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Article

Double Controlled Quasi Metric Like Spaces

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Abstract: In this article, we present a generalization of the double controlled metric like spaces, called quasi double controlled metric like spaces, by assuming that the symmetric condition is not necessary satisfied. Moreover, the self distance is not necessary zero.

Keywords: fixed point; double controlled metric type spaces; double controlled metric like spaces; quasi double controlled metric like spaces

MSC: 47H10; 54H25

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1. Introduction

Extension of metric spaces has been the focus of many researchers in mathematics, the reason behind this focus is that the more general the metric is the more fields can be applied on. All started by generalizing metric spaces to partial metric spaces and to b-metric spaces. For example in partial metric spaces the author assumed that the self distance may be not zero. In b-metric spaces the author add a constant bigger than 1 in the triangle inequality. In the next paragraph we present some more details.

Bakhtin in [1], initiated a generalization of metric spaces called b-metric spaces. Lately, several generalizations of the b-metric spaces were initiated such as extended b-metric spaces by Kamran et al. [2]. In 2018, Mlaiki et al. [3], introduced the concept of controlled metric type spaces which will be denoted in the sequel by $(CMTS)$. Few months later, Abdeljawad et al. in [4], initiated a more general metric type so called double controlled metric type spaces denoted $(DCMTS)$. In 2020, Mlaiki in [5], introduced a generalization of $(DCMTS)$ so called double controlled metric like spaces denoted $(DCMLS)$, where he assumed that the self distance is not necessary zero. Another type of extension of metric spaces, is where we assume that we do not necessary have the symmetry condition which called quasi metric spaces for more details we refer the reader to the following references [6–13]. In this work, we introduce the concept of double controlled quasi metric like spaces denoted $(DCQMLS)$, where we assume that there is no symmetric condition in the $(DCMLS)$. Since in this work we are introducing a new type of metric spaces, we are going to prove the existence of a fixed point for mapping that satisfying the Banach contraction principle, and nonlinear contraction principle. Also, we present an application in polynomial equations.

2. Preliminaries

The first extension of b-metric spaces was initiated by Kamran et al. in [2].

Definition 1 ([2])**.** *Consider the set $\mathbb{B} \neq \emptyset$ and $\mathbb{K} : \mathbb{B} \times \mathbb{B} \to [1, +\infty)$. If the function $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to R^+$ satisfies the following conditions for all $\varsigma, \varrho, \vartheta \in \mathbb{B}$,*

($\mathbb{E}_1$) $\mathbb{M}(\varsigma, \varrho) = 0$ *if and only if* $\varsigma = \varrho$;
($\mathbb{E}_2$) $\mathbb{M}(\varsigma, \varrho) = \mathbb{M}(\varrho, \varsigma)$;
($\mathbb{E}_3$) $\mathbb{M}(\varsigma, \varrho) \leq \mathbb{K}(\varsigma, \varrho)[\mathbb{M}(\varsigma, \vartheta) + \mathbb{M}(\vartheta, \varrho)]$,
then, the pair $(\mathbb{B}, \mathbb{M})$ *is called an extended b-metric spaces.*

Later, Mlaiki et al. [3] introduced the $(CMTS)$ an interesting extension to the space proposed by Kamran et al. in [2].

Definition 2 ([3])**.** *Given a nonempty set* $\mathbb{B}$ *and* $\mathbb{K} : \mathbb{B}^2 \to [1, +\infty)$. *The function* $\mathbb{M} : \mathbb{B}^2 \to [0, +\infty)$ *is a controlled metric type if*
($\mathbb{C}_1$) $\mathbb{M}(\varsigma, \varrho) = 0$ *if and only if* $\varsigma = \varrho$;
($\mathbb{C}_2$) $\mathbb{M}(\varsigma, \varrho) = \mathbb{M}(\varrho, \varsigma)$;
($\mathbb{C}_3$) $\mathbb{M}(\varsigma, \varrho) \leq \mathbb{K}(\varsigma, \vartheta)\mathbb{M}(\varsigma, \vartheta) + \mathbb{K}(\vartheta, \varrho)\mathbb{M}(\vartheta, \varrho)$,
for all $\varsigma, \varrho, \vartheta \in \mathbb{B}$. *The pair* $(\mathbb{B}, \mathbb{M})$ *is called a controlled metric type space.*

In the same year, a generalization of $(CMTS)$ was initiated in [4] as follows.

Definition 3 ([4])**.** $(DCMTS)$ *Consider the set* $\mathbb{B} \neq \emptyset$ *and* $\mathbb{K}, \mathbb{L} : \mathbb{B} \times \mathbb{B} \to [1, +\infty)$ *be non-comparable functions. The function* $\mathbb{M} : \mathbb{B}^2 \to [0, +\infty)$; *if we have*
($\mathbb{D}_1$) $\mathbb{M}(\varsigma, \varrho) = 0$ *if and only if* $\varsigma = \varrho$;
($\mathbb{D}_2$) $\mathbb{M}(\varsigma, \varrho) = \mathbb{M}(\varrho, \varsigma)$;
($\mathbb{D}_3$) $\mathbb{M}(\varsigma, \varrho) \leq \mathbb{K}(\varsigma, \vartheta)\mathbb{M}(\varsigma, \vartheta) + \mathbb{L}(\vartheta, \varrho)\mathbb{M}(\vartheta, \varrho)$,
for all $\varsigma, \varrho, \vartheta \in \mathbb{B}$, *then the pair* $(\mathbb{B}, \mathbb{M})$ *is called a double controlled metric type space.*

In 2020, as an extension of $(DCMTS)$, $(DCMLS)$ was introduced by assuming that the self distance is not necessary zero see [5].

Definition 4 ([5])**.** $(DCMLS)$ *Consider the set* $\mathbb{B} \neq \emptyset$ *and* $\mathbb{K}, \mathbb{L} : \mathbb{B} \times \mathbb{B} \to [1, +\infty)$ *be non-comparable functions. if the function* $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, +\infty)$ *satisfies the following conditions;*
($\mathbb{DL}_1$) $\mathbb{M}(\varsigma, \varrho) = 0$ *implies* $\varsigma = \varrho$;
($\mathbb{DL}_2$) $\mathbb{M}(\varsigma, \varrho) = \mathbb{M}(\varrho, \varsigma)$;
($\mathbb{DL}_3$) $\mathbb{M}(\varsigma, \varrho) \leq \mathbb{K}(\varsigma, \vartheta)\mathbb{M}(\varsigma, \vartheta) + \mathbb{L}(\vartheta, \varrho)\mathbb{M}(\vartheta, \varrho)$,
for all $\varsigma, \varrho, \vartheta \in \mathbb{B}$, *the pair* $(\mathbb{B}, \mathbb{M})$ *is called a double controlled metric like space.*

Now, we introduce the notion of $(DCQMLS)$ as follows.

Definition 5. $(DCQMLS)$ *Consider the set* $\mathbb{B} \neq \emptyset$ *and* $\mathbb{K}, \mathbb{L} : \mathbb{B} \times \mathbb{B} \to [1, +\infty)$ *be non-comparable functions. if the function* $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, +\infty)$ *satisfies the following conditions;*
($\mathbb{QDL}_1$) $\mathbb{M}(\varsigma, \varrho) = \mathbb{M}(\varsigma, \varrho) = 0$ *implies* $\varsigma = \varrho$;
($\mathbb{QDL}_2$) $\mathbb{M}(\varsigma, \varrho) \leq \mathbb{K}(\varsigma, \vartheta)\mathbb{M}(\varsigma, \vartheta) + \mathbb{L}(\vartheta, \varrho)\mathbb{M}(\vartheta, \varrho)$,
for all $\varsigma, \varrho, \vartheta \in \mathbb{B}$, *the pair* $(\mathbb{B}, \mathbb{M})$ *is called a double controlled quasi metric like space.*

Note that in Definition 5, we do not have the symmetry condition. Next, we present the topology of $(DCQMLS)$.

Definition 6. *Let* $(\mathbb{B}, \mathbb{M})$ *be a* $(DCQMLS)$, $b_0 \in \mathbb{B}$ *and* $r > 0$. *The upper closed ball of radius* r *centered* b_0 *and the lower closed ball of radius* r *centered* b_0 *are defined by,*

$$\overline{B^+}(b_0, r) = \{b \in \mathbb{B} : |\mathbb{M}(b, b_0) - \mathbb{M}(b_0, b_0)| \leq r\}$$

and

$$\overline{B^-}(b_0, r) = \{b \in \mathbb{B} : |\mathbb{M}(b_0, b) - \mathbb{M}(b_0, b_0)| \leq r\},$$

respectively. Now, we define the notions of a circle and a disc on a quasi-metric space $(\mathbb{B}, \mathbb{M})$ *as follows: Let* $r \geq 0$ *and* $b_0 \in \mathbb{B}$. *The circle* $C^{\mathbb{M}}_{b_0,r}$ *and the disc* $D^{\mathbb{M}}_{b_0,r}$ *are*

$$C^{\mathbb{M}}_{b_0,r} = \{b \in \mathbb{B} : |\mathbb{M}(b_0, b) - \mathbb{M}(b_0, b_0)| = |\mathbb{M}(b, b_0) - \mathbb{M}(b_0, b_0)| = r\}$$

and

$$D^{\mathbb{M}}_{b_0,r} = \overline{B^+}(b_0, r) \cap \overline{B^-}(b_0, r) = \{b \in \mathbb{B} : |\mathbb{M}(b_0, b) - \mathbb{M}(b_0, b_0)| \leq r \text{ and } |\mathbb{M}(b, b_0) - \mathbb{M}(b_0, b_0)| \leq r\}.$$

Definition 7.

- *Let* $(\mathbb{B}, \mathbb{M})$ *a* $(DCQMLS)$, *a sequence* b_n *is convergent to some* b *in* $\mathbb{B}$ *if and only if*

$$\lim_{n\to+\infty} \mathbb{M}(b_n, b) = \lim_{n\to+\infty} \mathbb{M}(b, b_n) = \mathbb{M}(b, b).$$

- *A sequence* b_n *is a left Cauchy sequence if and only if for all* $m > n$, *we have* $\lim_{n,m\to+\infty} \mathbb{M}(b_m, b_n)$ *exists and finite.*
- *A sequence* b_n *is a right Cauchy sequence if and only if for all* $m > n$, *we have* $\lim_{n,m\to+\infty} \mathbb{M}(b_n, b_m)$ *exists and finite.*
- *A sequence* b_n *is a Cauchy sequence if and only if* b_n *is left and right Cauchy.*
- *We say that* $(\mathbb{B}, \mathbb{M})$ *is left complete, right complete and dual complete if and only if each left-Cauchy, right Cauchy and Cauchy sequence in* $\mathbb{B}$ *is convergent respectively.*

Example 1. *Let* $\mathbb{B} = l_1$ *be defined by*

$$l_1 = \left\{ \{\xi_n\}_{n\geq1} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |\xi_n| < +\infty \right\}.$$

Consider $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, +\infty)$ *such that*

$$\mathbb{M}(\eta, \xi) = \sum_{n=1}^{+\infty} (\xi_n - \eta_n)^+$$

where $\alpha^+ := \max\{\alpha, 0\}$ *for* $\alpha \in \mathbb{R}$, *and* $\xi = \{\xi_n\}$ *and* $\eta = \{\eta_n\}$ *are in* $\mathbb{B}$. *Also, let* $\mathbb{K}(\xi, \eta) = \max\{\xi, \eta\} + 2$ *and* $\mathbb{L}(\xi, \eta) = \max\{\xi, \eta\} + 3$.

Note that, $(\mathbb{B}, \mathbb{M})$ *is a* $(DCQMLS)$ *with control functions* $\mathbb{K}$, $\mathbb{L}$.

It is not difficult to see that $(\mathbb{B}, \mathbb{M})$ *is a* $(DCQMLS)$, *but it is not a* $(DCMLS)$.

3. Main Results

Now, we prove the Banach contraction principle in $(DCQMLS)$.

Theorem 1. *Let* $(\mathbb{B}, \mathbb{M})$ *be a dual complete* $(DCQMLS)$ *defined by the functions* $\mathbb{K}, \mathbb{L} : \mathbb{B}^2 \to [1, +\infty)$. *Let* $\mathbb{H}\colon \mathbb{B} \to \mathbb{B}$ *be a mapping such that*

$$\mathbb{M}(\mathbb{H}\varsigma, \mathbb{H}\varrho) \leq k\mathbb{M}(\varsigma, \varrho), \tag{1}$$

for all $\varsigma, \varrho \in \mathbb{B}$, *where* $k \in (0, 1)$. *For* $\varsigma_0 \in \mathbb{B}$, *take* $\varsigma_n = \mathbb{H}^n\varsigma_0$. *Suppose that*

$$\sup_{m\geq1} \lim_{i\to+\infty} \frac{\mathbb{K}(\varsigma_{i+1}, \varsigma_{i+2})}{\mathbb{K}(\varsigma_i, \varsigma_{i+1})} \mathbb{L}(\varsigma_{i+1}, \varsigma_m) < \frac{1}{k}. \tag{2}$$

Also, assume that for every $\varsigma \in \mathbb{B}$, *we have*

$$\lim_{n\to+\infty} \mathbb{K}(\varsigma, \varsigma_n) \text{ and } \lim_{n\to+\infty} \mathbb{L}(\varsigma_n, \varsigma) \text{ exist and are finite.} \tag{3}$$

Then $\mathbb{H}$ *admits a unique fixed point.*

Proof. Let $\{\varsigma_n = \mathbb{H}^n \varsigma_0\}$ in $\mathbb{B}$ be the sequence that satisfies the conditions of our theorem. Fom (1), we obtain

$$\mathbb{M}(\varsigma_n, \varsigma_{n+1}) \leq k^n \mathbb{M}(\varsigma_0, \varsigma_1) \quad \text{for all} \quad n \geq 0. \tag{4}$$

Similarly, we can show that;

$$\mathbb{M}(\varsigma_{n+1}, \varsigma_n) \leq k^n \mathbb{M}(\varsigma_1, \varsigma_0) \quad \text{for all} \;\; n \geq 0. \tag{5}$$

Let $n, m \in \mathbb{N}$ such that $n < m$. Thus,

$$\begin{aligned}
\mathbb{M}(\varsigma_n, \varsigma_m) &\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})\mathbb{M}(\varsigma_n, \varsigma_{n+1}) + \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{M}(\varsigma_{n+1}, \varsigma_m) \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})\mathbb{M}(\varsigma_n, \varsigma_{n+1}) + \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{K}(\varsigma_{n+1}, \varsigma_{n+2})\mathbb{M}(\varsigma_{n+1}, \varsigma_{n+2}) \\
&+ \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{L}(\varsigma_{n+2}, \varsigma_m)\mathbb{M}(\varsigma_{n+2}, \varsigma_m) \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})\mathbb{M}(\varsigma_n, \varsigma_{n+1}) + \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{K}(\varsigma_{n+1}, \varsigma_{n+2})\mathbb{M}(\varsigma_{n+1}, \varsigma_{n+2}) \\
&+ \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{L}(\varsigma_{n+2}, \varsigma_m)\mathbb{K}(\varsigma_{n+2}, \varsigma_{n+3})\mathbb{M}(\varsigma_{n+2}, \varsigma_{n+3}) \\
&+ \mathbb{L}(\varsigma_{n+1}, \varsigma_m)\mathbb{L}(\varsigma_{n+2}, \varsigma_m)\mathbb{L}(\varsigma_{n+3}, \varsigma_m)\mathbb{M}(\varsigma_{n+3}, \varsigma_m) \\
&\leq \cdots \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})\mathbb{M}(\varsigma_n, \varsigma_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})\mathbb{M}(\varsigma_i, \varsigma_{i+1}) \\
&+ \prod_{k=n+1}^{m-1} \mathbb{L}(\varsigma_k, \varsigma_m)\mathbb{M}(\varsigma_{m-1}, \varsigma_m) \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})k^n\mathbb{M}(\varsigma_0, \varsigma_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})k^i\mathbb{M}(\varsigma_0, \varsigma_1) \\
&+ \prod_{i=n+1}^{m-1} \mathbb{L}(\varsigma_i, \varsigma_m)k^{m-1}\mathbb{M}(\varsigma_0, \varsigma_1) \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})k^n\mathbb{M}(\varsigma_0, \varsigma_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})k^i\mathbb{M}(\varsigma_0, \varsigma_1) \\
&+ \left(\prod_{i=n+1}^{m-1} \mathbb{L}(\varsigma_i, \varsigma_m) \right) k^{m-1}\mathbb{K}(\varsigma_{m-1}, \varsigma_m)\mathbb{M}(\varsigma_0, \varsigma_1) \\
&= \mathbb{K}(\varsigma_n, \varsigma_{n+1})k^n\mathbb{M}(\varsigma_0, \varsigma_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})k^i\mathbb{M}(\varsigma_0, \varsigma_1) \\
&\leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})k^n\mathbb{M}(\varsigma_0, \varsigma_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})k^i\mathbb{M}(\varsigma_0, \varsigma_1)
\end{aligned}$$

Note that, we are using the fact that $\mathbb{K}(\varsigma, \varrho) \geq 1$. Let

$$\mathbb{S}_p = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1})k^i.$$

Hence, we have

$$\mathbb{M}(\varsigma_n, \varsigma_m) \leq \mathbb{M}(\varsigma_0, \varsigma_1)[k^n\mathbb{K}(\varsigma_n, \varsigma_{n+1}) + (\mathbb{S}_{m-1} - \mathbb{S}_n)]. \tag{6}$$

Now, note that

$$\begin{aligned}|\frac{\mathbb{S}_{p+1}}{\mathbb{S}_p}| &= |\frac{\left(\prod_{j=0}^{i+1}\mathbb{L}(\varsigma_j,\varsigma_m)\right)\mathbb{K}(\varsigma_{i+1},\varsigma_{i+2})k^{i+1}}{\left(\prod_{j=0}^{i}\mathbb{L}(\varsigma_j,\varsigma_m)\right)\mathbb{K}(\varsigma_i,\varsigma_{i+1})k^{i}}| \\ &= |\frac{\mathbb{L}(\varsigma_{i+1},\varsigma_m)\mathbb{K}(\varsigma_{i+1},\varsigma_{i+2})k}{\mathbb{K}(\varsigma_i,\varsigma_{i+1})}| \\ &< \frac{1}{k}k = 1 \text{ by condition (2).}\end{aligned}$$

Thus, by the ratio test we deduce that $\lim_{n\to+\infty}\mathbb{S}_n$ exists and then the real sequence $\{\mathbb{S}_n\}$ is $\mathbb{M}$-Cauchy. Finally, if we take the limit in the inequality (6) as $n,m \to +\infty$, we deduce that

$$\lim_{n,m\to+\infty}\mathbb{M}(\varsigma_n,\varsigma_m) = 0, \tag{7}$$

that is, the sequence $\{\varsigma_n\}$ is right Cauchy in $(\mathbb{B},\mathbb{M})$. Similarly we can show by the use of (5) that the sequence $\{\varsigma_n\}$ is left Cauchy in $(\mathbb{B},\mathbb{M})$. Therefore, we deduce that $\{\varsigma_n\}$ is a Cauchy sequence in $(\mathbb{B},\mathbb{M})$. Since $(\mathbb{B},\mathbb{M})$ is dual complete, we can conclude that the sequence $\{\varsigma_n\}$ converges to some $\varsigma \in \mathbb{B}$, that is

$$\lim_{n\to+\infty}\mathbb{M}(\varsigma_n,\varsigma) = \mathbb{M}(\varsigma,\varsigma) = \lim_{n\to+\infty}\mathbb{M}(\varsigma,\varsigma_n). \tag{8}$$

Now, We prove that $\mathbb{H}\varsigma = \varsigma$. Note that,

$$\mathbb{M}(\varsigma,\varsigma_{n+1}) \leq \mathbb{K}(\varsigma,\varsigma_n)\mathbb{M}(\varsigma,\varsigma_n) + \mathbb{L}(\varsigma_n,\varsigma_{n+1})\mathbb{M}(\varsigma_n,\varsigma_{n+1}).$$

Taking the limit as $n \to +\infty$ we deduce that

$$\mathbb{M}(\varsigma,\varsigma) = 0. \tag{9}$$

Using again the triangle inequality and (1),

$$\begin{aligned}\mathbb{M}(\varsigma,\mathbb{H}\varsigma) &\leq \mathbb{K}(\varsigma,\varsigma_{n+1})\mathbb{M}(\varsigma,\varsigma_{n+1}) + \mathbb{L}(\varsigma_{n+1},\mathbb{H}\varsigma)\mathbb{M}(\varsigma_{n+1},\mathbb{H}\varsigma) \\ &\leq \mathbb{K}(\varsigma,\varsigma_{n+1})\mathbb{M}(\varsigma,\varsigma_{n+1}) + k\mathbb{L}(\varsigma_{n+1},\mathbb{H}\varsigma)\mathbb{M}(\varsigma_n,\varsigma).\end{aligned}$$

Taking the limit as $n \to +\infty$ and by (3) and (9), we deduce that $\mathbb{M}(\varsigma,\mathbb{H}\varsigma) = 0$, similarly, we can easly deduce that $\mathbb{M}(\mathbb{H}\varsigma,\varsigma) = 0$ that is, $\mathbb{H}\varsigma = \varsigma$. Finally, assume that $\mathbb{H}$ has two fixed points, say ς_1 and ς_2. Thus,

$$\mathbb{M}(\varsigma_1,\varsigma_2) = \mathbb{M}(\mathbb{H}\varsigma_1,\mathbb{H}\varsigma_2) \leq k\mathbb{M}(\varsigma_1,\varsigma_2),$$

which holds unless $\mathbb{M}(\varsigma_1,\varsigma_2) = 0$, using the same technique we can show that $\mathbb{M}(\varsigma_2,\varsigma_1) = 0$. Thus, $\varsigma_1 = \varsigma_2$. Hence $\mathbb{H}$ has a unique fixed point as required. □

Now, we illustrate Theorem 1 by the following example.

Example 2. *Let $\mathbb{B} = [0,1]$. For all $\zeta,\xi \in \mathbb{B}$ we have:*

$$\mathbb{M}(\zeta,\xi) = |\zeta - \xi|.$$

Now, let $\mathbb{K}(\zeta,\xi) = \max\{\zeta,\xi\} + 2$ and $\mathbb{L}(\zeta,\xi) = \max\{\zeta,\xi\} + 3$. It is easy to see that $(\mathbb{B},\mathbb{M})$ is a dual complete (DCQMLS). Now, let $\mathbb{H} : \mathbb{B} \to \mathbb{B}$, defined by

$$\mathbb{H}\zeta = \frac{\zeta^4 + 1}{255\zeta^4 + 256}$$

Thus,

$$
\begin{aligned}
\mathbb{M}(\mathbb{H}\zeta, \mathbb{H}\xi) &= \left|\frac{\zeta^4+1}{255\zeta^4+256} - \frac{\xi^4+1}{255\xi^4+256}\right| \\
&= \left|\frac{\zeta^4-\xi^4}{(255\zeta^4+256)(255\xi^4+256)}\right| \\
&\leq \frac{|\zeta-\xi|^4}{|(256\zeta^4+256)(255\xi^4+256)|};\ \ since\ \ \zeta,\xi \in (0,1) \\
&\leq \frac{|\zeta-\xi|}{256};\ \ since\ \ |(255\zeta^4+256)(255\xi^4+256)| > 256 \\
&\leq \frac{|\zeta-\xi|}{2} \\
&= \frac{1}{2}\mathbb{M}(\zeta,\xi)
\end{aligned}
$$

Hence,

$$\mathbb{M}(\mathbb{H}\zeta, \mathbb{H}\xi) \leq k\mathbb{M}(\zeta,\xi) \quad where \quad k = \frac{1}{2}$$

Note that it is not difficult to see that $\mathbb{H}$, $\mathbb{L}$ and $\mathbb{K}$ satisfies all the hypothesis of Theorem 1. Therefore, $\mathbb{H}$ has a unique fixed point in $\mathbb{B}$.

Definition 8. *Let $\mathbb{H} : \mathbb{B} \longrightarrow \mathbb{B}$. For some $\varsigma_0 \in \mathbb{B}$, let $O(\varsigma_0) = \{\varsigma_0, \mathbb{H}\varsigma_0, \mathbb{H}^2\varsigma_0, \dots\}$ be the orbit of ς_0. We say that the function $G : \mathbb{B} \longrightarrow \mathbb{R}$ is $\mathbb{H}$-orbitally lower semi-continuous at $\vartheta \in \mathbb{B}$ if for $\{\varsigma_n\} \subset O(\varsigma_0)$ such that $\varsigma_n \longrightarrow \vartheta$, we have $G(\vartheta) \leq \lim_{n\to+\infty} \inf G(\varsigma_n)$.*

Corollary 1. *Let $(\mathbb{B}, \mathbb{M})$ be a dual complete $(DCQMLS)$ defined by the functions $\mathbb{K}, \mathbb{L} : \mathbb{B}^2 \to [1, +\infty)$. Let $\mathbb{H}\colon \mathbb{B} \to \mathbb{B}$, Let $\varsigma_0 \in \mathbb{B}$ and $0 < k < 1$ such that*

$$\mathbb{M}(\mathbb{H}\vartheta, \mathbb{H}^2\vartheta) \leq k\mathbb{M}(\vartheta, \mathbb{H}\vartheta),\ for\ each\ \vartheta \in O(\varsigma_0). \tag{10}$$

Take $\varsigma_n = \mathbb{H}^n\varsigma_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\mathbb{K}(\varsigma_{i+1}, \varsigma_{i+2})}{\mathbb{K}(\varsigma_i, \varsigma_{i+1})} \mathbb{L}(\varsigma_{i+1}, \varsigma_m) < \frac{1}{k}. \tag{11}$$

Then $\lim_{n\to+\infty} \varsigma_n = \vartheta \in \mathbb{B}$. Also, $\mathbb{H}\vartheta = \vartheta$ if and only if $\varsigma \mapsto \mathbb{M}(\varsigma, \mathbb{H}\varsigma)$ is $\mathbb{H}$-orbitally lower semi-continuous at ϑ.

In the next theorem, we study the nonlinear case, but first we remind the reader of the following set Φ of comparison functions.

Definition 9 ([14])**.** *Define Φ to be the set of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies the following properties*

- *[i] Φ is monotone increasing;*
- *[ii] $\Phi(t) < t$ for all $t > 0$;*
- *[iii] $\Phi(0) = 0$;*
- *[iv] Φ is continuous;*
- *[v] $\Phi^n(t)$ converges to 0 for all $t \geq 0$;*
- *[vi] $\sum_{n=0}^{+\infty} \Phi^n(t)$ converges for all $t > 0$*

Next, we present the following lemma.

Lemma 1 ([14])**.** *1. [i] and [ii] $\Longrightarrow$ [iii];*
2. [ii] and [iv] $\Longrightarrow$ [iii];

3. $[i]$ *and* $[v] \Longrightarrow [ii]$.

Theorem 2. *Let* $(\mathbb{B}, \mathbb{M})$ *be a dual complete* $(DCQMLS)$ *defined by the functions* $\mathbb{K}, \mathbb{L} : \mathbb{B}^2 \to [1, +\infty)$. *Consider the map* $\mathbb{H} \colon \mathbb{B} \to \mathbb{B}$, *and assume that there exists* $\phi \in \Phi$ *such that*

$$\mathbb{M}(\mathbb{H}\varsigma, \mathbb{H}\varrho) \leq \phi(\Delta(\varsigma, \varrho)), \;\; \Delta(\varsigma, \varrho) = \max\{\mathbb{M}(\varsigma, \varrho), \mathbb{M}(\varsigma, \mathbb{H}\varsigma), \mathbb{M}(h, \mathbb{H}\varrho)\}, \tag{12}$$

for all $\varsigma, \varrho \in \mathbb{B}$. *Moreover, assume that for each* $\varsigma_0 \in \mathbb{B}$, *we have*

$$\sup_{m\geq 1} \lim_{i \to +\infty} \frac{\mathbb{K}(\varsigma_{i+1}, \varsigma_{i+2})}{\mathbb{K}(\varsigma_i, \varsigma_{i+1})} \mathbb{L}(\varsigma_{i+1}, \varsigma_m) \frac{\phi^{i+1}(\mathbb{M}(\varsigma_1, \varsigma_0))}{\phi^i(\mathbb{M}(\varsigma_1, \varsigma_0))} < 1, \tag{13}$$

where $\varsigma_n = \mathbb{H}^n \varsigma_0, \quad n \in \mathbb{N}$. *If the* $(DCQMLS)$ $\mathbb{M}$ *and* $\mathbb{H}$ *are continuous, then* $\mathbb{H}$ *has a unique fixed point* $\vartheta \in \mathbb{B}$ *with* $\mathbb{H}^n s \to \vartheta$ *for each* $\varsigma \in \mathbb{B}$.

Proof. Let $\{\varsigma_n\}$ and ς_0 be as in the hypothesis of the theorem. Suppose that there exists $m \in \mathbb{N}$, such that $\varsigma_m = \varsigma_{m+1} = \mathbb{H}\varsigma_m$, then clearly ς_m is the fixed point. Hence, we may assume that $\varsigma_{n+1} \neq \varsigma_n$ for each n. From the condition (12), we have

$$\mathbb{M}(\varsigma_n, \varsigma_{n+1}) = \mathbb{M}(\mathbb{H}\varsigma_n, \mathbb{H}\varsigma_{n-1}) \leq \phi(\Delta(\varsigma_{n-1}, \varsigma_n)), \tag{14}$$

where clearly $\Delta(\varsigma_{n-1}, \varsigma_n) = \max\{\mathbb{M}(\varsigma_{n-1}, \varsigma_n), \mathbb{M}(\varsigma_n, \varsigma_{n+1})\}$. If for some n, we accept that $\Delta(\varsigma_{n-1}, \varsigma_n) = \mathbb{M}(\varsigma_n, \varsigma_{n+1})$, then by (14) and the fact that for all $t > 0$ we have $\phi(t) < t$, we deduce that

$$0 < \mathbb{M}(\varsigma_n, \varsigma_{n+1}) \leq \phi(\mathbb{M}(\varsigma_n, \varsigma_{n+1})) < \mathbb{M}(\varsigma_n, \varsigma_{n+1}), \tag{15}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$ we obtain $\Delta(\varsigma_{n-1}, \varsigma_n) = \mathbb{M}(\varsigma_{n-1}, \varsigma_n)$. From which, it follows that $0 < \mathbb{M}(\varsigma_n, \varsigma_{n+1}) \leq \phi(\mathbb{M}(\varsigma_{n-1}, \varsigma_n))$. Then, we obtain by induction,

$$0 < \mathbb{M}(\varsigma_n, \varsigma_{n+1}) \leq \phi^n(\mathbb{M}(\varsigma_0, \varsigma_1)). \text{ for all } n \geq 0$$

By using the properties of ϕ, we obtain that

$$\mathbb{M}(\varsigma_n, \varsigma_{n+1}) \to 0 \;\text{ as }\; n \to +\infty.$$

Similarly, we can show that

$$\mathbb{M}(\varsigma_{n+1}, \varsigma_n) \to 0 \;\text{ as }\; n \to +\infty.$$

Now, using the same technique in the proof of Theorem 1, for $n, m \in \mathbb{N}$ where $n < m$. we can easily deduce that

$$\mathbb{M}(\varsigma_n, \varsigma_m) \leq \mathbb{K}(\varsigma_n, \varsigma_{n+1})\phi^n(\mathbb{M}(\varsigma_0, \varsigma_1)) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \mathbb{L}(\varsigma_j, \varsigma_m) \right) \mathbb{K}(\varsigma_i, \varsigma_{i+1}) \phi^i(\mathbb{M}(\varsigma_0, \varsigma_1)) \tag{16}$$

By condition (13) and by using the ratio test we deduce that $\{\varsigma_n\}$ is an $\mathbb{M}$-Cauchy sequence. The fact that $(\mathbb{B}, \mathbb{M})$ is a dual complete $(DCQMLS)$ implies that $\varsigma_n \to \vartheta \in \mathbb{B}$ as $n \to +\infty$ such that $\lim_{n \to +\infty} \mathbb{M}(\varsigma_n, \vartheta) = 0$. Hence, we can easily deduce that $\mathbb{H}\vartheta = \vartheta$. Finally, assume that ϑ and υ are two fixed points of $\mathbb{H}$ such that $\vartheta \neq \upsilon$. From the assumption (12), we have

$$\mathbb{M}(\vartheta, \upsilon) = \mathbb{M}(\mathbb{H}\vartheta, \mathbb{H}\upsilon) \leq \phi(\Delta(\vartheta, \upsilon)) = \phi(\mathbb{M}(\vartheta, \upsilon)) < \mathbb{M}(\vartheta, \upsilon),$$

which leads to a contradiction. Hence, $\mathbb{M}(\vartheta, \upsilon) = 0$. Similarly, we can show that $\mathbb{M}(\upsilon, \vartheta) = 0$. Therefore $\vartheta = \upsilon$ as desired. $\square$

4. Application

In closing, we present the following application for our results.

Theorem 3. *Let $m \geq 3$ a natural number the equation*

$$\zeta^m + 1 = (m^4 - 1)\zeta^{m+1} + m^4\zeta \tag{17}$$

has a unique real solution in $[0, 1]$.

Proof. Before anything else, note that if $|\zeta| > 1$, Equation (17), does not have a solution. So, let $\mathbb{B} = [0, 1]$ and for all $\zeta, \xi \in \mathbb{B}$ let $\mathbb{M}(\zeta, \xi) = |\zeta - \xi|$ if $\zeta \in (0, 1)$ and $\mathbb{M}(0, 1) = 1$, $\mathbb{M}(1, 0) = 2$. Now, let $\mathbb{K}(\zeta, \xi) = \max\{\zeta, \xi\} + 2$ and $\mathbb{L}(\zeta, \xi) = \max\{\zeta, \xi\} + 3$. It's clear that $(\mathbb{B}, \mathbb{M})$ is a dual complete $(DCQMLS)$. Now, let

$$\mathbb{H}\zeta = \frac{\zeta^m + 1}{(m^4 - 1)\zeta^m + m^4}$$

Notice that, since $m \geq 2$, we can deduce that $m^4 \geq 6$. Thus,

$$\begin{aligned}
\mathbb{M}(\mathbb{H}\zeta, \mathbb{H}\xi) &= |\frac{\zeta^m + 1}{(m^4 - 1)\zeta^m + m^4} - \frac{\xi^m + 1}{(m^4 - 1)\xi^m + m^4}| \\
&= |\frac{\zeta^m - \xi^m}{((m^4 - 1)\zeta^m + m^4)((m^4 - 1)\xi^m + m^4)}| \\
&\leq \frac{|\zeta - \xi|^m}{|((m^4 - 1)\zeta^m + m^4)((m^4 - 1)\xi^m + m^4)|}; \text{ since } \zeta, \xi \in (0, 1) \\
&\leq \frac{|\zeta - \xi|}{m^4}; \text{ since } |((m^4 - 1)\zeta^m + m^4)((m^4 - 1)\xi^m + m^4)| > m^4 \\
&\leq \frac{|\zeta - \xi|}{6}; \text{ since } m^4 > 6 \\
&= \frac{1}{6}\mathbb{M}(\zeta, \xi)
\end{aligned}$$

Hence,

$$\mathbb{M}(\mathbb{H}\zeta, \mathbb{H}\xi) \leq k\mathbb{M}(\zeta, \xi) \quad \text{where} \quad k = \frac{1}{6}$$

Finally, note that it is not difficult to see that $\mathbb{H}$, $\mathbb{L}$ and $\mathbb{K}$ satisfies all the hypothesis of Theorem 1. Therefore, $\mathbb{H}$ has a unique fixed point in $\mathbb{B}$, which implies that Equation (17) has a unique real solution as desired. □

5. Future Work

In this section, we propose to endowed the $(DCQMLS)$ by a graph in order to obtain the triplet $(\mathbb{B}, \mathbb{M}, G)$. Indeed, the concept of fixed point theory with a graph has been widely studied. We refer the reader to the paper of Jachymski [15] for more details on this problem. Inspired by several works in this field, we can introduce the concept of G-contraction where G is the graph associated to $\mathbb{B}$ and next we present a conjecture related to the existence and uniqueness of fixed point for such contractions in a $(DCQMLS)$ with a graph. First, in Figure 1, we present an example to illustrate this approach.

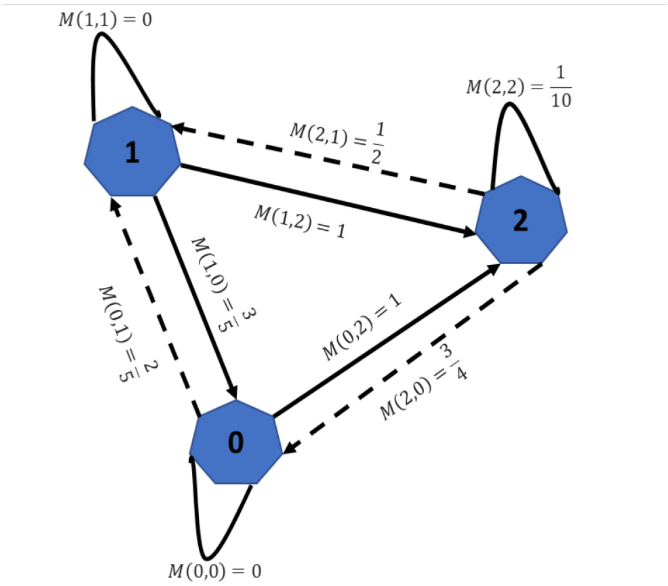

Figure 1. $(DCQMLS)$.

Let $(\mathbb{B}, \mathbb{M}, G)$ a $(DCQMLS)$ with graph where the set $\mathbb{B} = \{0, 1, 2\}$, and $\mathbb{M}$ defined on the figure and $\mathbb{L}, \mathbb{K} : \mathcal{F} \times \mathcal{F} \to [1, +\infty)$ to be defined by

$$\mathbb{L}(0,0) = \mathbb{L}(1,1) = \mathbb{L}(2,2) = 1, \quad \mathbb{L}(0,2) = \frac{151}{100}, \quad \mathbb{L}(1,2) = \frac{8}{5}, \quad \mathbb{L}(0,1) = \frac{6}{5},$$

and

$$\mathbb{K}(0,0) = \mathbb{K}(1,1) = \mathbb{K}(2,2) = 1, \quad \mathbb{K}(0,2) = \frac{8}{5}, \quad \mathbb{K}(1,2) = \frac{33}{20}, \quad \mathbb{K}(0,1) = \frac{6}{5},$$

Now, define the self mapping $\mathbb{H}$ on $\mathcal{F}$ as follows;

$$\mathbb{H}0 = 2 \quad \text{and} \quad \mathbb{H}1 = \mathbb{H}2 = 1.$$

It is easy to see that $\mathbb{H}$ has a unique fixed point. Let $\mathbb{M}$ be a $(DCQMLS)$ on a set $\mathbb{B} \neq \emptyset$. Let Δ be the diagonal of $\mathbb{B}^2$. A graph G is defined by the pair (V, E) where V is a set of vertices coinciding with $\mathbb{B}$ and E is the set of its edges with $\Delta \subset E$. From now on, assume that G has no parallel edges.

Definition 10. *Let t and g be two vertices in a graph G. A path in G from t to g of length q ($q \in \mathbb{N} \cup \{0\}$) is a sequence $(k_i)_{i=0}^{q}$ of $q+1$ distinct vertices so that $k_0 = t$, $k_n = g$ and $(k_i, k_{i+1}) \in E(G)$ for $i = 1, 2, \ldots, q$.*

The graph G may be converted to a weighted graph by assigning to each edge the distance given by the $(DCQMLS)$ between its vertices.

Notation: Let $\mathbb{B}^{\mathbb{H}} = \{x \in \mathbb{B} / (x, \mathbb{H}x) \in E(G) \text{ or } (\mathbb{H}x, x) \in E(G)\}$.

Definition 11. *Let $(\mathbb{B}, \mathbb{M})$ be a complete $(DCQMLS)$ endowed with a graph G. The mapping $\mathbb{H} \colon \mathbb{B} \to \mathbb{B}$ is said to be a G_ϕ-contraction if*

- $$\textit{for all } t, g \in \mathbb{B}, (t, g) \in E(G) \implies (\mathbb{H}t, \mathbb{H}g) \in E(G); \tag{18}$$
- *there is a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ so that*

$$\xi(\mathbb{H}t, \mathbb{H}^2 t) \leq \phi(\xi(t, \mathbb{H}t)) \quad \textit{for all } t \in \mathbb{B}^{\mathbb{H}}, \tag{19}$$

where ϕ is a nondecreasing function and $\{\phi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 for each $t > 0$.

Definition 12. *The mapping $\mathbb{H} : \mathbb{B} \longrightarrow \mathbb{B}$ is called orbitally G-continuous if for all $a, b \in X$ and any positive sequence $\{g_n\}_{n\in\mathbb{N}}$,*

$$\mathbb{H}^{g_n}a \longrightarrow b, \;\; (\mathbb{H}^{g_n}a, \mathbb{H}^{g_{n+1}}a) \in E(G) \Longrightarrow \mathbb{H}(\mathbb{H}^{g_n}a) \longrightarrow \mathbb{H}b \text{ as } n \to +\infty.$$

Conjecture 1. *Let $(\mathbb{B}, \mathbb{M}, G)$ be a complete $(DCQMLS)$ with a graph G. Let $\mathbb{H}\colon \mathbb{B} \to \mathbb{B}$ be a G_ϕ-contraction which is orbitally G-continuous. Suppose the following property (P): for any $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}$, if $t_n \longrightarrow t$ and $(t_n, t_{n+1}) \in E(G)$, then there is a subsequence $\{t_{k_n}\}_{n\in\mathbb{N}}$ with $(t_{k_n}, t) \in E(G)$, holds. Further, suppose that, for each $g \in \mathbb{B}$,*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\mathbb{L}(g_{i+1}, g_{i+2})}{\mathbb{L}(g_i, g_{i+1})} \mathbb{K}(g_{i+1}, g_m) < M; \;\; M > 1$$

Also, assume that for every $g \in \mathbb{B}$, we have

$$\lim_{n\to+\infty} \mathbb{L}(g, g_n) \text{ and } \lim_{n\to+\infty} \mathbb{K}(g_n, g) \text{ exist and are finite.}$$

Then the restriction of $\mathbb{H}_{|[g]_{\tilde{G}}}$ to $[g]_{\tilde{G}}$ possesses a fixed point.

6. Conclusions

We introduced the concept of double controlled quasi metric like spaces, we proved the existence and uniqueness of a fixed point for self mapping in such spaces that satisfies the Banach contraction principle. Also, we proved the same result for mapping that satisfies nonlinear type of contraction. We presented an application of our result to polynomial equations. In the last section we provided the reader with an idea about future work on $(DCQMLS)$ endowed with a graph.

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