$. The fractional integral of the Riemann–Liouville type with order $\delta > 0$ is defined by [2]*

$$I_{c^+}^{\delta} h(z) = \frac{1}{\Gamma(\delta)} \int_c^z (z - \tau)^{\delta-1} h(\tau) d\tau,$$

in which $\Gamma(\cdot)$ is the gamma function.

Definition 2. *Let $h \in C^n([c,d],\mathbb{R})$. The fractional derivative of Caputo type with order $\delta > 0$ of function h is defined by [2]*

$$^{C}D^{\delta}_{c^+}h(z) = \frac{1}{\Gamma(n-\delta)}\int_c^z (z-s)^{n-\delta-1}h^{(n)}(s)ds,$$

while the Riemann–Liouville type is defined by

$$^{RL}D^{\delta}_{c^+}h(z) = \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dz}\right)^n \int_c^z (z-s)^{n-\delta-1}h(s)ds,$$

where $n-1 < \delta < n$, $n \in \mathbb{N}$, provided the right-hand side of the two equations above exists.

Definition 3. *Let $\sigma \in (0,1]$ and $\delta \in \mathbb{C}$ with $Re(\delta) > 0$. Then, fractional operator [21]*

$$I^{\delta,\sigma}_{c^+}h(z) = \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{\delta-1}h(s)ds,\ z > c,$$

indicates the left-sided generalized proportional integral of order $\delta > 0$ of function h.

Definition 4. *The Riemann–Liouville type of proportional fractional derivative of order $\delta > 0$ and $\sigma \in (0,1]$ of function h is defined by [21]*

$$D^{\delta,\sigma}_{c^+}h(z) = \frac{D^{n,\sigma}}{\sigma^{n-\delta}\Gamma(n-\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{n-\delta-1}h(s)ds,$$

while the Caputo type is given by

$$^{C}D^{\delta,\sigma}_{c^+}h(z) = \frac{1}{\sigma^{n-\delta}\Gamma(n-\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{n-\delta-1}D^{n,\sigma}h(s)ds,$$

provided that the right-hand side of the two equations above exists, where $\Gamma(\cdot)$ indicates the gamma function and $n = [\delta]+1$, $[\delta]$ denotes the integer part of a real number δ. In addition, notation $D^{n,\sigma}$ is defined by

$$D^{n,\sigma} = \underbrace{D^{\sigma}D^{\sigma}\cdots D^{\sigma}}_{n-times},$$

where the proportional derivative of a function $h \in C^1(\mathbb{R},\mathbb{R})$ is defined as

$$D^{\sigma}h(z) = (1-\sigma)h(z) + \sigma\frac{d}{dz}h(z), \quad \sigma \in (0,1].$$

Remark 1. *If $\sigma = \frac{1}{2}$, then $D^{\frac{1}{2}}h(z)$ lies in a path of symmetrical distance between $h(z)$ and $h'(z)$.*

Remark 2. *If we set $D^{\sigma}h(z) = (1-\sigma)h(z) + \sigma h'(z) := f(z)$, this is a linear first-order differential equation that can be solved as*

$$h(z) = \frac{1}{\sigma}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}f(s)ds = I^{1,\sigma}_{c+}f(z).$$

Some properties of the generalized proportional fractional integral and derivative are given in the next lemmas.

Lemma 1. *Assume that $\delta, \overline{\delta} \in \mathbb{C}$, so that $Re(\delta) \geq 0$ and $Re(\overline{\delta}) > 0$. Then, for any $\sigma \in (0,1]$, [21]*

$$(I^{\delta,\sigma}_{c^+}e^{\frac{\sigma-1}{\sigma}s}(s-c)^{\overline{\delta}-1})(z) = \frac{\Gamma(\overline{\delta})}{\sigma^{\delta}\Gamma(\overline{\delta}+\delta)}e^{\frac{\sigma-1}{\sigma}z}(z-c)^{\overline{\delta}+\delta-1},$$

$$(D^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}s}(s-c)^{\overline{\delta}-1})(z) = \frac{\sigma^{\delta}\Gamma(\overline{\delta})}{\Gamma(\overline{\delta}-\delta)} e^{\frac{\sigma-1}{\sigma}z}(z-c)^{\overline{\delta}-\delta-1},$$

$$(I^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}(c-s)}(c-s)^{\overline{\delta}-1})(z) = \frac{\Gamma(\overline{\delta})}{\sigma^{\delta}\Gamma(\overline{\delta}+\delta)} e^{\frac{\sigma-1}{\sigma}(c-z)}(c-z)^{\overline{\delta}+\delta-1},$$

$$(D^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}(c-s)}(c-s)^{\overline{\delta}-1})(z) = \frac{\sigma^{\delta}\Gamma(\overline{\delta})}{\Gamma(\overline{\delta}-\delta)} e^{\frac{\sigma-1}{\sigma}(c-z)}(c-z)^{\overline{\delta}-\delta-1}.$$

Lemma 2. *Suppose that $\sigma \in (0,1]$, $Re(\delta_1) > 0$ and $Re(\delta_2) > 0$. If $h \in C([c,d],\mathbb{R})$, then [21]*

$$I^{\delta,\sigma}_{c^+}(I^{\overline{\delta},\sigma}h)(z) = I^{\overline{\delta},\sigma}_{c^+}(I^{\delta,\sigma}h)(z) = (I^{\delta+\overline{\delta},\sigma}_{c^+}h)(z), \;\; z \geq c. \tag{3}$$

Lemma 3. *Let $\sigma \in (0,1]$ and $0 \leq m < [\Re(\delta)] + 1$. If $h \in L^1([a,b])$ then [21]*

$$D^{m,\sigma}_{c^+}(I^{\delta,\sigma}_{c^+}h)(z) = \big(I^{\delta-m}_{c^+}h\big)(z), \;\; z > c.$$

The Hilfer generalized proportional fractional derivative is introduced.

Definition 5. *Let $n-1 < \delta < n$, $n \in \mathbb{N}$, $\sigma \in (0,1]$ and $0 \leq \eta \leq 1$. Then, the generalized proportional fractional derivative of the Hilfer type with order δ, parameter η, and proportional number σ of function h is defined by [19]*

$$(D^{\delta,\eta,\sigma}_{c^+}h)(z) = I^{\eta(n-\delta),\sigma}_{c^+}[D^{n,\sigma}(I^{(1-\eta)(n-\delta),\sigma}_{c^+}h)](z),$$

in which $D^{n,\sigma}$ is the proportional derivative of order n, and $I^{(\cdot),\sigma}$ is the generalized proportional fractional integral in Definition 3.

The Hilfer generalized proportional fractional derivative is equivalent to

$$(D^{\delta,\eta,\sigma}_{c^+}h)(z) = I^{\eta(n-\delta),\sigma}_{c^+}[D^{n,\sigma}(I^{(1-\eta)(n-\delta),\sigma}_{c^+}h)](z) = (I^{\eta(n-\delta),\sigma}_{c^+}D^{\gamma,\sigma}h)(z),$$

where $\gamma = \delta + \eta(n-\delta)$. Thus, operator $D^{\delta,\eta,\sigma}_{c^+}$ can be represented in terms of operators given in Definition 4. If $\eta \to 1$, then it is the Caputo-type proportional fractional derivative; $\eta \to 0$ can be reduced to a Reimann–Liouville type proportional fractional derivative. Parameter γ satisfies

$$1 < \gamma \leq 2, \;\; \gamma \geq \delta, \;\; \gamma > \eta, \;\; n - \gamma < n - \eta(n-\delta).$$

Inspired by the Lemma 3.9 of [19], the following lemma is introduced.

Lemma 4. *Let $n-1 < \delta < n$, $\sigma \in (0,1]$, $0 \leq \eta \leq 1$ and $\gamma = \delta + \eta(n-\delta) \in [\delta, n]$. If $h \in L^1(c,d)$ and $I^{n-\gamma,\sigma}_{c^+}h \in C^n([c,d],\mathbb{R})$, then*

$$I^{\delta,\sigma}_{c^+}D^{\delta,\eta,\sigma}_{c^+}h(z) = h(z) - \sum_{j=1}^{n} e^{\frac{\sigma-1}{\sigma}(z-c)} \frac{(z-c)^{\gamma-j}}{\sigma^{\gamma-j}\Gamma(\gamma+1-j)} \big(I^{j-\gamma,\sigma}h\big)(c^+).$$

In the following lemma, we solve the linear variant of Problem (1).

Lemma 5. *Let $1 < \delta_1, \delta_2 < 2$, $0 \leq \eta \leq 1$, $\gamma_1 = \delta_1 + \eta(2-\delta_1) \in [\delta_1, 2]$, $\gamma_2 = \delta_2 + \eta(2-\delta_2) \in [\delta_2, 2]$, $\sigma \in (0,1]$, $g, g_1 \in C([c,d], \mathbb{R})$ and $D \neq 0$. Then pair (r,s) is the solution of system*

$$\begin{cases} \left(D^{\delta_1,\eta,\sigma} + kD^{\delta_1-1,\eta,\sigma}\right) r(z) = g(z), \quad z \in [c,d] \\ \left(D^{\delta_2,\eta,\sigma} + k_1 D^{\delta_2-1,\eta,\sigma}\right) s(z) = g_1(z), \\ r(c) = 0, \quad r(d) = \sum_{j=1}^{m} \theta_j s(\xi_j), \\ s(c) = 0 \quad s(d) = \sum_{i=1}^{k} \varepsilon_i r(\lambda_i), \end{cases} \tag{4}$$

if and only if

$$\begin{aligned} r(z) &= I_{c^+}^{\delta_1,\sigma} g(z) - \frac{k}{\sigma}\int_c^z r(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du + \frac{(z-c)^{\gamma_1-1}}{D\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(z-c)} \Big[\Delta \Big(\sum_{j=1}^{m} \theta_j I_{c^+}^{\delta_2,\sigma} g_1(\xi_j) \\ &\quad - \frac{k_1}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} s(u) e^{\frac{\sigma-1}{\sigma}(\xi_j-u)} du - I_{c^+}^{\delta_1,\sigma} g(d) + \frac{k}{\sigma}\int_c^d r(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \Big) \\ &\quad + B \Big(\sum_{i=1}^{k} \varepsilon_i I_{c^+}^{\delta_1,\sigma} g(\lambda_i) - \frac{k}{\sigma} \sum_{i=1}^{n} \varepsilon_i \int_c^{\lambda_i} r(u) e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)} du - I_{c^+}^{\delta_2,\sigma} g_1(d) \\ &\quad + \frac{k_1}{\sigma} \int_c^d s(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \Big) \Big], \end{aligned} \tag{5}$$

$$\begin{aligned} s(z) &= I_{c^+}^{\delta_2,\sigma} g_1(z) - \frac{k_1}{\sigma}\int_c^z s(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du + \frac{(z-c)^{\gamma_2-1}}{D\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(z-c)} \Big[A \Big(\sum_{i=1}^{k} \varepsilon_i I_{c^+}^{\delta_1,\sigma} g(\lambda_i) \\ &\quad - \frac{k}{\sigma} \sum_{i=1}^{n} \varepsilon_i \int_c^{\lambda_i} r(u) e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)} du - I_{c^+}^{\delta_2,\sigma} g_1(d) + \frac{k_1}{\sigma}\int_c^d s(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \Big) \\ &\quad + \Gamma \Big(\sum_{j=1}^{m} \theta_j I_{c^+}^{\delta_2,\sigma} g_1(\xi_j) - \frac{k_1}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} s(u) e^{\frac{\sigma-1}{\sigma}(\xi_j-u)} du - I_{c^+}^{\delta_1,\sigma} g(d) \\ &\quad + \frac{k}{\sigma} \int_c^d r(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \Big) \Big], \end{aligned} \tag{6}$$

where

$$\begin{aligned} A &= \frac{(d-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(d-c)}, \qquad B = \sum_{j=1}^{m} \theta_j \frac{(\xi_j-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(\xi_j-c)}, \\ \Gamma &= \sum_{i=1}^{n} \varepsilon_i \frac{(\lambda_i-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(\lambda_i-c)}, \quad \Delta = \frac{(d-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(d-c)}, \\ D &= A\Delta - B\Gamma. \end{aligned} \tag{7}$$

Proof. Assume that pair (r,s) is the solution of Problem (2). Operating fractional integrals $I_{c^+}^{\delta_1,\sigma}$ and $I_{c^+}^{\delta_2,\sigma}$ to both sides of equations of (4) and using Lemma 4, we obtain

$$\begin{aligned} r(z) &= I_{c^+}^{\delta_1,\sigma} g(z) + c_0 \frac{(z-c)^{\gamma_1-2}}{\Gamma(\gamma_1-1)} e^{\frac{\sigma-1}{\sigma}(z-c)} + c_1 \frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(z-c)} \\ &\quad - \frac{k}{\sigma}\int_c^z r(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du, \end{aligned} \tag{8}$$

$$\begin{aligned} s(z) &= I_{c^+}^{\delta_2,\sigma} g_1(z) + d_0 \frac{(z-c)^{\gamma_2-2}}{\Gamma(\gamma_2-1)} e^{\frac{\sigma-1}{\sigma}(z-c)} + d_1 \frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(z-c)} \\ &\quad - \frac{k_1}{\sigma} \int_c^z s(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du. \end{aligned} \tag{9}$$

Applying $r(c) = s(c) = 0$, we get $c_0, d_0 = 0$. Hence,

$$r(z) = I_{c^+}^{\delta_1,\sigma} g(z) + c_1 \frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(z-c)} - \frac{k}{\sigma} \int_c^z r(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du, \tag{10}$$

$$s(z) = I_{c^+}^{\delta_2,\sigma} g_1(z) + d_1 \frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(z-c)} - \frac{k_1}{\sigma} \int_c^z s(u) e^{\frac{\sigma-1}{\sigma}(z-u)} du. \tag{11}$$

From $r(d) = \sum_{j=1}^{m} \theta_j s(\xi_j)$ and $s(d) = \sum_{i=1}^{k} \varepsilon_i r(\lambda_i)$, we obtain system

$$\begin{aligned} & I_{c^+}^{\delta_1,\sigma} g(d) + c_1 \frac{(d-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(d-c)} - \frac{k}{\sigma} \int_c^d r(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \\ =\; & \sum_{j=1}^{m} \theta_j I_{c^+}^{\delta_2,\sigma} g_1(\xi_j) + d_1 \sum_{j=1}^{m} \theta_j \frac{(\xi_j-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(\xi_j-c)} \\ & - \frac{k_1}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} s(u) e^{\frac{\sigma-1}{\sigma}(\xi_j-u)} du, \end{aligned} \tag{12}$$

$$\begin{aligned} & I_{c^+}^{\delta_2,\sigma} g_1(d) + d_1 \frac{(d-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(d-c)} - \frac{k_1}{\sigma} \int_c^d s(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du \\ =\; & \sum_{i=1}^{n} \varepsilon_i I_{c^+}^{\delta_1,\sigma} g(\lambda_i) + c_1 \sum_{i=1}^{n} \varepsilon_i \frac{(\lambda_i-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} e^{\frac{\sigma-1}{\sigma}(\lambda_i-c)} \\ & - \frac{k}{\sigma} \sum_{i=1}^{n} \varepsilon_i \int_c^{\lambda_i} r(u) e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)} du, \end{aligned} \tag{13}$$

or, using notation (7),

$$\begin{cases} Ac_1 - Bd_1 = P, \\ -\Gamma c_1 + \Delta d_1 = Q, \end{cases} \tag{14}$$

where

$$\begin{aligned} P &= \sum_{j=1}^{m} \theta_j I_{c^+}^{\delta_2,\sigma} g_1(\xi_j) - \frac{k_1}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} s(u) e^{\frac{\sigma-1}{\sigma}(\xi_j-u)} du \\ &\quad - I_{c^+}^{\delta_1,\sigma} g(d) + \frac{k}{\sigma} \int_c^d r(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du, \\ Q &= \sum_{i=1}^{n} \varepsilon_i I_{c^+}^{\delta_1,\sigma} g(\lambda_i) - \frac{k}{\sigma} \sum_{i=1}^{n} \varepsilon_i \int_c^{\lambda_i} r(u) e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)} du \\ &\quad - I_{c^+}^{\delta_2,\sigma} g_1(d) + \frac{k_1}{\sigma} \int_c^d s(u) e^{\frac{\sigma-1}{\sigma}(d-u)} du. \end{aligned}$$

By solving System (14), we have

$$c_1 = \frac{\Delta P + BQ}{D} \quad \text{and} \quad d_1 = \frac{AQ + \Gamma P}{D}.$$

Substituting the values of c_1 and d_1 in Equations (10) and (11), respectively, we obtain Solutions (5) and (6). The conversion can be proven by direct computation. The proof is finished. $\square$

3. Existence and Uniqueness Results

Let $\mathrm{X} = C([c,d],\mathbb{R})$ be the Banach space of all continuous functions from $[c,d]$ to $\mathbb{R}$ endowed with the sup norm $\|r\| = \sup\{|r(z)| : z \in [c,d]\}$. Let $\mathrm{Y} = C([c,d],\mathbb{R})$ be the Banach space endowed with the sup norm $\|s\| = \sup\{|s(z)| : z \in [c,d]\}$. Product space $(\mathrm{X}\times\mathrm{Y}, \|(r,s)\|)$ is a Banach space with norm $\|(r,s)\| = \|r\| + \|s\|$.

In view of Lemma 5, we define operator $\mathrm{P} : \mathrm{X}\times\mathrm{Y} \to \mathrm{X}\times\mathrm{Y}$ by

$$\mathrm{P}(r,s)(z) := (\mathrm{P}_1(r,s)(z), \mathrm{P}_2(r,s)(z)), \tag{15}$$

in which

$$\begin{aligned}
\mathrm{P}_1(r,s)(z) \;=\;& I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(z) - \frac{k}{\sigma}\int_c^z r(u)e^{\frac{\sigma-1}{\sigma}(z-u)}du \\
&+\frac{(z-c)^{\gamma_1-1}}{D\Gamma(\gamma_1)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[\Delta\Big(\sum_{j=1}^m \theta_j I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(\xi_j) \\
&-\frac{k_1}{\sigma}\sum_{j=1}^m \theta_j\int_c^{\xi_j} s(u)e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du - I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(d) \\
&+\frac{k}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big) + B\Big(\sum_{i=1}^n \varepsilon_i I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(\lambda_i) - I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(d) \\
&-\frac{k}{\sigma}\sum_{i=1}^n \varepsilon_i\int_c^{\lambda_i} r(u)e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du + \frac{k_1}{\sigma}\int_c^d s(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)\Big],
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\mathrm{P}_2(r,s)(z) \;=\;& I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(z) - \frac{k_1}{\sigma}\int_c^z s(u)e^{\frac{\sigma-1}{\sigma}(z-u)}du \\
&+\frac{(z-c)^{\gamma_2-1}}{|D|\Gamma(\gamma_2)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[A\Big(\sum_{i=1}^n \varepsilon_i I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(\lambda_i) \\
&-\frac{k}{\sigma}\sum_{i=1}^n \varepsilon_i\int_c^{\lambda_i} r(u)e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du - I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(d) + \frac{k_1}{\sigma}\int_c^d s(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big) \\
&+\Gamma\Big(\sum_{j=1}^m \theta_j I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(\xi_j) - \frac{k_1}{\sigma}\sum_{j=1}^m \theta_j\int_c^{\xi_j} s(u)e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du \\
&-I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(d) + \frac{k}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)\Big],
\end{aligned} \tag{17}$$

where $h_{1,r,s}(z) = h_1(z, r(z), s(z))$ and $h_{2,r,s}(z) = h_2(z, r(z), s(z))$.

For convenience, the following notations are applied:

$$\begin{aligned}
A_1 &= \frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + B\frac{\sum_{i=1}^n|\varepsilon_i|(\lambda_i-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\Big], \\
A_2 &= \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\frac{\sum_{j=1}^m|\theta_j|(\xi_j-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} + B\frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\Big], \\
A_3 &= \frac{|k|}{\sigma}(d-c) + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^m|\theta_j|(\xi_j-c) + \frac{|k|}{\sigma}(d-c)\Big) \\
&\quad+B\Big(\frac{|k|}{\sigma}\sum_{i=1}^n|\varepsilon_i|(\lambda_i-c) + \frac{|k_1|}{\sigma}(d-c)\Big)\Big],
\end{aligned} \tag{18}$$

$$\begin{aligned} B_1 &= \frac{(d-c)^{\gamma_2-1}}{|D|\Gamma(\gamma_2)}\Big[A\frac{\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}+\Gamma\frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\Big], \\ B_2 &= \frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}+\frac{(d-c)^{\gamma_2-1}}{|D|\Gamma(\gamma_2)}\Big[A\frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}+\Gamma\frac{\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\Big], \\ B_3 &= \frac{|k_1|}{\sigma}(d-c)+\frac{(d-c)^{\gamma_2-1}}{|D|\Gamma(\gamma_2)}\Big[A\Big(\frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)+\tfrac{|k_1|}{\sigma}(d-c)\Big) \\ &\quad +\Gamma\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)+\tfrac{|k|}{\sigma}(d-c)\Big)\Big]. \end{aligned} \tag{19}$$

Now, Banach's fixed-point theorem is applied to prove an existence and uniqueness result.

Lemma 6. (Banach fixed point theorem) *[22] Let G be a closed set in X and $H : G \to G$ satisfies*

$$|Hu_1 - Hu_2| \le \lambda|u_1 - u_2|, \text{ for some } \lambda \in (0,1), \text{ and for all } u_1, u_2 \in G.$$

Then, H admits a unique fixed point in G.

Theorem 1. *Let $D \neq 0$ and $h_1, h_2 : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be two functions satisfying*
(G_1) there exist $\ell_1, \ell_2 > 0$, such that, for all $z \in [c,d]$ and $r_i, s_i \in \mathbb{R}$, $i = 1,2$, we have

$$|h_1(z,r_1,s_1) - h_1(z,r_2,s_2)| \le \ell_1(|r_1-r_2| + |s_1-s_2|),$$
$$|h_2(z,r_1,s_1) - h_2(z,r_2,s_2)| \le \ell_2(|r_1-r_2| + |s_1-s_2|).$$

Then, a unique solution of Problem (1) is obtained provided that

$$\ell_1(A_1+B_1) + \ell_2(A_2+B_2) + A_3 + B_3 < 1, \tag{20}$$

where $A_i, B_i,\ i = 1,2,3$ are presented by (18) and (19), respectively.

Proof. We converted Problem (1) into fixed-point problem $(r,s) = \mathrm{P}(r,s)(z)$ in which operator P is defined as in (15). Using Banach's theorem, we obtain a unique fixed point of operator P, and this completes the proof. Let $\sup_{z\in[c,d]}|h_1(z,0,0)| = S_1 < \infty$ and $\sup_{z\in[c,d]}|h_2(z,0,0)| = S_2 < \infty$. Next, assume that $B_r = \{(r,s) \in \mathrm{X}\times\mathrm{Y}; \|(r,s)\| \le r\}$, in which

$$r \ge \frac{S_1(A_1+B_1) + S_2(A_2+B_2)}{1 - [\ell_1(A_1+B_1) + \ell_2(A_2+B_2) + A_3 + B_3]}. \tag{21}$$

B_r is a bounded, closed, and convex subset of $\mathrm{X}\times\mathrm{Y}$. First, we indicate that $\mathrm{P}(B_r) \subseteq B_r$. For all $(r,s) \in B_r$ and $z \in [c,d]$, applying condition (G_1), we obtain

$$\begin{aligned} |h_{1,r,s}(z)| = |h_1(z,r(z),s(z))| &\le |h_1(z,r(z),s(z)) - h_1(z,0,0)| + |h_1(z,0,0)| \\ &\le \ell_1(|r(z)| + |s(z)|) + S_1 \\ &\le \ell_1(\|r\| + \|s\|) + S_1 \le \ell_1 r + S_1, \end{aligned}$$

and

$$|h_{2,r,s}(z)| = |h_2(z,r(z),s(z))| \le \ell_2 r + S_2.$$

Thus for $(r,s) \in B_r$ and $z \in [c,d]$, and using the fact that $|e^{\frac{\sigma-1}{\sigma}z}| \le 1$, we have

$$
\begin{aligned}
& |\mathrm{P}_1(r,s)(z)| \\
\leq\ & I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(z)| + \frac{|k|}{\sigma}\int_c^z |r(u)|e^{\frac{\sigma-1}{\sigma}(z-u)}du \\
& + \frac{(z-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[\Delta\Big(\sum_{j=1}^{m}\theta_j I_{c^+}^{\delta_2,\sigma}|h_{2,r,s}(\xi_j)| \\
& + \frac{|k_1|}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}|s(u)|e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du + I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(d)| \\
& + \frac{|k|}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big) + B\Big(\sum_{i=1}^{n}|\varepsilon_i|I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(\lambda_i)| + I_{c^+}^{\delta_2,\sigma}|h_{2,r,s}(d)| \\
& + \frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|\int_c^{\lambda_i}|r(u)|e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du + \frac{|k_1|}{\sigma}\int_c^d |s(u)|e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)\Big] \\
\leq\ & \frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}(\ell_1 r + S_1) + \frac{|k|}{\sigma}(d-c)r \\
& + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}(\ell_2 r + S_2) + \frac{r|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c) \\
& + \frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}(\ell_1 r + S_1) + \frac{r|k|(d-c)}{\sigma}\Big) + B\Big(\frac{\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}(\ell_1 r + S_1) \\
& + \frac{r|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c) + \frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}(\ell_2 r + S_2) + \frac{r|k_1|(d-c)}{\sigma}\Big)\Big] \\
=\ & (\ell_1 r + S_1)\Big\{\frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + B\frac{\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\Big]\Big\} \\
& + (\ell_2 r + S_2)\Big\{\frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\frac{\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} + B\frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\Big]\Big\} \\
& + r\Big\{\frac{|k|}{\sigma}(d-c) + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c) + \frac{|k|}{\sigma}(d-c)\Big) \\
& + B\Big(\frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c) + \frac{|k_1|}{\sigma}(d-c)\Big)\Big\} \\
=\ & (\ell_1 r + S_1)A_1 + (\ell_2 r + S_2)A_2 + rA_3.
\end{aligned}
$$

Consequently, we have

$$\|\mathrm{P}_1(r,s)\| \leq (\ell_1 r + S_1)A_1 + (\ell_2 r + S_2)A_2 + rA_3.$$

Similarly, we obtain that

$$\|\mathrm{P}_2(r,s)\| \leq (\ell_1 r + S_1)B_1 + (\ell_2 r + S_2)B_2 + rB_3.$$

Hence, we have

$$
\begin{aligned}
\|P(r,s)\| \quad \leq \quad & \Big[\ell_1(A_1+B_1) + \ell_2(A_2+B_2) + A_3 + B_3\Big]r \\
& + (A_1+B_1)S_1 + (A_2+B_2)S_2 \leq r.
\end{aligned}
$$

Therefore, $\mathrm{P}(B_r) \subseteq B_r$.

Now, we indicate that $\mathrm{P}: \mathrm{X} \times \mathrm{Y} \longrightarrow \mathrm{X} \times \mathrm{Y}$ is a contraction mapping. Due to condition (G_1) for all $(r_1, s_1), (r_2, s_2) \in \mathrm{X} \times \mathrm{Y}$ and $z \in [c, d]$, we obtain

$$
\begin{aligned}
& |\mathrm{P}_1(r_1,s_1)(z) - \mathrm{P}_2(r_2,s_2)(z)| \\
\leq \; & I_{c^+}^{\delta_1,\sigma}|h_{1,r_1,s_1} - h_{1,r_2,s_2}|(z) + \frac{|k|}{\sigma}\int_c^z |r_1(u) - r_2(u)|du \\
& + \frac{(d-c)^{\gamma_1 - 1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\sum_{j=1}^m |\theta_j| I_{c^+}^{\delta_2,\sigma}|h_{2,r_1,s_1} - h_{1,r_2,s_2}|(\xi_j) \\
& + \frac{|k_1|}{\sigma}\sum_{j=1}^m |\theta_j| \int_c^{\xi_j} |s_1(u) - s_2(u)|du + I_{c^+}^{\delta_1,\sigma}|h_{1,r_1,s_1} - h_{1,r_2,s_2}|(d) \\
& + \frac{|k|}{\sigma}\int_c^d |r_1(u) - r_2(u)|du\Big) + B\Big(\sum_{i=1}^n \varepsilon_i I_{c^+}^{\delta_1,\sigma}|h_{1,r_1,s_1} - h_{1,r_2,s_2}|(\lambda_i) \\
& + \frac{|k|}{\sigma}\sum_{i=1}^n |\varepsilon_i| \int_c^{\lambda_i} |r_1(u) - r_2(u)|du + I_{c^+}^{\delta_2,\sigma}|h_{2,r_1,s_1} - h_{2,r_2,s_2}|(d) \\
& + \frac{|k_1|}{\sigma}\int_c^d |s_1(u) - s_2(u)|du\Big)\Big] \\
\leq \; & \frac{\ell_1(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}(\|r_1 - r_2\| + \|s_1 - s_2\|) + \frac{|k|(d-c)}{\sigma}\|r_1 - r_2\| \\
& + \frac{(d-c)^{\gamma_1 - 1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{\ell_2(\|r_1 - r_2\| + \|s_1 - s_2\|)}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\sum_{j=1}^m |\theta_j|(\xi_j - c)^{\delta_2} \\
& + \frac{|k_1|}{\sigma}\sum_{j=1}^m |\theta_j|(\xi_j - c)\|s_1 - s_2\| + \frac{(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\ell_1(\|r_1 - r_2\|) + \|s_1 - s_2\|) \\
& + \frac{|k|(d-c)}{\sigma}\|r_1 - r_2\|\Big) + B\Big(\frac{\ell_1(\|r_1 - r_2\| + \|s_1 - s_2\|)}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\sum_{i=1}^n \varepsilon_i(\lambda_i - c)^{\delta_1} \\
& + \frac{|k|}{\sigma}\sum_{i=1}^n |\varepsilon_i|(\lambda_i - c)\|r_1 - r_2\| + \frac{(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\ell_2(\|r_1 - r_2\| + \|s_1 - s_2\|) \\
& + \frac{|k_1|(d-c)}{\sigma}\|s_1 - s_2\|\Big) \\
\leq \; & (\ell_1 A_1 + \ell_2 A_2)(\|r_1 - r_2\| + \|s_1 - s_2\|) + A_3(\|r_1 - r_2\| + \|s_1 - s_2\|) \\
= \; & (\ell_1 A_1 + \ell_2 A_2 + A_3)(\|r_1 - r_2\| + \|s_1 - s_2\|);
\end{aligned}
$$

hence,

$$\|\mathrm{P}_1(r_1,s_1) - \mathrm{P}_1(r_2,s_2)\| \leq (\ell_1 A_1 + \ell_2 A_2 + A_3)(\|r_1 - r_2\| + \|s_1 - s_2\|). \tag{22}$$

Similarly, we obtain

$$\|\mathrm{P}_2(r_1,s_1) - \mathrm{P}_2(r_2,s_2)\| \leq (\ell_1 B_1 + \ell_2 B_2 + B_3)(\|r_1 - r_2\| + \|s_1 - s_2\|). \tag{23}$$

Combining (22) and (23), we have

$$
\begin{aligned}
\|\mathrm{P}(r_1,s_1) - \mathrm{P}(r_2,s_2)\| \leq \; & \Big[\ell_1(A_1 + B_1) + \ell_2(A_2 + B_2) + A_3 + B_3\Big] \\
& \times(\|r_1 - r_2\| + \|s_1 - s_2\|).
\end{aligned}
$$

Since $\ell_1(A_1 + B_1) + \ell_2(A_2 + B_2) + A_3 + B_3 < 1$, operator P is contraction mapping. Consequently, applying Banach's fixed-point theorem, a unique fixed point of operator P is obtained that is a solution of Problem (1). The proof is completed. □

Now, we apply the Leray–Schauder alternative to obtain our first existence result.

Lemma 7. *(Leray–Schauder nonlinear alternative [23]). Let set Ω be closed bounded convex in X, and O an open set contained in Ω with $0 \in O$. Then, for continuous and compact $T, \bar{U} \to \Omega$ either*

($\mathfrak{a}$) *T admits a fixed–point in $\bar{U}$ or*

($\mathfrak{aa}$) *$\exists\, \mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.*

Theorem 2. *Assume that $D \neq 0$ and $h_1, h_2 : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions satisfying*

(G_2) There exist $r_i, s_i \geq 0$ for $i = 1,2$ and $r_0, s_0 > 0$ such that for any $r, s \in \mathbb{R}$, we have

$$|h_1(z,r,s)| \leq r_0 + r_1|r| + r_2|s|,$$
$$|h_2(z,r,s)| \leq s_0 + s_1|r| + s_2|s|.$$

If $(A_1 + B_1)r_1 + (A_2 + B_2)s_1 + A_3 + B_3 < 1$ and $(A_1 + B_1)r_2 + (A_2 + B_2)s_2 + A_3 + B_3 < 1$, where A_i, B_i for $i = 1,2,3$ are presented by (18) and (19), respectively, then Problem (1) contains at least one solution on $[c,d]$.

Proof. Since functions h_1, h_2 are continuous on $[c,d] \times \mathbb{R}^2$, operator P is also continuous. Now, the completely continuous property of operator P is shown. Let $B_\varepsilon = \{(r,s) \in \mathrm{X} \times \mathrm{Y} : \|(r,s)\| \leq \varepsilon\}$. Thus, for all $(r,s) \in B_\varepsilon$, there exist $D_1, D_2 > 0$, such that $|h_{1,r,s}(z)| = |h_1(z, r(z), s(z))| \leq D_1$ and $|h_{2,r,s}(z)| = |h_2(z, r(z), s(z))| \leq D_2$. Hence, for all $(r,s) \in B_\varepsilon$, we have

$$\begin{aligned}
|\mathrm{P}_1(r,s)(z)| \leq\; & I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(z)| + \frac{|k|}{\sigma}\int_c^z |r(u)|du \\
& + \frac{(d-c)^{\gamma_1 - 1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\sum_{j=1}^m |\theta_j| I_{c^+}^{\delta_2,\sigma}|h_{2,r,s}(\xi_j)| + \frac{|k_1|}{\sigma}\sum_{j=1}^m |\theta_j|\int_c^{\xi_j} |s(u)|du \\
& + I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(d)| + \frac{|k|}{\sigma}\int_c^d |r(u)du\Big) + B\Big(\sum_{j=1}^m |\varepsilon_i| I_{c^+}^{\delta_1,\sigma}|h_{1,r,s}(\lambda_i)| \\
& + \frac{|k|}{\sigma}\sum_{i=1}^n |\varepsilon_i|\int_c^{\lambda_i} |r(t)|dt + I_{c^+}^{\delta_2,\sigma}|h_{2,r,s}(d)| + \frac{|k_1|}{\sigma}\int_c^d |s(t)|dt\Big)\Big] \\
\leq\; & \frac{D_1(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{|k|(d-c)r}{\sigma} \\
& + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{D_2}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\sum_{j=1}^m |\theta_j|(\xi_j - c)^{\delta_2} + \frac{|k_1|r}{\sigma}\sum_{j=1}^m |\theta_j|(\xi_j - c) \\
& + \frac{D_1(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{r|k|(d-c)}{\sigma}\Big) + B\Big(\frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\sum_{i=1}^n |\varepsilon_i|(\lambda_i - c)^{\delta_1} \\
& + \frac{D_2(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} + r\frac{|k|}{\sigma}\sum_{i=1}^n |\varepsilon_i|(\lambda_i - c) + \frac{r|k_1|(d-c)}{\sigma}\Big)\Big] \\
=\; & A_1 D_1 + A_2 D_2 + rA_3,
\end{aligned}$$

which implies that

$$\|\mathrm{P}_1(r,s)\| \leq A_1 D_1 + A_2 D_2 + rA_3.$$

Similarly, we obtain that

$$\|\mathrm{P}_2(r,s)\| \leq B_1 D_1 + B_2 D_2 + rB_3.$$

Consequently,

$$\|\mathrm{P}(r,s)\| \leq (A_1 + B_1)D_1 + (A_2 + B_2)D_2 + r(A_3 + B_3),$$

and we conclude that P is uniformly bounded.

Next, the equicontinuous property of operator P is proven. Let $z_1, z_2 \in [c,d]$ with $z_1 < z_2$. Thus, we have

$$\begin{aligned}
& |\mathrm{P}_1(r,s)(z_2) - \mathrm{P}_1(r,s)(z_1)| \\
\leq\ & \frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)} \int_c^{z_2} [(z_2-u)^{\delta_1-1} - (z_1-u)^{\delta_1-1}]|h_1(u,r(u),s(u))|du \\
& + \frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)} \int_{z_1}^{z_2} (z_2-u)^{\delta_1-1}|h_1(u,r(u),s(u))|du + \frac{|k|}{\sigma}\|r\|(z_2-z_1) \\
& + \frac{(z_2-c)^{\gamma_1-1} - (z_1-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)} \Big[\Delta\Big(\frac{D_2}{\sigma^{\delta_2}\Gamma(\delta_2+1)} \sum_{j=1}^{m} |\theta_j|(\xi_j-c)^{\delta_2} \\
& + \frac{|k_1|r}{\sigma} \sum_{j=1}^{m} |\theta_j|(\xi_j-c) + \frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{r|k|(d-c)}{\sigma}\Big) \\
& + B\Big(\frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)} \sum_{i=1}^{n} \varepsilon_i(\lambda_i-c) + \frac{|k|r}{\sigma} \sum_{i=1}^{n} |\varepsilon_i|(\lambda_i-c) \\
& + \frac{D_2(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} + \frac{r|k_1|(d-c)}{\sigma}\Big)\Big] \\
\leq\ & \frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)} \Big[|(z_2-c)^{\delta_1} - (z_1-c)^{\delta_1} + 2(z_2-z_1)^{\delta_1}\Big] + \frac{|k|}{\sigma}\|r\|(z_2-z_1) \\
& + \frac{(z_2-c)^{\gamma_1-1} - (z_1-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)} \Big[\Delta\Big(\frac{D_2}{\sigma^{\delta_2}\Gamma(\delta_2+1)} \sum_{j=1}^{m} |\theta_j|(\xi_j-c)^{\delta_2} \\
& + \frac{|k_1|r}{\sigma} \sum_{j=1}^{m} |\theta_j|(\xi_j-c) + \frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{r|k|(d-c)}{\sigma}\Big) \\
& + B\Big(\frac{D_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)} \sum_{i=1}^{n} |\varepsilon_i|(\lambda_i-c) + \frac{|k|r(\lambda_i-c)}{\sigma} + \frac{D_2(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} + \frac{r|k_1|(d-c)}{\sigma}\Big)\Big].
\end{aligned}$$

Thus, we have

$$|\mathrm{P}_1(r,s)(t_2) - \mathrm{P}_1(r,s)(t_1)| \longrightarrow 0, \text{ as } t_1 \to t_2.$$

Similarly, we have

$$|\mathrm{P}_2(r,s)(t_2) - \mathrm{P}_2(r,s)(t_1)| \to 0.$$

Hence, set $\mathrm{P}(B_r)$ is equicontinuous. Consequently, due to the Arzelá-Ascoli theorem, we obtain that P is completely continuous.

Lastly, the boundedness property of set $\Theta = \{(r,s) \in \mathrm{X} \times \mathrm{Y} : (r,s) = \lambda \mathrm{P}(r,s),\ 0 \leq \lambda \leq 1\}$ is showed. Let $(r,s) \in \Theta$, then $(r,s) = \lambda \mathrm{P}(r,s)$. Hence, for all $z \in [c,d]$ we have

$$r(z) = \lambda \mathrm{P}_1(r,s)(z), \ \ s(z) = \lambda \mathrm{P}_2(r,s)(z).$$

Thus, we have

$$\begin{aligned}
\|r\| &\leq (r_0 + r_1\|r\| + r_2\|s\|)A_1 + (s_0 + s_1\|r\| + s_2\|s\|)A_2 + (\|r\| + \|s\|)A_3, \\
\|s\| &\leq (r_0 + r_1\|r\| + r_2\|s\|)B_1 + (s_0 + s_1\|r\| + s_2\|s\|)B_2 + (\|r\| + \|s\|)B_3.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|r\| + \|s\| \ \leq\ & (A_1+B_1)r_0 + (A_2+B_2)s_0 + \Big[(A_1+B_1)r_1 + (A_2+B_2)s_1 \\
& + A_3 + B_3\Big]\|r\| + \Big[(A_1+B_1)r_2 + (A_2+B_2)s_2 + A_3 + B_3\Big]\|s\|.
\end{aligned}$$

Consequently, we have

$$\|(r,s)\| \leq \frac{(A_1+B_1)r_0+(A_2+B_2)s_0}{D^*}, \tag{24}$$

where $D^* = \min\{1-[(A_1+B_1)r_1-(A_2+B_2)s_1-(A_3+B_3)], 1-[(A_1+B_1)r_2-(A_2+B_2)s_2-(A_3+B_3)]\}$ Hence, the set Θ is bounded, and via the Leray–Schader alternative, at least one fixed point of operator P is obtained, and this completes the proof. □

Krasnosel'skiǐ's fixed-point theorem is applied to obtain our second existence result.

Lemma 8. *(Krasnosel'skiǐ fixed-point theorem) [24] Let N indicates a closed, bounded, convex and nonempty subset of a Banach space Y, and C, D are operators, such that (i) $Cx + Dy \in N$ where $x, y \in N$, (ii) C is compact and continuous, and (iii) D is contraction mapping. Then, there exists $z \in N$, such that $z = Cz + Dz$.*

Theorem 3. *Assume that $D \neq 0$ and $h_1, h_2 : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. In addition, we suppose that*

(G_3) There exist non-negative functions $\phi_1, \phi_2 \in C([c,d], \mathbb{R}^+)$, such that $|h_1(z,r,s)| \leq \phi_1(z)$, $|h_2(z,r,s)| \leq \phi_2(z)$ for all $(z,r,s) \in [c,d] \times \mathbb{R} \times \mathbb{R}$.

Then, at least one solution of Problem (1) is obtained on $[c,d]$, provided that

$$A_3 + B_3 < 1. \tag{25}$$

Proof. First, we decompose operator P into four operators:

$$\begin{aligned}
\mathrm{M}(r,s)(z) \;=\;& -\frac{k}{\sigma}\int_c^z r(u)e^{\frac{\sigma-1}{\sigma}(z-u)}du \\
&+\frac{(z-c)^{\gamma_1-1}}{D\Gamma(\gamma_1)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[\Delta\Big(-\frac{k_1}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}s(u)e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du \\
&+\frac{k}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-c)}du\Big)+B\Big(-\frac{k}{\sigma}\sum_{i=1}^{n}\varepsilon_i\int_c^{\lambda_i}r(u)e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du \\
&+\frac{k_1}{\sigma}\int_c^d s(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)\Big],
\end{aligned} \tag{26}$$

$$\begin{aligned}
\mathrm{N}(r,s)(z) \;=\;& I_{c^+}^{\delta_1,\sigma}h_{1,r,s}(z) \\
&+\frac{(z-c)^{\gamma_1-1}}{D\Gamma(\gamma_1)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[\Delta\Big(\sum_{j=1}^{m}\theta_j I_{c^+}^{\delta_2,\sigma}h_{2,r,s}(\xi_j)-I_{c^+}^{\delta_1,\sigma}h_{1,r,s}(d)\Big) \\
&+B\Big(\sum_{i=1}^{n}\varepsilon_i I_{c^+}^{\delta_1,\sigma}h_{1,r,s}(\lambda_i)-I_{c^+}^{\delta_2,\sigma}h_{2,r,s}(d)\Big)\Big],
\end{aligned} \tag{27}$$

$$\begin{aligned}
\mathrm{S}(r,s)(z) \;=\;& -\frac{k_1}{\sigma}\int_c^z s(u)e^{\frac{\sigma-1}{\sigma}(z-u)}du \\
&+\frac{(z-c)^{\gamma_2-1}}{D\Gamma(\gamma_2)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[A\Big(-\frac{k}{\sigma}\sum_{i=1}^{n}\varepsilon_i\int_c^{\lambda_i}r(u)e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du\Big) \\
&+\frac{k_1}{\sigma}\int_c^d s(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)+\Gamma\Big(-\frac{k_1}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}s(u)e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du \\
&+\frac{k}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du\Big)\Big],
\end{aligned} \tag{28}$$

$$\begin{aligned} \mathrm{R}(r,s)(z) &= I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(z) \\ &+ \frac{(z-c)^{\gamma_2-1}}{D\Gamma(\gamma_2)} e^{\frac{\sigma-1}{\sigma}(z-c)} \Big[A\Big(\sum_{i=1}^{n} \varepsilon_i I_{c^+}^{\delta_1,\sigma} h_{1,r,s}(\lambda_i) - I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(d) \Big) \\ &+ \Gamma\Big(\sum_{j=1}^{m} \theta_j I_{c^+}^{\delta_2,\sigma} h_{2,r,s}(\xi_j) - I_{c^+}^{\delta_1,\sigma} h_{2,r,s}(d) \Big) \Big], \end{aligned} \tag{29}$$

Thus, $\mathrm{P}_1(r,s)(z) = \mathrm{M}(r,s)(z) + N(r,s)(z)$ and $\mathrm{P}_2(r,s)(z) = \mathrm{S}(r,s)(z) + R(r,s)(z)$. Let $B_\varepsilon = \{(r,s) \in \mathrm{X} \times \mathrm{Y}; \|(r,s)\| \leq \varepsilon\}$, in which

$$\varepsilon \geq \frac{(A_1+B_1)\|\phi_1\| + (A_2+B_2)\|\phi_2\|}{1-(A_3+B_3)}.$$

First, we indicate that $\mathrm{P}_1(r,s) + \mathrm{P}_2(r,s) \in B_\varepsilon$, where $(r,s) \in B_\varepsilon$. In view of the proof of Theorem 2, we have

$$\begin{aligned} &|\mathrm{M}(r,s)(z) + \mathrm{N}(r,s)(z)| \leq A_1\|\phi_1\| + A_2\|\phi_2\| + \varepsilon A_3, \\ &|\mathrm{S}(r,s)(z) + \mathrm{R}(r,s)(z)| \leq B_1\|\phi_1\| + B_2\|\phi_2\| + \varepsilon B_3, \end{aligned}$$

which leads to the fact that

$$\|\mathrm{P}_1(r,s) + \mathrm{P}_2(r,s)\| \leq (A_1+B_1)\|\phi_1\| + (A_2+B_2)\|\phi_2\| + \varepsilon(A_3+B_3) \leq \varepsilon.$$

Hence, $\mathrm{P}_1(r,s) + \mathrm{P}_2(r,s) \in B_\varepsilon$ and condition (i) of Lemma 8 is obtained.

Now, it is proven that operator (M,S) is contraction mapping. For $(r_1,s_1),(r_2,s_2) \in B_\varepsilon$, we have

$$\begin{aligned} & |\mathrm{M}(r_1,s_1)(z) - \mathrm{M}(r_2,s_2)(z)| \\ \leq\; & \frac{|k|}{\sigma}\int_c^d |r_1(u)-r_2(u)|du + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|\int_c^{\xi_j}|s_1(u)-s_2(u)|du \\ & + \frac{|k|}{\sigma}\int_c^d |r_1(u)-r_2(u)|du\Big) + B\Big(\frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|\int_c^{\lambda_i}|r_1(u)-r_2(u)|du \\ & + \frac{|k_1|}{\sigma}\int_c^d |s_1(u)-s_2(u)|du\Big)\Big] \\ \leq\; & \frac{|k|(d-c)}{\sigma}\|r_1-r_2\| + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)\|s_1-s_2\| \\ & + \frac{|k|(d-c)}{\sigma}\|r_1-r_2\|\Big) + B\Big(\frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)\|r_1-r_2\| + \frac{|k_1|(d-c)}{\sigma}\|s_1-s_2\|\Big)\Big] \\ \leq\; & \Big\{\frac{|k|(d-c)}{\sigma} + \frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c) + \frac{|k|(d-c)}{\sigma}\Big) \\ & + B\Big(\frac{|k|}{\sigma}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c) + \frac{|k_1|(d-c)}{\sigma}\Big)\Big]\Big\}(\|r_1-r_2\| + \|s_1-s_2\|) \\ =\; & A_3(\|r_1-r_2\| + \|s_1-s_2\|), \end{aligned}$$

and hence

$$\|\mathrm{M}(r_1,s_1) - \mathrm{M}(r_2,s_2)\| \leq A_3(\|r_1-r_2\| + \|s_1-s_2\|).$$

Similarly, we have

$$\|\mathrm{S}(r_1,s_1) - \mathrm{S}(r_2,s_2)\| \leq B_3(\|r_1-r_2\| + \|s_1-s_2\|).$$

Consequently, we obtain

$$\|(\mathrm{M},\mathrm{S})(r_1,s_1)-(\mathrm{M},\mathrm{S})(r_2,s_2)\|\leq(A_3+B_3)(\|r_1-r_2\|+\|s_1-s_2\|),$$

which, by (25), implies that (M,S) is a contraction, and condition (iii) of Lemma 8 is obtained.

In the next step, condition (ii) of Lemma 8 is considered for operator (N,R). By applying the continuity property of functions h_1 and h_2, we obtain that operator (NR) is continuous. For all $(r,s)\in B_\varepsilon$, due to the proof of Theorem 2, we have

$$\begin{aligned}|\mathrm{N}(r,s)(z)| &\leq \frac{\|\phi\|_1(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}+\frac{(d-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{\|\phi\|_2}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2}\\ &\quad+\frac{\|\phi\|_1(d-c)^{\delta_1}}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\Big)+B\Big(\frac{\|\phi\|_1}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1}\\ &\quad+\frac{\|\phi\|_2(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\Big)\Big]\\ &= A_1\|\phi_1\|+A_2\|\phi_2\|,\end{aligned}$$

and hence

$$\|\mathrm{N}(r,s)\|\leq A_1\|\phi_1\|+A_2\|\phi_2\|.$$

Similarly,

$$\|\mathrm{R}(r,s)\|\leq B_1\|\phi_1\|+B_2\|\phi_2\|.$$

Consequently, we have

$$\|(N,R)(r,s)\|\leq(A_1+B_1)\|\phi_1\|+(A_2+B_2)\|\phi_2\|,$$

thus, set $(N,R)B_\varepsilon$ is uniformly bounded.

Lastly, we show that set $(\mathrm{N},\mathrm{R})B_\varepsilon$ is equicontinuous. Let $z_1,z_2\in[c,d]$, such that $z_1<z_2$. For all $(r,s)\in B_\varepsilon$, due to the equicontinuous property of operators P_1 and P_2, we can show that $|\mathrm{N}(r,s)(z_2)-\mathrm{N}(r,s)(z_1)|\longrightarrow 0$, $|R(r,s)(z_2)-R(r,s)(z_1)|\longrightarrow 0$ az $z_1\longrightarrow z_2$. Consequently, set $(\mathrm{N},\mathrm{R})B_\varepsilon$ is equicontinuous. Now, using Arzelá-Ascoli theorem, the compactness property of operator (N,R) on B_ε is obtained. Hence, by using Lemma 8, at least one solution of Problem (1) is obtained on $[c,d]$. The proof is finished. □

4. Existence Results in Banach Space

In this section, the technique of measure of noncompactness is applied to construct an existence result concerning Problem (1). First, some elementary concepts about the notion of the measure of noncompactness are recalled.

Definition 6. *Assume that E is a Banach space, and M_E indicates the set of all bounded subsets of E. Mapping $\Omega: M_E\longrightarrow[0,\infty)$ defined via [25]*

$$\Omega(N)=\inf\Big\{\varepsilon>0: N\subseteq\cup_{i=1}^{m}N_i,\ \mathrm{diam}(N_i)\leq\varepsilon\Big\},$$

is called the Kuratowski measure of noncompactness.

Measure of noncompactness Ω comprises the following properties [25]:

(1) $\Omega(N)=0\Leftrightarrow\overline{N}$ is compact.
(2) $\Omega(N)=\Omega(\overline{N})$.
(3) $N_1\subset N_2\Rightarrow\Omega(N_1)\leq\Omega(N_2)$.
(4) $\Omega(N_1+N_2)\leq\Omega(N_1)+\Omega(N_2)$.
(5) $\Omega(\lambda N)=|\lambda|\Omega(N)$, $\lambda\in\mathbb{R}$.

(6) $\Omega(\text{conv}N) = \Omega(N)$.

Lemma 9. *Assume that $G \subseteq C([c,d],E)$ is a bounded and equicontinuous subset. Then, function $z \longrightarrow \Omega(G(z))$ is continuous on $[c,d]$ [26]:*

$$\Omega_C(G) = \max_{z\in[c,d]} \Omega(G(z)),$$

and

$$\Omega\Big(\int_c^d u(s)ds\Big) \le \int_c^d \Omega(G(s))ds,$$

where $G(s) = \{u(s) : u \in G\}, s \in [c,d]$.

Theorem 4. *(Mönch's fixed point theorem) Let set V be a closed, bounded, and convex subset in a Banach space Y, such that $0 \in Y$, and let $TV \longrightarrow V$ be continuous mapping satisfying [27]*

$$\overline{V} = \overline{conv}T(\overline{V}),\ or\ \overline{V} = T(\overline{V}) \cup \{0\} \Rightarrow \Omega(\overline{V}) = 0, \tag{30}$$

for all subsets $\overline{V}$ of V. Then, T contains a fixed point.

Definition 7. *Function $h : [c,d] \times E \longrightarrow E$ satisfies Carathéodory conditions if [28]:*

(i) *$h(z,u)$ is measurable with respect to z for all $u \in E$,*
(ii) *$h(z,u)$ is continuous with respect to $u \in E$ for $z \in [c,d]$.*

Theorem 5. *Assume that $A_3 + B_3 < 1$. Moreover, assume that*

(L_1) *Caratheodory conditions are satisfied by functions $h_1, h_2 : [c,d] \times E \times E \longrightarrow E$;*
(L_2) *There exist $\Omega_{h_1}, \Omega_{h_2} \in C([c,d], \mathbb{R}_+)$ and $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with ψ that is nondecreasing, such that*

$$\|h_i(z,r,s)\| \le \Omega_{h_i}(z)\psi(\|(r,s)\|),\ \ i = 1,2\ for\ a.e\ z \in [c,d]\ and\ r,s \in E.$$

(L_3) *For each bounded set $G \subseteq E \times E$ and for all $z \in [c,d]$, we have*

$$\Omega(h_i(z,G)) \le \Omega_{h_i}(z)\Omega(G).$$

If

$$\max\Big\{\Omega^*_{h_1}A_1 + \Omega^*_{h_2}A_2 + A_3,\ \ \Omega^*_{h_1}B_1 + \Omega^*_{h_2}B_2 + B_3\Big\} < 1,$$

where $\Omega^*_{h_i} = \sup_{z\in[c,d]} \Omega_{h_i}(z), i = 1,2$; then, Problem (1) has at least one solution on $[c,d]$.

Proof. Let operators $\mathrm{P}, \mathrm{P}_1, \mathrm{P}_2 : \mathrm{X} \times \mathrm{Y} \longrightarrow \mathrm{X} \times \mathrm{Y}$ be defined by (15)–(17), respectively. One can see that the fixed point of operator P is a solution of Problem (1). Define

$$B_r = \{(s,r) \in \mathrm{X} \times \mathrm{Y} : \|(s,r)\|_\infty \le r\},$$

in which $r \ge \dfrac{\Omega^*_{h_1}\psi(r)(A_1 + B_1) + \Omega^*_{h_2}\psi(r)(A_2 + B_2)}{1 - (A_3 + B_3)}$.

Step 1. *F maps B_r into itself.*

For all $(r,s) \in B_r$ and $z \in [c,d]$, we obtain

$$\begin{aligned} &\|\mathrm{P}_1(r,s)(z)\| \\ \le\ & I^{\delta_1,\sigma}_{c^+}\|h_{1,r,s}(z)\| + \frac{|k|}{\sigma}\int_c^z \|u(s)\|ds + \frac{(d-c)^{\gamma_1 - 1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\sum_{j=1}^m |\theta_j| I^{\delta_2,\sigma}_{c^+}\|h_{2,r,s}(\xi_j)\| \end{aligned}$$

$$
\begin{aligned}
& +\frac{|k_1|}{\sigma}\sum_{j=1}^{m}|\theta_j|\int_c^{\xi_j}\|s(t)\|dt + I_{c^+}^{\delta_1,\sigma}\|h_{1,r,s}(d)\| + \frac{|k|}{\sigma}\int_c^d\|r(t)\|dt\Big) \\
& +B\Big(\sum_{j=1}^{m}|\varepsilon_i|I_{c^+}^{\delta_1,\sigma}\|h_{1,r,s}(\lambda_i)\| + \frac{|k|}{\sigma}\sum_{j=1}^{m}|\varepsilon_i|\int_c^{\lambda_i}\|r(t)\|dt + I_{c^+}^{\delta_2,\sigma}\|h_{2,r,s}(d)\| \\
& +\frac{|k_1|}{\sigma}\int_c^d\|s(t)\|dt\Big)\Big] \\
\leq\ & \Omega_{h_1}^*\psi(r)A_1 + \Omega_{h_2}^*\psi(r)A_2 + rA_3,
\end{aligned}
$$

and

$$
\|\mathrm{P}_2(r,s)(z)\| \leq \Omega_{h_1}^*\psi(r)B_1 + \Omega_{h_2}^*\psi(r)B_2 + rB_3.
$$

Hence,

$$
\begin{aligned}
\|\mathrm{P}(r,s)\|_\infty &\leq \Omega_{h_1}^*\psi(r)(A_1+B_1) + \Omega_{h_2}^*\psi(r)(A_2+B_2) + r(A_3+B_3) \\
&\leq r.
\end{aligned}
$$

Hence, operator P maps ball B_r into itself.

Step 2. *The operator* P *is continuous.*

Let $\{(r_n,s_n)\} \in B_r$ such that $(r_n,s_n) \longrightarrow (r,s)$ as $n \longrightarrow \infty$. We indicate that $\|\mathrm{P}(r_n,s_n) - \mathrm{P}(r,s)\| \longrightarrow 0$. Since functions h_1 and h_2 satisfy Carathéodory conditions, we conclude that $\mathrm{P}_1(r_n,s_n) \longrightarrow \mathrm{P}_1(r,s)$ and $\mathrm{P}_2(r_n,s_n) \longrightarrow \mathrm{P}_2(r,s)$ as $n \longrightarrow \infty$. Now, due to condition (L_2) and the Lebesgue dominated convergence theorem, we obtain that $\|\mathrm{P}_1(r_n,s_n) - \mathrm{P}_1(r,s)\|, \|\mathrm{P}_2(r_n,s_n) - \mathrm{P}_2(r,s)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, $\|\mathrm{P}(r_n,s_n) - \mathrm{P}(r,s)\|_\infty \longrightarrow 0$, which is implies that P is continuous on B_r.

Step 3. *Operator* P *is equicontinuous.*

Let $l_1, l_2 \in [c,d]$ with $l_1 < l_2$ and $(r,s) \in B_r$. Thus, we have

$$
\begin{aligned}
& \|\mathrm{P}_1(r,s)(l_2) - \mathrm{P}_1(r,s)(l_1)\| \\
\leq\ & \frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)}\int_c^{l_2}[(l_2-u)^{\delta_1-1} - (l_1-u)^{\delta_1-1}]\|h_1(u,r(u),s(u))\|du \\
& +\frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)}\int_{l_1}^{l_2}(l_2-u)^{\delta_1-1}\|h_1(u,r(u),s(u))\|du + \frac{|k|}{\sigma}r(l_2-l_1) \\
& +\frac{(l_2-c)^{\gamma_1-1} - (l_1-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{\Omega_{h_2}^*\psi(r)}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2} \\
& +\frac{|k_1|r}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c) + \frac{\Omega_{h_1}^*\psi(r)}{\sigma^{\delta_1}\Gamma(\delta_1+1)} + \frac{r|k|(d-c)}{\sigma}\Big) \\
& +B\Big(\frac{\Omega_{h_2}^*\psi(r)}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1} + \frac{|k|r(\lambda_i-c)}{\sigma}|\varepsilon_i| + \frac{\Omega_{h_1}^*\psi(r)(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)} \\
& +\frac{r|k_1|(d-c)}{\sigma}\Big)\Big] \\
\leq\ & \frac{\Omega_{h_1}^*\psi(r)}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\Big[|(l_2-c)^{\delta_1} - (l_1-c)^{\delta_1} + 2(l_2-l_1)^{\delta_1}\Big] + \frac{|k|}{\sigma}r(l_2-l_1) \\
& +\frac{(l_2-c)^{\gamma_1-1} - (l_1-c)^{\gamma_1-1}}{|D|\Gamma(\gamma_1)}\Big[\Delta\Big(\frac{\Omega_{h_2}^*\psi(r)}{\sigma^{\delta_2}\Gamma(\delta_2+1)}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)^{\delta_2}
\end{aligned}
$$

$$+\frac{|k_1|r}{\sigma}\sum_{j=1}^{m}|\theta_j|(\xi_j-c)+\frac{\Omega^*_{h_1}\psi(r)}{\sigma^{\delta_1}\Gamma(\delta_1+1)}+\frac{r|k|(d-c)}{\sigma}\Big)$$
$$+B\Big(\frac{\Omega^*_{h_1}\psi(r)}{\sigma^{\delta_1}\Gamma(\delta_1+1)}\sum_{i=1}^{n}|\varepsilon_i|(\lambda_i-c)^{\delta_1}+\frac{|k|r(\lambda_i-c)}{\sigma}|\varepsilon_i|+\frac{\Omega^*_{h_2}\psi(r)(d-c)^{\delta_2}}{\sigma^{\delta_2}\Gamma(\delta_2+1)}$$
$$+\frac{r|k_1|(d-c)}{\sigma}\Big)\Big].$$

As $l_2\longrightarrow l_1$, we obtain that $\|\mathrm{P}_1(r,s)(l_2)-\mathrm{P}_1(r,s)(l_1)\|\longrightarrow 0$. Similarly, $\|\mathrm{P}_2(r,s)(l_2)-\mathrm{P}_2(r,s)(l_1)\|\longrightarrow 0$ as $l_2\longrightarrow l_1$. Consequently, $\|\mathrm{P}(r,s)(l_2)-\mathrm{P}(r,s)(l_1)\|_\infty\longrightarrow 0$, as l_2 tends to l_1. Thus P is equicontinuous.

Step 4. *Condition (30) of Theorem 4 is satisfied.*

Let $V_1=M_1\times M_2\subseteq\overline{conv}(\mathrm{P}_1(V_1)\cup\{0\})$ and $V_2=N_1\times N_2\subseteq\overline{conv}(\mathrm{P}_2(V_2)\cup\{0\})$ be two bounded and equicontinuous subsets. Thus, $T_1(z)=\Omega(V_1(z))$ and $T_2(z)=\Omega_2(V_2(z))$ are continuous on $[c,d]$. Now, in view of Lemma 9 and (L_3), we have

$$\begin{aligned}
T_1(z) &= \Omega(V_1(z))\le\Omega(\overline{conv}(\mathrm{P}_1(V_1)(z)\cup\{0\}))\le\Omega(\mathrm{P}_1(V_1)(z)))\\
&\le \Omega\Big\{\frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-u)}(z-u)^{\delta_1-1}h_1(u,r(u),s(u))du:(r,s)\in V_1\Big\}\\
&\quad+\Omega\Big\{\frac{k}{\sigma}\int_c^z s(u)e^{\frac{\sigma-1}{\sigma}(z-u)}du:s\in M_2\Big\}\\
&\quad+\frac{(z-c)^{\gamma_1-1}}{D\Gamma(\gamma_1)}e^{\frac{\sigma-1}{\sigma}(z-c)}\Big[\Delta\Big(\sum_{j=1}^{m}\theta_j\Omega\Big\{I_{c^+}^{\delta_2,\sigma}h_{1,r,s}(\xi_j):(r,s)\in V_1\Big\}\\
&\quad+\Omega\Big\{\frac{k_1}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}s(u)e^{\frac{\sigma-1}{\sigma}(\xi_j-u)}du:s\in M_2\Big\}\\
&\quad+\Omega\Big\{\frac{1}{\sigma^{\delta_1}\Gamma(\delta_1)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-u)}(d-u)^{\delta_1-1}h_1(u,r(u),s(u))du:(r,s)\in V_1\Big\}\\
&\quad+\Omega\Big\{\frac{k}{\sigma}\int_c^d r(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du:r\in M_1\Big\}\Big)\\
&\quad+B\Big(\sum_{i=1}^{n}\Omega\Big\{\varepsilon_i I_{c^+}^{\delta_1,\sigma}h_{1,r,s}(\lambda_i):(r,s)\in V_1\Big\}\\
&\quad+\Omega\Big\{\frac{k}{\sigma}\sum_{i=1}^{n}\Big\{\varepsilon_i\int_c^{\lambda_i}r(u)e^{\frac{\sigma-1}{\sigma}(\lambda_i-u)}du:r\in M_1\Big\}\\
&\quad+\Omega\Big\{\frac{1}{\sigma^{\delta_2}\Gamma(\delta_2)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-u)}(d-u)^{\delta_2-1}h_2(u,r(u),s(u))du:(r,s)\in V_1\Big\}\\
&\quad+\Omega\Big\{\frac{k_1}{\sigma}\int_c^d s(u)e^{\frac{\sigma-1}{\sigma}(d-u)}du:s\in M_2\Big\}\Big)\Big]\\
&\le \|T_1\|\Big\{\Omega^*_{h_1}A_1+\Omega^*_{h_2}A_2+A_3\Big\}.
\end{aligned}$$

Hence,

$$\|T_1\|_\infty\le\|T_1\|_\infty\Big\{\Omega^*_{h_1}A_1+\Omega^*_{h_2}A_2+A_3\Big\}.$$

It follows that $\|T_1\|_\infty=0$. Hence, for all $z\in[c,d]$ $T_1(z)=0$. Similarly, $T_2(z)=0$. Consequently, $\Omega((V_1\cap V_2)(z))\le\Omega(V_1(z))=0$ and $\Omega((V_1\cap V_2)(z))\le\Omega(V_2(z))=0$. Thus, $V(z)=(V_1\cap V_2)(z)$ is relatively compact in $E\times E$ and by Arzelá-Ascoli theorem V is relatively compact in $B_r\times B_r$. Now by applying Theorem 4, P contains a fixed point on $B_r\times B_r$ which is a solution of the problem (1). The proof is completed. □

5. Some Examples

Now, we present some illustration cases to show the benefits of our theorems.

Example 1. *Consider the following nonlocal boundary value problems of a coupled system of Hilfer generalized proportional fractional differential equations of form*

$$\begin{cases} \left(D^{\frac{3}{2},\frac{2}{3},\frac{3}{4}}_{\frac{1}{11}} + \dfrac{1}{12}D^{\frac{1}{2},\frac{2}{3},\frac{3}{4}}_{\frac{1}{11}}\right)r(z) = h_1(z,r(z),s(z)), \qquad z \in \left[\dfrac{1}{11},\dfrac{13}{11}\right], \\ \left(D^{\frac{5}{4},\frac{2}{3},\frac{3}{4}}_{\frac{1}{11}} + \dfrac{1}{15}D^{\frac{1}{4},\frac{2}{3},\frac{3}{4}}_{\frac{1}{11}}\right)s(z) = h_2(z,r(z),s(z)), \qquad z \in \left[\dfrac{1}{11},\dfrac{13}{11}\right], \\ r\left(\dfrac{1}{11}\right) = 0, \qquad r\left(\dfrac{13}{11}\right) = \dfrac{2}{21}s\left(\dfrac{3}{11}\right) + \dfrac{4}{23}s\left(\dfrac{7}{11}\right) + \dfrac{6}{25}s\left(\dfrac{10}{11}\right), \\ s\left(\dfrac{1}{11}\right) = 0, \ \ s\left(\dfrac{13}{11}\right) = \dfrac{1}{31}r\left(\dfrac{2}{11}\right) + \dfrac{3}{41}r\left(\dfrac{5}{11}\right) + \dfrac{5}{61}r\left(\dfrac{9}{11}\right) + \dfrac{7}{71}r\left(\dfrac{12}{11}\right). \end{cases} \tag{31}$$

Setting constants from boundary value problem (31) as $\delta_1 = 3/2$, $\delta_2 = 5/4$, $\eta = 2/3$, $\sigma = 3/4$, $k = 1/12$, $k_1 = 1/15$, $c = 1/11$, $d = 13/11$, $m = 3$, $\theta_1 = 2/21$, $\theta_2 = 4/23$, $\theta_3 = 6/25$, $\xi_1 = 3/11$, $\xi_2 = 7/11$, $\xi_3 = 10/11$, $n = 4$, $\varepsilon_1 = 1/31$, $\varepsilon_2 = 3/41$, $\varepsilon_3 = 5/61$, $\varepsilon_4 = 7/71$, $\lambda_1 = 2/11$, $\lambda_2 = 5/11$, $\lambda_3 = 9/11$, $\lambda_4 = 12/11$. Then we can calculate $\gamma_1 = 11/6$, $\gamma_2 = 7/4$, $A \approx 0.7945741588$, $B \approx 0.2983181613$, $\Gamma \approx 0.1617113429$, $\Delta \approx 0.8073671886$, $D \approx 0.5932716742$, $A_1 \approx 3.483107763$, $A_2 \approx 0.6362573874$, $A_3 \approx 0.3699010850$, $B_1 \approx 0.3281306250$, $B_2 \approx 1.755337743$, $B_3 \approx 0.2927192099$

(i) Consider unbounded Lipschitz functions $h_1, h_2 : [(1/11),(13/11)] \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$h_1(z,r,s) = \frac{e^{-(11z-1)^2}}{40}\left(\frac{r^2+2|r|}{1+|r|}\right) + \frac{\tan^{-1}|s|}{11z+20} + \frac{1}{3}z^2 + 2z, \tag{32}$$

$$h_2(z,r,s) = \frac{\sin^2 \pi z}{17}\sin|r| + \frac{2s^2+3|s|}{54(1+|s|)} + \frac{1}{4}z^3 + \frac{1}{2}. \tag{33}$$

Then, we can compute that

$$|h_1(z,r_1,s_1) - h_1(z,r_2,s_2)| \leq \frac{1}{20}|r_1 - r_2| + \frac{1}{21}|s_1 - s_2|$$

and

$$|h_2(z,r_1,s_1) - h_2(z,r_2,s_2)| \leq \frac{1}{17}|r_1 - r_2| + \frac{1}{18}|s_1 - s_2|,$$

for all $r_1, r_2, s_1, s_2 \in \mathbb{R}$ and $z \in [(1/11),(13/11)]$. By choosing $\ell_1 = 1/20$, $\ell_2 = 1/17$, we obtain $\ell_1(A_1 + B_1) + \ell_2(A_2 + B_2) + A_3 + B_3 \approx 0.9938642808 < 1$. By the benefit of Theorem 1, we deduce that the nonlocal boundary value problem of a coupled system of Hilfer generalized proportional fractional differential equations in (31) with h_1, h_2 given by (32) and (33), respectively, has a unique solution on $[(1/11),(13/11)]$.

(ii) Let nonlinear functions $h_1, h_2 : [(1/11),(13/11)] \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$h_1(z,r,s) = \frac{1}{2}\cos^{28}(zrs) + \frac{|r|^{153}}{21(1+r^{152})}e^{-s^{12}} + \frac{s^{184}}{20(1+|s|^{183})}\sin^{16} r, \tag{34}$$

$$h_2(z,r,s) = \tan^{-1}(\pi z) + \frac{r\cos^{14} s}{11z+17} + \frac{s^{36}}{19(1+|s|^{35})}e^{-r^{72}}. \tag{35}$$

Then we have

$$|h_1(z,r,s)| \leq \frac{1}{2} + \frac{1}{21}|r| + \frac{1}{20}|s|, \quad |h_2(z,r,s)| \leq \frac{\pi}{2} + \frac{1}{18}|r| + \frac{1}{19}|s|.$$

By setting $r_0 = 1/2, r_1 = 1/21, r_2 = 1/20, s_0 = \pi/2, s_1 = 1/18, s_2 = 1/19$, we obtain $(A_1 + B_1)r_1 + (A_2 + B_2)s_1 + A_3 + B_3 \approx 0.9769742333 < 1$ and $(A_1 + B_1)r_2 + (A_2 + B_2)s_2 + A_3 + B_3 \approx 0.9790556422 < 1$. By Theorem 2, we conclude that boundary value problem (31) with h_1, h_2 given by (34) and (35), respectively, has at least one solution on interval $[(1/11), (13/11)]$.

(iii) Suppose that nonlinear functions $h_1, h_2 : [(1/11), (13/11)] \times \mathbb{R}^2 \to \mathbb{R}$ are appeared by

$$h_1(z, r, s) = \frac{r^8 s^8}{1 + r^8 s^8} + \tan^{-1}\left(\sqrt{1 + r^4 s^6}\right) + \sin z + \frac{1}{3}, \tag{36}$$

$$h_2(z, r, s) = \frac{1}{2}z^2 + 2z + \frac{3}{4}e^{-r^2} + \frac{1}{4}\cos^4 s. \tag{37}$$

Then, we have

$$|h_1(z, r, s)| \le \frac{8 + 3\pi}{6} + \sin z := \phi_1(z) \quad \text{and} \quad |h_2(z, r, s)| \le \frac{1}{2}z^2 + 2z + 1 := \phi_2(z).$$

Since $A_3 + B_3 \approx 0.6626202949 < 1$, then, from Theorem 3, the nonlocal boundary value problem of a coupled system (31) with h_1, h_2 given bhy (36) and (37) respectively, has at least one solution on an interval $[(1/11), (13/11)]$.

(iv) Let $E = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0\}$ be a Banach space of real sequences converging to zero, endowed with norm $\|u\| = \sup_{n \ge 1} |u_n|$. Suppose that $h_1, h_2 : [(1/11), (13/11)] \times E^2 \to E$ appear through

$$h_1(z, r, s) = \frac{1}{11z + 7}\left(\frac{r_n^2 s_n^2}{1 + |r_n s_n|} + 1\right), \tag{38}$$

$$h_2(z, r, s) = \frac{1}{5}e^{-z^2}(1 + \sin(r_n s_n)). \tag{39}$$

It is easy to see that (L_1) in Theorem 5 holds. In addition, for $z \in [(1/11), (13/11)]$, we have

$$\|h_1(z, r, s)\| \le \frac{1}{11z + 7}\psi(\|(r, s)\|) \qquad \text{and} \qquad \|h_2(z, r, s)\| \le \frac{1}{5}e^{-z^2}\psi(\|(r, s)\|),$$

where $\psi(u) = u + 1$. Therefore, we get $\Omega^*_{h_1} = 1/8$ and $\Omega^*_{h_2} = 1/5$ and consequently

$$\begin{aligned} &\max\left\{\Omega^*_{h_1}A_1 + \Omega^*_{h_2}A_2 + A_3,\ \Omega^*_{h_1}B_1 + \Omega^*_{h_2}B_2 + B_3\right\} \\ \approx\ &\max\{0.9325410329, 0.6848030866\} = 0.9325410329 < 1. \end{aligned}$$

The application of Theorem 5 yields that boundary value problem (31) with h_1, h_2 given by (38) and (39), respectively, has at least one solution on an interval $[(1/11), (13/11)]$.

6. Conclusions

In the present research work, we investigated the existence and uniqueness of solutions for a new class of coupled system of Hilfer-type generalized proportional fractional differential equations supplemented with nonlocal multipoint boundary conditions. First, we proved an auxiliary result concerning a linear variant of the given problem, helping us in transforming the problem at hand into a fixed-point problem. Then, we proved the existence and uniqueness results in the scalar case by applying Banach's contraction mapping principle, Krasnosel'skiĭ's fixed-point theorem, and the Leray–Schauder alternative. Next, we studied the Banach space case, and established an existence result on the basis of Mönch's fixed-point theorem and the technique of the measure of noncompactness. All obtained results are well-illustrated by numerical examples. Our results are new and enrich the literature on coupled systems of Hilfer-type generalized proportional fractional differential equations.

Author Contributions: Conceptualization, A.S., S.K.N. and J.T.; methodology, A.S., S.K.N. and J.T.; validation, A.S., S.K.N. and J.T.; formal analysis, A.S., S.K.N. and J.T.; writing—original draft preparation, A.S., S.K.N. and J.T.; funding acquisition, J.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with contract no. KMUTNB-FF-65-36.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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