

On Nonlinear Biharmonic Problems on the Heisenberg Group

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Abstract: We investigate the boundary value problem for biharmonic operators on the Heisenberg group. The inherent features of \mathbb{H}^n make it an appropriate environment for studying symmetry rules and the interaction of analysis and geometry with manifolds. The goal of this paper is to prove that a weak solution for a biharmonic operator on the Heisenberg group exists. Our key tools are a version of the Mountain Pass Theorem and the classical variational theory. This paper will be of interest to researchers who are working on biharmonic operators on \mathbb{H}^n .

Keywords: Heisenberg group \mathbb{H}^n ; bi-Kohn Laplacian; Mountain Pass Theorem; variational theory

MSC: 35R03; 35A15



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1. Introduction

Recently, the embedded symmetry of the Heisenberg group \mathbb{H}^n has been used to combine geometric understanding and analytic calculations to establish a new sharp Stein–Weiss inequality on the line of duality with the mixed homogeneity. The Riesz potentials and $SL(2, \mathbb{R})$ invariance yield a natural bridge for the encoded information connecting these geometric structures.

Let us take a look at a few of the apparently disparate fields where \mathbb{H}^n emerges as a key player. We mention topics such as nilpotent Lie group representation theory, the structure theory of finite groups, homological algebra, the moduli of abelian varieties, abelian harmonic analysis foundations, partial differential equations, quantum physics, etc.

The automorphisms or symmetries of the finite Heisenberg group are crucial in the study of Lie algebras [1,2] and quantum mechanics in finite dimensions [3,4]. The concept of the quotient group of a specific normalizer properly expresses these symmetries [5]. One of the reasons might be that the role of \mathbb{H}^n is understated in many instances.

One could achieve what one wants out of a situation by ignoring extra structure imposed by \mathbb{H}^n , as did Hermann Weyl, one of the pioneers of incorporating \mathbb{H}^n into quantum mechanics (see Weyl [6]). We also think that Mumford's key contributions to the study included an understanding of the importance of \mathbb{H}^n in rigidifying abelian varieties (see Mumford [7–9]).

Another element that contributes to its relative obscurity is that \mathbb{H}^n is actually a collection of comparable objects, analogous to a functor or an algebraic geometry scheme, or even a combination of numerous overlapping functors. Howe [10] provided examples in harmonic analysis, which are largely attempts to bridge the gap between abelian and nonabelian harmonic analysis by illustrating how certain significant abelian harmonic analysis conclusions might be enriched through a Heisenberg group interpretation.

Suppose that $\Omega \subset \mathbb{H}^n$ is a bounded domain with smooth boundary $\partial\Omega$. The following is the boundary value problem for a biharmonic operator with a nonlinear source on \mathbb{H}^n :

$$\begin{cases} -\Delta_{\mathbb{H}^n}^2 u = f(x, u), & x \in \Omega \subset \mathbb{H}^n, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where ν denotes the outward unit normal on the boundary $\partial\Omega$.

We shall consider Carathéodory functions $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfy the following conditions:

(f₁) For some $c_1, c_2 > 0$,

$$|f(x, \xi)| \leq c_1 + c_2|\xi|^s \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R},$$

where Q is the homogeneous dimension of \mathbb{H}^n and

$$2 < s < \frac{2Q}{Q-2} - 1;$$

(f₂) $\lim_{|\xi| \rightarrow 0} \frac{f(x, \xi)}{|\xi|} = 0$ uniformly in $x \in \Omega$;

(f₃) For some $\mu > 2$ and $r > 0$,

$$0 < \mu F(x, \xi) < \xi f(x, \xi) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R},$$

where $|\xi| > r$ and $F(x, \xi) = \int_0^\xi f(x, t) dt$;

(f₄) $f(x, \xi) \in C(\bar{\Omega}, \mathbb{R})$.

Example 1. Let $n \geq 3$ and introduce the following notation:

$$t^* := \frac{nt}{n-t}.$$

Then $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, given by

$$f(x, \xi) := a(x)|\xi|^{s-2}\xi, \quad s \in [2, 2^*),$$

where $a \in L^\infty(\Omega)$, is an example of a Carathéodory function satisfying conditions (f₁) – (f₄).

Recently, the existence of at least one radial solution has been proved by Safari and Razani [11] for the following problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + R(\xi)u = a(|\xi|_{\mathbb{H}^n})|u|^{p-2}u - b(|\xi|_{\mathbb{H}^n})|u|^{q-2}u, & \xi \in \Omega, \\ u > 0, & \xi \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & \xi \in \partial\Omega, \end{cases}$$

where Ω denotes the Korányi ball in \mathbb{H}^n .

On the other hand, Kumar [12] considered the following problem:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u - \mu \frac{g(\xi)u}{(|z|^4+t^2)^{1/2}} = \frac{\lambda f(\xi)}{u^r} + h(\xi)u^p & \text{in } \Omega_r, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and showed that there exists a solution $u \in H_0^1(\Omega, \mathbb{H}^n) \cap L^\infty(\Omega)$.

Related to problem (1), Huang [13] studied eigenvalue problem (2) for the bi-Kohn Laplacian

$$\begin{cases} \Delta_{\mathbb{H}^n}^2 u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

and obtained universal bounds on the $(k + 1)$ -th eigenvalue in terms of the first k eigenvalues independent of the domains.

The purpose of our paper is to investigate \mathbb{H}^n , which is the most well-known non-abelian nilpotent Lie group. Many researchers have investigated the existence of solutions to semilinear equations in recent years – see, for example, Citti’s [14] pioneering work on \mathbb{H}^n . Problem (1) on \mathbb{H}^n is a natural generalization of the classical problem on \mathbb{R}^d (see An and Liu [15], Benci and Cerami [16], Cerami, Solimini and Struwe [17], Chaudhuri and Ramaswamy [18], Chen [19,20], and the references therein).

Our paper was inspired by recent work on \mathbb{H}^n by Bordoni, Filippucci and Pucci [21], D’Onofrio and Molica Bisci [22], Kassymov [23], Molica Bisci and Repovš [24], and Pucci and Temperini [25], where the authors established the existence of solutions to the elliptic equations (systems).

The following is the main result of our paper.

Theorem 1. *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the conditions $(f_1) - (f_4)$. Then, problem (1) has a nontrivial weak solution.*

To prove this result, in order to establish the existence, we shall invoke the Mountain Pass Theorem, as well as variational tools (see Ambrosetti and Rabinowitz [26]). As usual, one must prove that the Euler functional I associated with problem (1) satisfies the Palais–Smale compactness criterion and has appropriate geometric features.

2. Preliminaries

The definitions and notations associated with \mathbb{H}^n are discussed in this section. For further information, see Garofalo and Lanconelli [27] and Loiudice [28]. The Heisenberg group $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \cdot)$ is the space \mathbb{R}^{2n+1} with the following composition law

$$(\tilde{z}, t) \circ (z', t') := (\tilde{z} + z', t + t' + 2 \operatorname{Im}\langle \tilde{z}, z' \rangle).$$

Here,

$$\tilde{\xi} := (\tilde{z}, t) = (\tilde{x}, \tilde{y}, t) \in \mathbb{R}^{2n+1}, \tilde{x} \in \mathbb{R}^n, \tilde{y} \in \mathbb{R}^n, t \in \mathbb{R},$$

and \mathbb{R}^{2n} is identified by \mathbb{C}^n . The family of dilations has the following form:

$$\delta_\lambda(\tilde{\xi}) := (\lambda\tilde{x}, \lambda\tilde{y}, \lambda^2 t) \quad \text{for all } \lambda > 0.$$

The homogeneous dimension of \mathbb{H}^n is $Q = 2n + 2$, whereas its topological dimension is $2n + 1$.

The Lie algebra of \mathbb{H}^n is generated by the left-invariant vector fields

$$T := \frac{\partial}{\partial t}, \quad X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, \dots, n,$$

and for the generators, the following noncommutative formula holds

$$[X_i, Y_j] = -4\delta_{ij}T, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0. \tag{3}$$

The parabolic dilation given by

$$\delta_\lambda \xi := (\lambda x, \lambda y, \lambda^2 t),$$

has the property

$$\delta_\lambda(\xi_0 \cdot \xi) = \delta_\lambda \xi \cdot \delta_\lambda \xi_0,$$

where

$$z = (x, y) \in \mathbb{R}^{2n}, \quad \xi = (z, t) \in \mathbb{H}^n.$$

We define the homogeneous norm on \mathbb{H}^n by

$$|\xi| := \left(|z|^4 + t^2\right)^{\frac{1}{4}}$$

and we can write it as follows

$$|(x, y, t)| = \left(\left(x^2 + y^2\right)^2 + t^2\right)^{\frac{1}{4}},$$

which is also known as the Korányi gauge norm $N(z, t)$, i.e.,

$$\rho(\xi) = \left(|z|^4 + t^2\right)^{\frac{1}{4}}$$

is the Heisenberg distance between ξ and the origin.

In the same way, the distance between (z, t) and (z', t') on \mathbb{H}^n is given by

$$\rho(z, t; z', t') := \rho\left((z', t')^{-1} \cdot (z, t)\right).$$

The vector field T , on the other hand, is homogeneous of order 2.

$$T(f \circ \delta_r) = r^2 T(f) \circ \delta_r.$$

The Heisenberg gradient and the Kohn Laplacian are explicitly given by

$$\nabla_{\mathbb{H}^n} := (X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n)$$

and

$$\Delta_{\mathbb{H}^n} := \sum_{i=1}^n X_i^2 + Y_i^2 = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2} \right).$$

Next, recall the definitions of spaces $D^{1,p}(\Omega)$, $D_0^{1,p}(\Omega)$, $D^{2,p}(\Omega)$, and $D_0^{2,p}(\Omega)$

$$D^{1,p}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \mid u, |\nabla_{\mathbb{H}^n} u| \in L^p(\Omega)\},$$

$$D^{2,p}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \mid u, |\nabla_{\mathbb{H}^n} u|, |\Delta_{\mathbb{H}^n} u| \in L^p(\Omega)\},$$

where $\Omega \subseteq \mathbb{H}^n$ and $1 < p < \infty$, equipped with the norms

$$\|u\|_{D^{1,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and

$$\|u\|_{D^{2,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} + \|\Delta_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

Note that $D_0^{1,p}(\Omega)$ and $D_0^{2,p}(\Omega)$ are the closures of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_{D_0^{1,p}(\Omega)} := \left(\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}} \text{ and } \|u\|_{D_0^{2,p}(\Omega)} := \left(\int_{\Omega} |\Delta_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$

Theorem 2 (Dwivedi and Tyagi [29]). *Suppose that $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then the following properties hold:*

- (i) *If $k < \frac{Q}{p}$, then $D_0^{k,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ for $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{Q}$;*
- (ii) *If $k = \frac{Q}{p}$, then $D_0^{k,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for all $r \in [1, \infty)$;*
- (iii) *If $k > \frac{Q}{p}$, then $D_0^{k,p}(\Omega)$ is continuously embedded in $C^{0,\gamma}(\bar{\Omega})$ for all $0 \leq \gamma < k - \frac{Q}{p}$.*

We denote $W := D_0^{2,2}(\Omega)$ and define the energy functional $I : W \rightarrow \mathbb{R}$ in order to introduce the variational framework for problem (1):

$$I(u) := \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 dx - \int_{\Omega} F(x, u) dx, \tag{4}$$

where

$$F(x, u) = \int_0^u f(x, s) ds.$$

Let f be a Carathéodory function satisfying conditions (f1) – (f4). Then, it follows that for any $\zeta \in \mathbb{R}$,

- (i) There exist $c_3, c_4 > 0$ such that

$$c_3 |\zeta|^\mu - c_4 \leq F(x, \zeta) \quad \text{for all } x \in \Omega, \tag{5}$$

where $\mu > 2$;

- (ii) For any $\zeta \in \mathbb{R}$, we have

$$|f(x, \zeta)| \leq \varepsilon |\zeta| + (s + 1) \kappa(\varepsilon) |\zeta|^s \tag{6}$$

and

$$|F(x, \zeta)| \leq \varepsilon |\zeta|^2 + \kappa(\varepsilon) |\zeta|^{s+1}, \tag{7}$$

where ε and $\kappa(\varepsilon)$ are sufficiently small positive numbers.

Throughout this article, we shall denote $\| \cdot \|_{D_0^{2,2}(\Omega)}$ by $\| \cdot \|$. Next, $\| \cdot \|_p$ will represent the standard L^p norm. Furthermore, I will be assumed to be Fréchet differentiable for $u \in W$ and any $\varphi \in W$. Hence,

$$I'(u)\varphi = \int_{\Omega} \Delta_{\mathbb{H}^n} u \cdot \Delta_{\mathbb{H}^n} \varphi dx - \int_{\Omega} f(x, u)\varphi(x) dx.$$

For more background details, we recommend the comprehensive treatise by Papageorgiou, Rădulescu, and Repovš [30].

3. Proof of Theorem 1

Definition 1. *A function $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of problem (1) if $u \in W$ and*

$$\int_{\Omega} \Delta_{\mathbb{H}^n} u \cdot \Delta_{\mathbb{H}^n} \varphi dx = \int_{\Omega} f(x, u)\varphi(x) dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Definition 2. *A sequence $\{u_n\}$ in Banach space E is a $(PS)_c$ sequence for a functional $\Phi \in (\Phi, \mathbb{R})$ if every $\{u_n\} \subset E$ satisfies the following property*

$$\Phi(u_n) \rightarrow c \quad \text{for } n \rightarrow \infty \quad \text{and} \quad \Phi'(u_n) \rightarrow 0 \quad \text{for } n \rightarrow \infty \text{ in } E^*,$$

where $'$ denotes the Fréchet differential and E^* is the dual space of E .

Theorem 3 (Jabri [31]). *Let X be a Banach space, $\phi : X \rightarrow \mathbb{R}$ a C -functional with a $(PS)_c$ sequence, and Γ a class of paths joining $u = 0$ to $u = \omega$,*

$$\Gamma := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, \gamma(1) = \omega\},$$

where $\omega \in X, \|\omega\| > r > 0, \phi$ is bounded from below on

$$S(0, \rho) := \{u \in X \mid \|u\| \leq \rho\},$$

i.e.,

$$\alpha := \max\{\phi(0), \phi(\omega)\} < \beta := \inf_{u \in S(0, \rho)} \phi(u).$$

Then, ϕ has a critical value

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma(0,1)} \Phi(u) \geq \beta.$$

To prove our main result, we must first establish the following lemma.

Lemma 1. *Let Ω be a measurable subset of \mathbb{H}^n and f a Carathéodory function satisfying conditions (f_1) and (f_2) . Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$.*

Proof. Using the relation (7), we obtain

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} |\Delta_{\mathbb{H}^n} u(x)|^2 dx - \int_{\Omega} F(x, u(x)) dx \\ &\geq \frac{1}{2} \int_{\Omega} |\Delta_{\mathbb{H}^n} u(x)|^2 dx - \varepsilon \int_{\Omega} |u(x)|^2 dx - \kappa(\varepsilon) \int_{\Omega} |u(x)|^{s+1} dx \\ &\geq \frac{1}{2} \|u\|^2 - \varepsilon \|u\|^2 - \kappa(\varepsilon) \|u\|_s^{s+1}, \end{aligned}$$

since $D_0^{2,2}(\Omega)$ is continuously embedded in $L^s(\Omega)$, for all $1 \leq s < \infty$ (see Theorem 2). Therefore,

$$\|u\|_s \leq C \|u\|. \tag{8}$$

Consequently,

$$I(u) \geq \frac{1}{2} \|u\|^2 - \varepsilon \|u\|^2 - \kappa(\varepsilon) C \|u\|^{s+1} \geq \|u\|^2 \left(\frac{1}{2} - \varepsilon - \kappa(\varepsilon) C \|u\|^{s-1} \right).$$

Let $u \in W$ and $\|u\| = \rho$. By hypothesis, we know that $s > 1$. Choosing ρ sufficiently small and ε such that

$$\alpha := \rho^2 \left(\frac{1}{2} - \varepsilon - \kappa(\varepsilon) C \rho^{s-1} \right) > 0,$$

we obtain $I(u) \geq \alpha > 0$. \square

A second condition of the Mountain Pass Theorem will be provided by following lemma.

Lemma 2. *Let f be a Carathéodory function satisfying conditions $(f1)$ – $(f4)$ and let $\alpha, \rho > 0$ be the constants from Lemma 1. Then there exists $v > 0$ a.e. in W , such that $\|v\| > \rho$ and $I(v) < \alpha$.*

Proof. Fix $\|u\| = 1$ and $u \geq 0$ a.e. in \mathbb{H}^n with $t > 0$. Invoking (5), we can obtain

$$\begin{aligned}
I(tu) &= \frac{1}{2} \int_{\Omega} |\Delta_{\mathbb{H}^n}(tu)|^2 dx - \int_{\Omega} F(x, tu(x)) dx \\
&\leq \frac{t^2}{2} \int_{\Omega} |\Delta_{\mathbb{H}^n} u|^2 dx - c_4 t^\mu \int_{\Omega} |u|^\mu dx + c_3 |\Omega| \\
&= \frac{t^2}{2} - c_4 t^\mu \int_{\Omega} |u|^\mu dx + c_3 |\Omega|.
\end{aligned}$$

Since $\mu > 2$ and $t \rightarrow +\infty$, we have $I(tu) \rightarrow -\infty$. Then, setting $v = \beta u$, we get the assertion when β is large enough. \square

Lemma 3. Let f be a Carathéodory function satisfying conditions (f_1) – (f_4) and $\{u_n\} \subset W$ a sequence satisfying $I(u_n) \rightarrow c$ and

$$\sup\{|I'(u_n)\varphi| : \varphi \in W, \|\varphi\| = 1\} \rightarrow 0, \quad n \rightarrow \infty. \quad (9)$$

Then, the sequence $\{u_n\}$ is bounded in W .

Proof. Let $\{u_n\} \subset W$ be a $(PS)_c$ sequence. Then, for every $\varphi \in W$, we have

$$I'(u_n)\varphi = \int_{\Omega} \Delta_{\mathbb{H}^n} u_n \cdot \Delta_{\mathbb{H}^n} \varphi dx - \int_{\Omega} f(x, u_n(x)) \varphi(x) dx$$

and

$$I(u_n) = \frac{1}{2} \int_{\Omega} |\Delta_{\mathbb{H}^n} u_n|^2 dx - \int_{\Omega} F(x, u_n(x)) dx.$$

Therefore,

$$\begin{aligned}
I(u) - \frac{1}{\mu} I'(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\Omega} |\Delta_{\mathbb{H}^n} u_n|^2 dx - \int_{\Omega} \left(F(x, u(x)) - \frac{f(x, u(x))u(x)}{\mu}\right) dx \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\Omega} |\Delta_{\mathbb{H}^n} u_n|^2 dx - \int_{\Omega \cap |u_n| \leq r} \left(F(x, u(x)) - \frac{f(x, u(x))u(x)}{\mu}\right) dx \\
&\quad - \int_{\Omega \cap |u| > r} \left(F(x, u(x)) - \frac{f(x, u(x))u(x)}{\mu}\right) dx,
\end{aligned}$$

where $\mu > 2$.

Invoking (7), we can calculate

$$\left| \int_{\Omega \cap |u| \leq r} F(x, u(x)) - \frac{f(x, u(x))u(x)}{\mu} dx \right| \leq \left(\varepsilon r^2 + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1}) \right) |\Omega|. \quad (10)$$

For simplicity, we denote

$$\tilde{\theta} := \left(\varepsilon r^2 + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1}) \right) |\Omega|.$$

Using (10) and assumption (f_3) , we get

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\Omega} |\Delta_{\mathbb{H}^n} u_n|^2 dx - \tilde{\theta}. \quad (11)$$

Invoking (9) with $\varphi := u_n / \|u_n\|_W$, we know that for any n there exists $\lambda > 0$ such that

$$\left| I'(u_n) \left(\frac{u_n}{\|u_n\|} \right) \right| \leq \lambda,$$

with $I(u_n) \leq \lambda$.

Thus, we can get

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \leq \lambda(1 + \|u_n\|).$$

Combining this with (11), we arrive at

$$\left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \leq \lambda(1 + \|u_n\|) + \tilde{\theta}.$$

Finally,

$$\|u_n\|^2 \leq \left(\frac{1}{2} - \frac{1}{\mu}\right)^{-1} (\lambda(1 + \|u_n\|) + \tilde{\theta}) \leq \left(\frac{1}{2} - \frac{1}{\mu}\right)^{-1} C_1(1 + \|u_n\|) \leq C(1 + \|u_n\|),$$

where $C > 0$. \square

Next we shall prove that the $(PS)_c$ sequence of I has a strongly convergent subsequence.

Lemma 4. *Let f be a Carathéodory function satisfying conditions (f_1) – (f_4) and suppose that $\{u_n\} \subset W$ is a $(PS)_c$ sequence of I . Then $\{u_n\}$ has a strong convergent subsequence in W .*

Proof. Since W is a reflexive Banach space, we have $u_n \rightarrow u$ weakly in W . Thus,

$$I'(u_n)(u_n - u) = \int_{\Omega} \Delta_{\mathbb{H}^n} u_n \cdot \Delta_{\mathbb{H}^n} (u_n - u) dx - \int_{\Omega} f(x, u_n)(u_n - u) dx \rightarrow 0, \quad n \rightarrow \infty. \quad (12)$$

Also, $u_n \rightarrow u$ is strongly convergent in $L^{s+1}(\Omega)$. Therefore

$$f(x, u_n)(u_n - u) \rightarrow 0, \text{ a.e. in } \Omega, \quad n \rightarrow \infty.$$

Using the Vitali convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n)(u_n - u) dx = 0. \quad (13)$$

Plugging (13) in (12), we get

$$\int_{\Omega} \Delta_{\mathbb{H}^n} u_n \cdot \Delta_{\mathbb{H}^n} (u_n - u) dx \rightarrow 0, \quad n \rightarrow \infty, \quad (14)$$

which yields

$$\|u_n - u\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof of Lemma 4. \square

We can now prove Theorem 1.

Proof. We begin the proof by observing that by virtue of Lemma 4, every $(PS)_c$ subsequence of I is strongly convergent in W . It is now easy to verify that $I(0) = 0$.

On the other hand, by Lemma 2, there exists $\alpha > 0$ such that

$$I(u) \geq \alpha > I(0), \text{ where } u \in W, \|u\| = \rho.$$

The existence of a critical point of the functional I , which is a nontrivial weak solution of problem (1), can now be deduced by invoking the Mountain Pass Theorem. \square

Remark 1. *We have proved the main result under the (AR) condition. It is a natural question if one can omit the (AR)-condition (see Choudhuri [32]).*

4. Epilogue

The biharmonic operators on the Heisenberg group \mathbb{H}^n were investigated in depth in this paper. Our findings support some previously published research. We believe that researchers working in this field will be inspired by our work and that our results will stimulate more research in this area.

In conclusion, we believe that our results can be generalized to other biharmonic operators. This problem is left as an open problem for researchers who are interested in this topics. For example, one might consider the following relevant open problem

$$\begin{cases} -\Delta_{H,p}^2 u(x) = f(x, u), & x \in \Omega \subset \mathbb{H}^n, \\ u(x) = \Delta_H u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (15)$$

where $\Omega \subset \mathbb{H}^n$ is a measurable set with sufficiently smooth boundary $\partial\Omega$ and the p -biharmonic is given by

$$\Delta_{H,p}^2 := \Delta_H \cdot (|\Delta_H|^{p-2} \Delta_H).$$

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