symmetry

MDPI

Article

Applications of the Atangana–Baleanu Fractional Integral Operator

Alina Alb Lupaş *,† **and Adriana Cătaş** †

Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; acatas@uoradea.ro

* Correspondence: dalb@uoradea.ro or alblupas@gmail.com

† These authors contributed equally to this work.

Abstract: Applications of the Atangana–Baleanu fractional integral were considered in recent studies related to geometric function theory to obtain interesting differential subordinations. Additionally, the multiplier transformation was used in many studies, providing elegant results. In this paper, a new operator is defined by combining those two prolific functions. The newly defined operator is applied for introducing a new subclass of analytic functions, which is investigated concerning certain properties, such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and radii of starlikeness, convexity and close-to-convexity. This class may have symmetric or asymmetric properties. The results could prove interesting due to the new applications of the Atangana–Baleanu fractional integral and of the multiplier transformation. Additionally, the univalence properties of the new subclass of functions could inspire researchers to conduct further investigations related to this newly defined class.

Keywords: analytic functions; univalent functions; radii of starlikeness and convexity; neighborhood property; multiplier transformation; Atangana–Baleanu fractional integral

MSC: 30C45; 30A20; 34A40

Citation: Alb Lupaş, A.; Cătaş, A. Applications of the Atangana–Baleanu Fractional Integral Operator. *Symmetry* **2022**, *14*, 630. https://doi.org/10.3390/sym14030630

Academic Editor: Şahsene Altınkaya

Received: 3 March 2022
Accepted: 19 March 2022
Published: 21 March 2022

1. Introduction

Fractional calculus has many applications in diverse fields of research. The papers [1,2] discuss the history of fractional calculus and provide references to its many applications in science and engineering.

Owa [3] and Owa and Srivastava [4] applied a fractional integral for a function which gave new possibilities for studying properties of functions and for defining new operators. A fractional integral was considered on a confluent hypergeometric function in recently published papers (see [5]) and on Ruscheweyh and Sălăgean operators in [6]. Applications of fractional derivatives with Mittag–Leffler kernels were considered in [7,8] and with non-local and non-singular kernels in [9].

Atangana and Baleanu [10] used the Riemann–Liouville fractional integral for introducing a new fractional integral studied by many researchers in recent years. The Atangana–Baleanu fractional integral of Bessel functions was used in studies [11,12]. Nice results were recently obtained regarding Ostrowski-type integral inequalities [13] and Hermite–Hadamard-type inequalities [14] involving an Atangana–Baleanu fractional integral operator. The definition given by Atangana–Baleanu can be extended to complex values of the order of differentiation ν by using analytic continuation.

Multiplier transformation has also been used for recent studies, as can be seen in [15,16].

Inspired by the nice results seen in the papers published considering the Atangana—Baleanu fractional integral and multiplier transformation separately, we have decided to merge them and to define a new operator, which will be given below. This operator is used to introduce a new subclass of analytic functions, since introducing and studying new

classes of univalent functions generates very interesting results, and we can remind only a few very recent results, such as new subclasses for bi-univalent functions [17,18] and classes of functions introduced using operators [19,20].

The class of analytic function is denoted by $\mathcal{H}(U)$ in $U = \{z \in \mathbb{C} : |z| < 1\}$. The open unit disc of the complex plane is denoted by $\mathcal{H}(a, n)$. The subclass of $\mathcal{H}(U)$ of functions is denoted with the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$, where $\mathcal{A} = \mathcal{A}_1$.

The special class of starlike functions of order α is defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, 0 \leq \alpha < 1 \right\}$$

and the class of convex functions of order α is defined as

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, 0 \leq \alpha < 1 \right\}.$$

For introducing the new operator, the following previously known results are necessary.

Definition 1 ([21]). *For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, the multiplier transformation $I(m, \alpha, l)f(z)$ is defined by the following infinite series:*

$$I(m, \alpha, l)f(z) := z + \sum_{k=2}^{\infty} \left(\frac{1 + \alpha(k-1) + l}{1 + l} \right)^m a_k z^k.$$

We recall that the Riemann–Liouville fractional integral ([22]) is defined by the following relationship:

$${}_c^{RL}I_z^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_c^z (z - w)^{\nu - 1} f(w) dw, \ \ \mathrm{Re}\ (\nu) > 0,$$

which is used in the Atangana–Baleanu fractional integral.

In paper [23], symmetric and anti-symmetric derivatives of the Riemann–Liouville and Caputo type were defined, and the reflection symmetry properties of fractional differentiation were studied.

Definition 2 ([24]). *Let c be a fixed complex number and f be a complex function which is analytic on an open star-domain D centered at c. The extended Atangana–Baleanu integral, denoted by ${}_c^{AB}I_z^{\nu} f(z)$, is defined for any $\nu \in \mathbb{C}$ and any $z \in D \backslash \{c\}$ by:*

$${}_c^{AB}I_z^{\nu} f(z) = \frac{1 - \nu}{B(\nu)} f(z) + \frac{\nu}{B(\nu)} {}_c^{RL}I_z^{\nu} f(z).$$

Proposition 1 ([24]). *The extended Atangana–Baleanu integral from Definition 2 is an analytic function of both $z \in D \backslash \{c\}$ and $\nu \in \mathbb{C}$, provided f and B are analytic and B is nonzero. This is identical to the original formula in the real case when $0 < \nu < 1$ and $c < z$ in $\mathbb{R}$.*

Therefore, it provides the analytic continuation of the original Atangana–Baleanu integral to complex values of z and ν.

Applying the Atangana–Baleanu fractional integral for $c = 0$ to multiplier transformation, we define a new operator:

Definition 3. *Let $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $\nu \in \mathbb{C}$, and any $z \in D \backslash \{0\}$. The Atangana–Baleanu fractional integral associated with the multiplier transformation $I(m, \alpha, l)f$ is defined by:*

$${}_0^{AB}I_z^{\nu}(I(m, \alpha, l)f(z)) = \frac{1 - \nu}{B(\nu)} I(m, \alpha, l)f(z) + \frac{\nu}{B(\nu)} {}_0^{RL}I_z^{\nu} I(m, \alpha, l)f(z).$$

After a simple calculation, the following form is obtained for this operator:

$$ {}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) = \frac{1-\nu}{B(\nu)}z + \frac{\nu}{B(\nu)\Gamma(\nu+2)}z^{\nu+1} +$$

$$\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m a_k z^k + \frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \frac{\Gamma(k+1)}{\nu+k+1}a_k z^{k+\nu},$$

for the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$.

The new class is defined using the new operator.

Definition 4. *A function $f \in \mathcal{A}$ is said to be in the class ${}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ if it satisfies the following criterion:*

$$\left|\frac{1}{d}\left(\frac{z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))' + \gamma z^2({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))''}{(1-\gamma){}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) + \gamma z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))'} - 1\right)\right| < \beta,$$

where $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \geq 0$, $\nu \in \mathbb{C}, d \in \mathbb{C} - \{0\}$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $z \in U\backslash\{0\}$.

In this section, a new subclass of analytic functions was introduced in Definition 4 after we presented the notations and definitions used during our investigation. The properties regarding the coefficient inequalities for the functions contained in the newly introduced class are obtained in Section 2 of the paper. Distortion bounds for functions from the class and for their derivative are given in Section 3, and properties regarding closure of the class are proven in Section 4, considering partial sums of functions from the class, with extreme points of the class being also provided. In Section 5, inclusion relations are obtained for certain values of the parameters involved, and neighborhood properties are discussed, while radii of starlikeness, convexity and close-to-convexity of the class are obtained in Section 6 of the paper.

To shorten the formulas, we have to make the notation $A = \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m$ throughout the paper.

2. Coefficient Inequality

Theorem 1. *The function $f \in \mathcal{A}$ belongs to the class ${}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) +$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_k \leq$$

$$\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}. \tag{1}$$

where $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \geq 0$, $\nu \in \mathbb{C}, d \in \mathbb{C} - \{0\}$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $z \in U\backslash\{0\}$.

Proof. Let $f \in \mathcal{A}$. Consider the inequality (1). After making an easy calculus, we get:

$$\left|\frac{z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))' + \gamma z^2({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))''}{(1-\gamma){}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) + \gamma z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)))'} - 1\right| =$$

$$\left|\frac{\frac{\nu^2(1+\gamma\nu)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1} + \frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A(k-1)[\gamma(k-1)+1]a_k z^k + \frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}(k+\nu-1)[\gamma(k+\nu-1)+1]a_k z^{k+\nu}}{\frac{1-\nu}{B(\nu)}z + \frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1} + \frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_k z^k + \frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_k z^{k+\nu}}\right| \leq$$

and applying properties of modulus function, we obtain:

$$\frac{\frac{\nu^2(1+\gamma\nu)}{B(\nu)\Gamma(\nu+2)}|z|^\nu+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A(k-1)[\gamma(k-1)+1]a_k|z|^{k-1}+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}(k+\nu-1)[\gamma(k+\nu-1)+1]a_k|z|^{k+\nu-1}}{\frac{1-\nu}{B(\nu)}-\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}|z|^\nu-\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_k|z|^{k-1}-\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_k|z|^{k+\nu-1}}$$
$$\leq \beta|d|.$$

Taking values of z on a real axis and for $z \to 1^-$, we find:

$$\sum_{k=2}^{\infty}A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$
$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_k \leq$$
$$\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}.$$

Conversely, consider $f \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$. Then, we obtain the inequality

$$Re\left\{\frac{z({}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z)))'+\gamma z^2({}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z)))''}{(1-\gamma){}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z))+\gamma z({}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z)))'}-1\right\}>-\beta|d|,$$

written as

$$Re\left\{\frac{\frac{\nu^2(1+\gamma\nu)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1}+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A(k-1)[\gamma(k-1)+1]a_kz^k+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}(k+\nu-1)[\gamma(k+\nu-1)+1]a_kz^{k+\nu}}{\frac{1-\nu}{B(\nu)}z+\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1}+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_kz^k+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_kz^{k+\nu}}\right.$$
$$\left.+\beta|d|\right\}>0$$

and equivalently with:

$$Re\left\{\frac{\frac{(1-\nu)\beta|d|}{B(\nu)}z+\frac{\nu(1+\gamma\nu)(\nu+\beta|d|)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1}+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A(k-1+\beta|d|)[\gamma(k-1)+1]a_kz^k}{\frac{1-\nu}{B(\nu)}z+\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1}+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_kz^k+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_kz^{k+\nu}}+\right.$$
$$\left.\frac{\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}(k+\nu-1+\beta|d|)[\gamma(k+\nu-1)+1]a_kz^{k+\nu}}{\frac{1-\nu}{B(\nu)}z+\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}z^{\nu+1}+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_kz^k+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_kz^{k+\nu}}\right\}>0.$$

Considering that $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the inequality becomes:

$$\frac{\frac{(1-\nu)\beta|d|}{B(\nu)}-\frac{\nu(1+\gamma\nu)(\nu+\beta|d|)}{B(\nu)\Gamma(\nu+2)}r^\nu-\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A(k-1+\beta|d|)[\gamma(k-1)+1]a_kr^{k-1}}{\frac{1-\nu}{B(\nu)}-\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}r^\nu-\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_kr^{k-1}-\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_kr^{k+\nu-1}}-$$
$$\frac{\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}(k+\nu-1+\beta|d|)[\gamma(k+\nu-1)+1]a_kr^{k+\nu-1}}{\frac{1-\nu}{B(\nu)}-\frac{\nu(\gamma\nu+1)}{B(\nu)\Gamma(\nu+2)}r^\nu-\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}A[\gamma(k-1)+1]a_kr^{k-1}-\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}A\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}[\gamma(k+\nu-1)+1]a_kr^{k+\nu-1}}>0.$$

Applying the mean value theorem when $r \to 1^-$ we get the inequality (1), and the proof is complete. □

Corollary 1. *Coefficients of the function $f \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ have the property:*

$$a_k \leq \frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\Big[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]},$$

for $k \geq 2$.

3. Distortion Theorems

Theorem 2. *Thefunction $f \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$, for $|z| = r < 1$, has the property:*

$$r-\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\Big[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\Big]}r^2 \leq |f(z)|$$

$$\leq r+\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r^2.$$

The equality holds for the function

$$f(z)=z+\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}z^2,\ z\in U. \quad (2)$$

Proof. Let $f\in{}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$. Then, considering the relation (1) and that the sequence

$$\sum_{k=2}^{\infty}A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]$$

is increasing and positive for $k\geq 2$, we obtain the inequality

$$\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]\sum_{k=2}^{\infty}a_k\leq$$

$$\sum_{k=2}^{\infty}A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_k\leq$$

$$\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)},$$

equivalently with

$$\sum_{k=2}^{\infty}a_k\leq\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}. \quad (3)$$

Using the properties of modulus function for

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k,$$

we get

$$r-r^2\sum_{k=2}^{\infty}a_k\leq r-\sum_{k=2}^{\infty}a_kr^k\leq|z|-\sum_{k=2}^{\infty}a_k|z|^k\leq|f(z)|$$

$$\leq|z|+\sum_{k=2}^{\infty}a_k|z|^k\leq r+\sum_{k=2}^{\infty}a_kr^k\leq r+r^2\sum_{k=2}^{\infty}a_k,$$

and considering relation (3), we obtain

$$r-\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r^2\leq|f(z)|$$

$$\leq r+\frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1)+\frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r^2,$$

completing the proof. □

Theorem 3. *The function $f \in {}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$, for $|z| = r < 1$, has the property:*

$$1 - \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r \leq |f'(z)|$$

$$\leq 1 + \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r.$$

The equality holds for the function given by relation (2).

Proof. Using the properties of the modulus function for

$$f'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1},$$

we obtain:

$$1 - \sum_{k=2}^{\infty} k a_k |z| \leq 1 - \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq |f'(z)| \leq 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1 + \sum_{k=2}^{\infty} k a_k |z|.$$

Applying relation (3), we get

$$1 - \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r \leq |f'(z)|$$

$$\leq 1 + \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}r,$$

and the proof is complete. □

4. Closure Theorems

Theorem 4. *The functions $f_p \in {}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ of the following form*

$$f_p(z) = z + \sum_{k=2}^{\infty} a_{k,p} z^k, \quad a_{k,p} \geq 0,\ z \in U, \tag{4}$$

$p = 1, 2, \ldots, q$, define the function h by relation

$$h(z) = \sum_{p=1}^{q} \mu_p f_p(z), \quad \mu_p \geq 0,\ z \in U,$$

which belongs to the class ${}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$, when

$$\sum_{p=1}^{q} \mu_p = 1.$$

Proof. The function h has the following form:

$$h(z) = \sum_{p=1}^{q} \mu_p z + \sum_{p=1}^{q}\sum_{k=2}^{\infty} \mu_p a_{k,p} z^k = z + \sum_{k=2}^{\infty}\sum_{p=1}^{q} \mu_p a_{k,p} z^k.$$

Regarding the functions f_p, $p = 1, 2, \ldots, q$, being contained in the class ${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$, by applying Theorem 1 we obtain:

$$\sum_{k=2}^{\infty} A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_{k,p} \leq$$

$$\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}.$$

In this condition, we have to prove that:

$$\sum_{k=2}^{\infty} A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]\left(\sum_{p=1}^{q}\mu_p a_{k,p}\right) \leq$$

$$\sum_{p=1}^{q}\mu_p \sum_{k=2}^{\infty} A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_{k,p} \leq$$

$$\sum_{p=1}^{q}\mu_p\left(\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right) =$$

$$\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}.$$

Hence the proof is complete. □

Corollary 2. *Let the functions f_p, $p = 1, 2$, as defined by relation (4) from the class ${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$. Then, the function h given by*

$$h(z) = (1-\xi)f_1(z) + \xi f_2(z), \quad 0 \leq \xi \leq 1,\ z \in U,$$

belongs to the class ${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$, as well.

Theorem 5. *Let*

$$f_k(z) = z+$$

$$\frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}z^k,$$

$k \geq 2$, $z \in U$, and

$$f_1(z) = z.$$

The function f belongs to the class ${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$ if and only if it can be written as

$$f(z) = \sum_{k=1}^{\infty}\mu_k f_k(z), \quad z \in U,$$

with $\mu_k \geq 0$, $k \geq 1$ and $\sum_{k=1}^{\infty}\mu_k = 1$.

Proof. Consider the function

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) =$$

$$z + \sum_{k=2}^{\infty} \frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]} \mu_k z^k;$$

we get:

$$\sum_{k=2}^{\infty} \frac{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}} \cdot$$

$$\frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]} \mu_k$$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

Therefore, $f \in_0^{AB} \mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$.
Conversely, considering $f \in_0^{AB} \mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$ and the setting

$$\mu_k = \frac{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}} a_k,$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k,$$

we obtain

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z).$$

Hence, the proof is complete. □

Corollary 3. *The class* ${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$ *has as the extreme points the functions*

$$f_1(z) = z,$$

and

$$f_k(z) = z +$$

$$\frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]} z^k,$$

$k \geq 2, z \in U$.

5. Inclusion and Neighborhood Results

The δ- neighborhood for a function $f \in \mathcal{A}$ is given by

$$N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta\}, \tag{5}$$

and for $e(z) = z$, we obtain

$$N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|b_k| \leq \delta\}. \tag{6}$$

A function $f \in \mathcal{A}$ belongs to the class ${}_0^{AB}\mathcal{I}^{\zeta}(m, \alpha, l, \nu, \gamma, d, \beta)$ if there exists a function $h \in {}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1 - \zeta, \ \ z \in U, \ \ 0 \leq \zeta < 1. \tag{7}$$

Theorem 6. *We have the inclusion*

$${}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta) \subset N_\delta(e)$$

for

$$\delta = \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}.$$

Proof. Consider $f \in {}_0^{AB}\mathcal{I}(m, \alpha, l, \nu, \gamma, d, \beta)$. Using Theorem 1 and using the fact that the sequence

$$\sum_{k=2}^{\infty} A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]$$

is increasing and positive for $k \geq 2$, as we say in Theorem 2, we obtain

$$\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]\sum_{k=2}^{\infty} a_k \leq$$

$$\sum_{k=2}^{\infty} A[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+$$

$$\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\Big]a_k \leq$$

$$\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)},$$

which implies

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}. \tag{8}$$

Applying Theorem 1 in conjunction with (8), we get

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}{\left(\frac{1+\alpha+l}{l+1}\right)^m\left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]} = \delta.$$

By virtue of (5), we obtain $f \in N_\delta(e)$, which completes the proof. □

Theorem 7. *If* $h \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ *and*

$$\zeta = 1 - \frac{\delta}{2\left(1 - \frac{\frac{(\beta-1)(\lambda+(1-\lambda)\mu)}{(\beta+1)\nu} - \frac{\beta\gamma B(\nu)}{(\beta+1)(1-\nu)\nu} - \frac{\lambda+(1+\nu-\lambda)\mu}{(1-\nu)\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m \left[\frac{\lambda+(2-\lambda)\mu}{\nu} + \frac{2(\lambda+(2+\nu-\lambda)\mu)}{(1-\nu)\Gamma(\nu+3)}\right]}\right)}, \tag{9}$$

then

$$N_\delta(h) \subset_0^{AB} \mathcal{I}^\zeta(m,\alpha,l,\nu,\gamma,d,\beta).$$

Proof. Consider $f \in N_\delta(h)$. Relation (5)

$$\sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta$$

implies

$$\sum_{k=2}^{\infty} |a_k - b_k| \leq \frac{\delta}{2}. \tag{10}$$

Using relation (8) for $h \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$, we get:

$$\sum_{k=2}^{\infty} b_k \leq \frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m \left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}. \tag{11}$$

Using (10) and (11), we have:

$$\left|\frac{f(z)}{h(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty} |a_k - b_k|}{1 - \sum_{k=2}^{\infty} b_k} \leq$$

$$\frac{\delta}{2\left(1 - \frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{\left(\frac{1+\alpha+l}{l+1}\right)^m \left[(1-\nu)(\gamma+1)(\beta|d|+1) + \frac{2\nu[\gamma(\nu+1)+1](\nu+1+\beta|d|)}{\Gamma(\nu+3)}\right]}\right)} = 1 - \zeta.$$

By relation (7), we obtain $f \in_0^{AB} \mathcal{I}^\zeta(m,\alpha,l,\nu,\gamma,d,\beta)$, where ζ is given by (9). □

6. Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 8. *The function* $f \in_0^{AB} \mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ *is univalent starlike of order* δ, $0 \leq \delta < 1$, *in* $|z| < r_1$, *with*

$$r_1 = \inf_k \left\{ \frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{(k-\delta)\left[\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]} \right\}^{\frac{1}{k-1}}.$$

For the function of the form

$$f_k(z) = z + \frac{\beta|d|(1-\gamma) - \frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|) + \frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]} z^k, \tag{12}$$

$k \geq 2$, *the result is sharp.*

Proof. For the function f to be univalent starlike of order δ, we have to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \delta, \quad |z| < r_1.$$

For $f \in \mathcal{A}$, we can write

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}}|\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} a_k|z|^{k-1}},$$

and it remains to be seen that

$$\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} a_k|z|^{k-1}} \leq 1-\delta$$

equivalently to

$$\sum_{k=2}^{\infty}(k-\delta)a_k|z|^{k-1} \leq 1-\delta.$$

Applying Theorem 1, we get

$$|z|^{k-1} \leq \frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{(k-\delta)\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]},$$

or

$$|z| \leq \left\{\frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{(k-\delta)\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}\right\}^{\frac{1}{k-1}}.$$

Hence, the proof is complete. □

Theorem 9. *The function $f \in {}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ is univalent convex of order δ, $0 \leq \delta \leq 1$, in $|z| < r_2$, with*

$$r_2 = \inf_k\left\{\frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k(k-\delta)\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}\right\}^{\frac{1}{k-1}}.$$

For the function of the form

$$f_k(z) = z + \frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}z^k,\ k \geq 2,$$

the result is sharp.

Proof. For the function f to be univalent convex of order δ, we have to prove that

$$\left|\frac{zf''(z)}{f'(z))}\right| \leq 1-\delta, \quad |z| < r_2.$$

For $f \in \mathcal{A}$, we can write

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}k(k-1)a_k z^{k-1}}{1+\sum_{k=2}^{\infty}ka_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}ka_k|z|^{k-1}},$$

and it remains to be seen that

$$\frac{\sum_{k=2}^{\infty}k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}ka_k|z|^{k-1}} \leq 1-\delta,$$

$$\sum_{k=2}^{\infty}k(k-\delta)a_k|z|^{k-1} \leq 1-\delta.$$

Using Theorem 1, we obtain

$$|z|^{k-1} \leq \frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k(k-\delta)\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]},$$

or

$$|z| \leq \left\{\frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k(k-\delta)\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}\right\}^{\frac{1}{k-1}},$$

and the proof is complete. □

Theorem 10. *The function* $f \in {}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ *is univalent close-to-convex of order* δ, $0 \leq \delta < 1$, *in* $|z| < r_3$, *with*

$$r_3 = \inf_k \left\{\frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}\right\}^{\frac{1}{k-1}}.$$

For the function of the form

$$f_k(z) = z + \frac{\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}}{A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}z^k,\ k \geq 2,$$

the result is sharp.

Proof. For the function f to be univalent close-to-convex of order δ, we have to prove that

$$|f'(z)-1| \leq 1-\delta, \quad |z| < r_3.$$

For $f \in \mathcal{A}$, we can write

$$|f'(z)-1| = \left|\sum_{k=2}^{\infty} k a_k z^{k-1}\right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1-\delta$$

if $\sum_{k=2}^{\infty} \frac{k a_k}{1-\delta}|z|^{k-1} \leq 1$. Using Theorem 1, the inequality holds true if

$$|z|^{k-1} \leq \frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}$$

or

$$|z| \leq \left\{\frac{(1-\delta)A\left[(1-\nu)[\gamma(k-1)+1](k-1+\beta|d|)+\frac{\nu[\gamma(k+\nu-1)+1](k+\nu-1+\beta|d|)\Gamma(k+1)}{\Gamma(\nu+k+1)}\right]}{k\left[\beta|d|(1-\gamma)-\frac{\nu(\gamma\nu+1)(\nu+\beta|d|)}{\Gamma(\nu+2)}\right]}\right\}^{\frac{1}{k-1}}.$$

Hence, the proof is complete. □

7. Conclusions

In this paper, a new operator ${}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f)$ is introduced in Definition 3, applying the Atangana–Baleanu fractional integral for $c = 0$ to a multiplier transformation. A new class of analytic functions ${}_0^{AB}\mathcal{I}(m,\alpha,l,\nu,\gamma,d,\beta)$ is defined in Definition 4 and follows a study of this class regarding coefficient inequality, distortion and closure theorems, inclusion and neighborhood results, radii of starlikeness, convexity and close-to-convexity.

The class introduced in this paper is interesting due to the operator used for introducing it, since this operator is part of the celebrated family of fractional integral operators,

which have been much investigated in the recent years. In addition to symmetry properties, which could be investigated related to the newly defined operator, algebraic properties could also be added after further studies in this regard. Considering the starlikeness and convexity properties of the newly defined class, symmetry properties could be found for this class, having in mind the connection between convexity and symmetry. Additionally, subordination and superordination properties could be obtained by using the means of the theories of differential subordination and superordination on this class due to its univalence properties. The results obtained in this paper could be adapted in view of quantum calculus aspects as seen in [25,26].

Author Contributions: Conceptualization, A.A.L. and A.C.; methodology, A.C.; software, A.A.L.; validation, A.A.L. and A.C.; formal analysis, A.A.L. and A.C.; investigation, A.A.L.; resources, A.C.; data curation, A.C.; writing—original draft preparation, A.A.L.; writing—review and editing, A.A.L. and A.C.; visualization, A.A.L.; supervision, A.C.; project administration, A.A.L.; funding acquisition, A.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baleanu, D.; Agarwal, R.P. Fractional calculus in the sky. *Adv. Differ. Equ.* **2021**, *117*, 105. [CrossRef]
2. Srivastava, H.M. An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions. *J. Adv. Eng. Comput.* **2021**, *5*, 135–166. [CrossRef]
3. Owa, S. On the distortion theorems I. *Kyungpook Math. J.* **1978**, *18*, 53–59.
4. Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. *Can. J. Math.* **1987**, *39*, 1057–1077. [CrossRef]
5. Alb Lupaş, A.; Oros, G.I. Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 327. [CrossRef]
6. Alb Lupaş, A.; Oros, G.I. On Special Differential Subordinations Using Fractional Integral of Sălăgean and Ruscheweyh Operators. *Symmetry* **2021**, *13*, 1553. [CrossRef]
7. Abdeljawad, T.; Baleanu, D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1098–1107. [CrossRef]
8. Baleanu, D.; Fernandez, A. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Commun. Nonlinear Sci. Numer Simul.* **2018**, *59*, 444–462. [CrossRef]
9. Djida, J.-D.; Atangana, A.; Area, I. Numerical computation of a fractional derivative with non-local and non-singular kernel. *Math. Model. Nat. Phenom.* **2017**, *12*, 413. [CrossRef]
10. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
11. Alb Lupaş, A.; Cătaş, A. An application of the principle of differential subordination to analytic functions involving Atangana-Baleanu fractional integral of Bessel functions. *Symmetry* **2021**, *13*, 971. [CrossRef]
12. Cătaş, A.; Alb Lupaş, A. Some Subordination Results for Atangana–Baleanu Fractional Integral Operator Involving Bessel Functions. *Symmetry* **2022**, *14*, 358. [CrossRef]
13. Ahmad, H.; Tariq, M.; Sahoo, S.K.; Askar, S.; Abouelregal, A.E.; Khedher, K.M. Refinements of Ostrowski Type Integral Inequalities Involving Atangana–Baleanu Fractional Integral Operator. *Symmetry* **2021**, *13*, 2059. [CrossRef]
14. Set, E.; Akdemir, A.O.; Karaoglan, A.; Abdeljawad, T.; Shatanawi, W. On New Generalizations of Hermite-Hadamard Type Inequalities via Atangana-Baleanu Fractional Integral Operators. *Axioms* **2021**, *10*, 223. [CrossRef]
15. Darweesh, A.M.; Atshan, W.G.; Battor, A.H. Fourth-order differential subordination and superordination results of meromorphic multivalent functions defined by multiplier transformation. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 2297–2313. [CrossRef]
16. Oluwayemi, M.O.; Alb Lupaş, A.; Cătaş, A. Results on a class of analytic functions with finitely many fixed coefficients related to a generalised multiplier transformation. *Sci. Afr.* **2022**, *15*, e01115. [CrossRef]
17. Atshan, W.G.; Rahman, I.A.R.; Alb Lupaş, A. Some Results of New Subclasses for Bi-Univalent Functions Using Quasi-Subordination. *Symmetry* **2021**, *13*, 1653. [CrossRef]
18. Cotîrlă, L.I. New classes of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 10642–10651. [CrossRef]

19. Oluwayemi, M.O.; Vijaya, K.; Cătaş, A. Certain Properties of a Class of Functions Defined by Means of a Generalized Differential Operator. *Mathematics* **2022**, *10*, 174. [CrossRef]
20. Guney, H.O.; Oros, G.I.; Owa, S. An Application of Sălăgean Operator Concerning Starlike Functions. *Axioms* **2022**, *11*, 50. [CrossRef]
21. Cătaş, A. On certain class of p-valent functions defined by a new multiplier transformations. In Proceedings of the Book of the International Symposium GFTA, Istanbul, Turkey, 20–24 August 2007; Istanbul Kultur University: Istanbul, Turkey, 2007; pp. 241–250.
22. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Taylor & Francis: London, UK, 2002.
23. Klimek, M.; Lupa, M. On reflection symmetry in fractional mechanics. *Sci. Res. Inst. Math. Comput. Sci.* **2011**, *1*, 109–121.
24. Fernandez, A. A complex analysis approach to Atangana–Baleanu fractional calculus. *Math. Methods Appl. Sci.* **2021**, *44*, 8070–8087. [CrossRef]
25. Aouf, M.K.; Mostafa, A.O.; Elmorsy, R.E. Certain subclasses of analytic functions with varying arguments associated with q-difference operator. *Afr. Mat.* **2021**, *32*, 621–630. [CrossRef]
26. Srivastava, H.M.; Arif, M.; Raza, M. Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator. *AIMS Math.* **2021**, *6*, 5869–5885. [CrossRef]