symmetry

Article

Symmetric Properties of Routh–Hurwitz and Schur–Cohn Stability Criteria

Ziad Zahreddine

Mathematics Division, College of Engineering, Kuwait College of Science and Technology, Al Jahraa 13133, Kuwait; z.zahreddine@kcst.edu.kw or ziadzahr@hotmail.com

Abstract: It is often noticed in the literature that some key results on the stability of discrete-time systems of difference equations are obtained from their corresponding results on the stability of continuous-time systems of differential equations using suitable conformal mappings or bilinear transformations. Such observations lead to the search for a unified approach to the study of root distribution for real and complex polynomials, with respect to the left-half plane for continuous-time systems (Routh–Hurwitz stability) and with respect to the unit disc for discrete-time systems (Schur–Cohn stability). This paper is a further contribution toward this objective. We present, in a systematic way, the similarities, and yet, the differences between these two types of stability, and we highlight the symmetry that exists between them. We also illustrate how results on the stability of continuous-time systems are conveyed to the stability of discrete-time systems through the proposed techniques. It should be mentioned that the results on Schur–Cohn stability are known to be harder to obtain than Routh–Hurwitz stability ones, giving more credibility to the proposed approach.

Keywords: Routh–Hurwitz stability; Schur–Cohn stability; continuous-time systems of differential equations; discrete-time systems of difference equations

Citation: Zahreddine, Z. Symmetric Properties of Routh–Hurwitz and Schur–Cohn Stability Criteria. *Symmetry* **2022**, *14*, 603. https://doi.org/10.3390/sym14030603

Academic Editor: Jan Awrejcewicz

Received: 1 March 2022
Accepted: 15 March 2022
Published: 18 March 2022

1. Introduction

The problem of eigenvalue distribution of systems of differential equations with respect to a given curve in the complex plane has been intensively studied. The curves mainly used are the imaginary axis and the unit circle. The Routh–Hurwitz stability test for the imaginary axis and the Schur–Cohn stability test for the unit circle are the most celebrated ones, and very efficient algorithms have been explored to handle these two types.

The Routh–Hurwitz criterion addresses the stability of continuous-time systems of differential equations, which requires that the eigenvalues of the system lie in the left-half of the complex plane. It was thoroughly investigated by Hermite, Routh and Hurwitz. Their contributions were further advanced by Lienard and Chipart (see [1–3]. Howland uses quadratic forms to achieve similar objectives [4]. The Routh–Hurwitz stability criterion remains the backbone of stability analysis of linear systems and has been tremendously applied to resolve several issues in mathematics and its applications, especially in the design of digital filters and other networks.

On the other hand, the Schur–Cohn criterion addresses the stability of discrete-time systems of difference equations, which requires that the eigenvalues lie inside the unit circle. It was explored by Schur, Cohn and others (look at [2,5–7]). The Schur–Cohn stability criterion is essential in various areas, such as digital signal processing, control theory, spectral analysis, numerical computations and many others. The root distribution of polynomials in other sub-regions of the complex plane has also been investigated by many authors [8].

It is often noticed in the literature that some interesting result about stability in the Hurwitz sense, for example, triggers an interest in the corresponding question in the Schur sense, and vice versa (see [9] for example).

The current work is motivated by several attempts to give a common interpretation to the algorithms for testing the stability for continuous-time (RH) and discrete-time (SC) systems, by invoking the intimate relationships that might prevail between these stability structures. For example, the notion of positive para-oddness is playing an increasingly effective role in the stability analysis of continuous-time systems. For discrete-time systems, the notion of para-oddness is mirrored by complex discrete reactance functions, which are the discrete-time counterpart of positive para-odd functions. For some recent work in this direction, see Ref. [10].

In Section 2 of the current work, we highlight in a systematic way the symmetries that exist between these two types of stability. In Section 3, we provide detailed proof of the key theorems presented in Section 2. In Section 4, we provide numerical illustrations. We end in Section 5 with some concluding remarks.

2. Symmetry between Schur–Cohn and Routh–Hurwitz

We shall list the Routh–Hurwitz related definitions and theorems in the left column and their Schur–Cohn definitions and theorems counterparts in the right one. The properties listed below highlight in a striking manner the strong correlations between these two types of stability. At least in the opinion of the author, such correlations would form a firm basis for any further investigations into the common nature of the Routh–Hurwitz and the Schur–Cohn stability criteria.

Definition 1. *A linear continuous-time system of differential equations is stable if and only if all its eigenvalues lie in the left half-plane. If*

$$f(s) = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + a_n s^n \tag{1}$$

with real or complex coefficients and n a non-negative integer, is the characteristic polynomial of the system, then the system is stable if all zeros of f (s) lie in the left-half plane. Such polynomials are said to be Routh–Hurwitz stable.

Definition 2. *The paraconjugate of f is defined by* $f^*(s) = \overline{f(-\bar{s})}$. *Then,* f^* *can be written as* $f^*(s) = \bar{a}_0 - \bar{a}_1 s + \bar{a}_2 s^2 + \cdots + (-1)^n \bar{a}_n s^n$ *where* $\bar{a}_k$ *denotes the complex conjugate of* a_k *for* $k = 0, 1, \ldots, n$.

Definition 3. *The test function of the given continuous-time*

$$\text{system is defined by } \Phi(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)} \tag{2}$$

Definition 4. *A rational function* H(s) *with complex coefficients is said to be positive if* $\mathrm{Re}[\mathrm{H}(s)] > 0$ *whenever* $\mathrm{Re}\, s > 0$.

Theorem 1. *The linear continuous-time system of differential equations characterized by (1) is stable if and only if the test function Φ(s) defined by (2) is a positive function.*

Definition 5. *A linear discrete-time system of difference equations is stable if and only if all its eigenvalues lie inside the unit disc. If*

$$g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \tag{3}$$

with real or complex coefficients and n a non-negative integer, is the characteristic polynomial of the system, then the system is stable if all zeros of g (z) lie inside the unit disc. Such polynomials are said to be Schur–Cohn stable.

Definition 6. *The reciprocal of g is defined by* $g^{\tau}(z) = z^n\overline{g(1/\overline{z})}$. *Then,* g^{τ} *can be written as* $g^{\tau}(z) = \overline{a}_n + \overline{a}_{n-1}z + \overline{a}_{n-2}z^2 + \cdots + \overline{a}_0 z^n$ *where* $\overline{a}_k$ *denotes the complex conjugate of* a_k *for* $k = 0, 1, \ldots, n$.

Definition 7. *The test function of the given discrete-time*

$$\text{system is defined by } \Psi(z) = \frac{g(z) - g^{\tau}(z)}{g(z) + g^{\tau}(z)} \tag{4}$$

Definition 8. *A rational function* K(z) *with complex coefficients is said to be a discrete reactance function if* Re[K(z)] > 0 *whenever* ∣z∣ > 1.

Theorem 2. *The linear discrete-time system of difference equations characterized by (3) is stable if and only if the test function* Ψ(z) *defined by (4) is a discrete reactance function.*

Theorem 3. *The linear continuous-time system of differential equations characterized by (1) is stable if and only if the test function* Φ(s) *defined by (2) can be written in the continued fraction expansion*

$$\Phi(s) = a_0 + b_0 s + \cfrac{1}{a_1 + b_1 s + \cfrac{1}{a_2 + b_2 s + \begin{matrix}\vdots \\ + \cfrac{1}{a_{n-2} + b_{n-2}s + \cfrac{1}{a_{n-1} + b_{n-1}s}}\end{matrix}}}$$

where $\mathrm{Re}a_j = 0$, and $b_j > 0$ for $0 \leq j \leq n-1$.

Theorem 4. *The linear discrete-time system of difference equations characterized by (3) is stable if and only if the test function* Ψ(z) *defined by (4) can be written in the continued fraction expansion*

$$\Psi(z) = h_0\frac{z-1}{z+1} + k_0 + \cfrac{1}{h_1\frac{z-1}{z+1} + k_1 + \begin{matrix}\vdots \\ + \cfrac{1}{h_n\frac{z-1}{z+1} + k_n}\end{matrix}}$$

where $h_0 \geq 0,\ h_1 > 0, \ldots, h_n > 0$ and k_j are imaginary or zero for $0 \leq j \leq n$.

3. Proof of the Key Results

The proofs of the above four key theorems will now be laid out.

To prove Theorem 1, the following two lemmas are needed.

Lemma 1. *Suppose* $\mathrm{Re}s_j < 0,$ then $|s - s_j| > |s + \overline{s}_j|$ whenever $\mathrm{Re}s > 0$.

Proof .

If $\mathrm{Re}s_j < 0,$ and $\mathrm{Re}s > 0,$ then $\mathrm{Re}s_j \cdot \mathrm{Re}s < 0$.

Hence,

$$(s_j + \overline{s}_j)(s + \overline{s}) < 0$$

which can be written as

$$-s\overline{s}_j - s_j\overline{s} > ss_j + \overline{s}\overline{s}_j.$$

By adding the expression $s\overline{s} + s_j\overline{s}_j$ to both sides, we obtain

$$s\overline{s} - s\overline{s}_j - s_j\overline{s} + s_j\overline{s}_j > s\overline{s} + ss_j + \overline{s}\overline{s}_j + s_j\overline{s}_j,$$

which can be written as

$$\left(s - s_j\right)\left(\bar{s} - \bar{s}_j\right) > \left(s + \bar{s}_j\right)\left(\bar{s} + s_j\right),$$

which is equivalent to

$$\left|s - s_j\right|^2 > \left|s + \bar{s}_j\right|^2$$

implying that

$$\left|s - s_j\right| > \left|s + \bar{s}_j\right|.$$

□

Lemma 2. *If f and f* have no common roots and* $g(s) = \frac{f^*(s)}{f(s)}$ then f is a Hurwitz polynomial if and only if g maps the right-half plane into the unit circle.

Proof .

First, assume that f is a Hurwitz polynomial. The factored forms of f and f^* can be written as

$$f(s) = (s - s_1)(s - s_2)\cdots(s - s_n) \text{ and } f^*(s) = (-1)^n(s + \bar{s}_1)(s + \bar{s}_2)\cdots(s + \bar{s}_n)$$

Since f is a Hurwitz polynomial, then $Res_j < 0,$ for all $1 \le j \le n$. by Lemma 1

$$\left|s - s_j\right| > \left|s + \bar{s}_j\right| \text{ for all } 1 \le j \le n, \text{whenever } Res > 0.$$

Hence

$$|f(s)| > |f^*(s)| \text{ whenever} Res > 0.$$

which is equivalent to

$$|g(s)| < 1.$$

Therefore, g maps the right-half plane into the unit circle.

To prove the converse, assume $|g(s)| < 1$ whenever $Res > 0$, then

$$|f(s)| > |f^*(s)| \text{ for} Res > 0.$$

So, f has no roots for Re $s > 0$, which implies the only possible roots of f when Re $s \ge 0$ are purely imaginary.

From the factored forms of f of f^*, any purely imaginary root of f is also a root of f^*, which contradicts the hypothesis that f and f^* have no roots in common.

Therefore, f has only roots with negative real parts, and f is a Hurwitz polynomial. □

Proof of Theorem 1.

$$\Phi(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)}$$

is equivalent to

$$\Phi(s) = \frac{1 - g(s)}{1 + g(s)} \text{ where } g(s) = \frac{f^*(s)}{f(s)}.$$

Clearly

$$\Phi(s) = \frac{1 - g(s)}{1 + g(s)}$$

is equivalent to

$$g(s) = \frac{1 - \Phi(s)}{1 + \Phi(s)}$$

Direct calculations lead to

$$\Phi(s)+\overline{\Phi(s)}=\frac{2\left[1-g(s)\,\overline{g(s)}\right]}{|g(s)+1|^2}$$

and

$$1-g(s)\overline{g(s)}=\frac{2\left[\Phi(s)+\overline{\Phi(s)}\right]}{|\Phi(s)+1|^2}.$$

by Lemma 2, f is a Hurwitz polynomial if and only if

$$|g(s)|<1 \text{ whenever } \mathrm{Re}s>0.$$

But

$$|g(s)|<1\Leftrightarrow g(s)\overline{g(s)}<1\Leftrightarrow 1-g(s)\overline{g(s)}>0$$

which is equivalent to

$$\Phi(s)+\overline{\Phi(s)}=\frac{2\left(1-g(s)\overline{g(s)}\right)}{|g(s)+1|^2}>0\Leftrightarrow \mathrm{Re}\Phi(s)>0.$$

The conclusion is:

f is a Hurwitz polynomial if and only if Re Φ (s) > 0 whenever Re s > 0.

By definition, Φ is a positive function if and only if Re Φ (s) > 0 whenever Re s > 0.

Therefore, f is a Hurwitz polynomial if and only if Φ is a positive function and the proof is complete. □

Proof of Theorem 2. Obviously, the relation $s=\frac{z-1}{z+1}$ is equivalent to $z=\frac{1+s}{1-s}$, and $|z|<1$ equivalent to $Res<0$.

Defining the function

$$f(s)=(1-s)^n g\left(\frac{1+s}{1-s}\right).$$

It follows that

$$f(s)=\sum_{k=0}^{n} a_k(1-s)^{n-k}(1+s)^k.$$

From the definition of Hurwitzness Hurwitz, f is a Hurwitz polynomial if and only if $g(z)\neq 0$ for all $|z|\geq 1$.

Consider the paraconjugate of f

$$f^*(s)=\sum_{k=0}^{n} \overline{a}_k(1+s)^{n-k}(1-s)^k.$$

If $g^\tau(z)=\overline{a}_n+\overline{a}_{n-1}z+\overline{a}_{n-2}z^2+\cdots+\overline{a}_0 z^n$ as defined in Definition 6, then

$$(1-s)^n g^\tau\left(\frac{1+s}{1-s}\right)=\sum_{k=0}^{n}\overline{a}_k(1+s)^{n-k}(1-s)^k.$$

So, if

$$f(s)=(1-s)^n g\left(\frac{1+s}{1-s}\right),$$

then

$$f^*(s)=(1-s)^n g^\tau\left(\frac{1+s}{1-s}\right).$$

It follows that

$$f(s)+f^*(s)=(1-s)^n(g+g^\tau)\left(\frac{1+s}{1-s}\right),$$

and
$$f(s) - f^*(s) = (1-s)^n (g - g^\tau)\left(\frac{1+s}{1-s}\right).$$

Defining the function
$$P(s) = \frac{(g - g^\tau)\left(\frac{1+s}{1-s}\right)}{(g + g^\tau)\left(\frac{1+s}{1-s}\right)},$$

Then
$$P(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)}.$$

By definition, the function $\frac{(g-g^\tau)(z)}{(g+g^\tau)(z)}$ is a complex discrete reactance function if and only if
$$Re\left[\frac{(g - g^\tau)(z)}{(g + g^\tau)(z)}\right] > 0 \text{ whenever } |z| > 1,$$
and that is equivalent to Re $[P(s)] > 0$ whenever $Res > 0$

Which is in turn equivalent to the fact that P is a positive function.

g, g^τ have no zeros in common, if and only f and f^* also have no zeros in common.

By [11] (Theorem 5.1, p. 300), P is positive if and only if f is Hurwitz, which is equivalent to $g(z) \neq 0$ for all $|z| \geq 1$, and that completes the proof. □

The proof of Theorem 3 was established in [12] (Theorem 3.2, p. 65).

The continued fraction expansion of the above theorem led to the construction of the Extended Routh Array in [12], which generalized the Routh Array to polynomials with complex coefficients.

Proof of Theorem 4. We can assume that $a_n = 1$ in $g(z)$.

The function $g(z)$ is Schur stable if and only if $f(s)$ as defined in (1) is a Hurwitz polynomial. By [2] (p. 78),the function $\Phi(s) = \frac{f(s)-f^*(s)}{f(s)+f^*(s)}$ defined in (2) can be written in the form
$$\Phi(s) = \cfrac{1}{it_1 + d_1 s + \cfrac{1}{it_2 + d_2 s + \cfrac{}{\ddots + \cfrac{1}{it_n + d_n s}}}}$$
where t_k real and $d_k > 0$ for $1 \leq k \leq n$.

Since $\Psi(z) = \Phi\left(\frac{z-1}{z+1}\right)$, we obtain
$$\Psi(z) = \cfrac{1}{it_1 + d_1 \frac{z-1}{z+1} + \cfrac{1}{it_2 + d_2 \frac{z-1}{z+1} + \cfrac{}{\ddots + \cfrac{1}{it_n + d_n \frac{z-1}{z+1}}}}}$$

Substitute d_j by h_j and t_j by k_j for $j = 1, \ldots, n$ to obtain the form for Ψ, as in the statement of the theorem.

h_0 and k_0 can be assumed to be zero, since in the rational function Φ as defined in (2), the degree of the numerator can be 1 less than the degree of denominator, which occurs when the degree of f defined in (1) is even. □

Note how Theorem 2 was obtained from Theorem 1 by using the conformal mapping $s = \frac{z-1}{z+1}$ which is equivalent to $z = \frac{1+s}{1-s}$.

Additionally, in the proof of Theorem 4, the conformal mapping $s = \frac{z-1}{z+1}$ was successfully used again to deliver Theorem 4 from the stability test theorem of Ref. [2] (p. 78).

An interesting connection between continued fraction expansions and systems that are stable with respect to the left-half plane has been established in Ref. [12]. Theorem 3.2 of [12] played a key role in the derivation of the ERA, which is the complex counterpart of the Routh Array. Additionally, for discrete-time systems, testing the stability requires the expansion of a discrete reaction function in a continued fraction form. For an excellent survey on continued fraction expansions in stability contexts, we refer to Ref. [13].

4. Numerical Illustrations

Example 1. *Consider the Hurwitz polynomial*

$$f(s) = s^3 + 4s^2 + 6s + 4$$

whose zeros are -2*,* $-1 + I$ *and* $-1 -- i$ *having all negative real parts.*

The paraconjugate of f is

$$f^*(s) = -s^3 + 4s^2 - 6s + 4.$$

Therefore, the test function can be written as

$$\Phi(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)} = \frac{s^3 + 6s}{4s^2 + 4}.$$

By long division, we get

$$\Phi(s) = \frac{1}{4}s + \frac{5s}{4s^2 + 4},$$

which can be written as

$$\Phi(s) = \frac{1}{4}s + \frac{1}{\frac{4s^2+4}{5s}}.$$

Another long division leads to

$$\Phi(s) = \frac{1}{4}s + \frac{1}{\frac{4}{5}s + \frac{4}{5s}},$$

which finally can be written as

$$\Phi(s) = \frac{1}{4}s + \frac{1}{\frac{4}{5}s + \frac{1}{\frac{5}{4}s}}.$$

This is exactly the continued fraction expansion of Φ*, as expressed in Theorem 3 with*

$$a_0 = a_1 = a_2 = 0,\ b_0 = \frac{1}{4} > 0,\ b_1 = \frac{4}{5} > 0,\ b_2 = \frac{5}{4} > 0.$$

Example 2. *Consider the Schur polynomial*

$$g(z) = 4z^3 - 6z^2 + 4z - 1$$

whose zeros are

$$\frac{1}{2},\ \frac{1}{2} + \frac{1}{2}i,\ \frac{1}{2} - \frac{1}{2}i$$

all lying inside the unit disc.

The reciprocalof g is

$$g^{\tau}(z) = z^n \overline{g(1/\overline{z})} = -z^3 + 4z^2 - 6z + 4$$

Therefore, the test function can be written as

$$\Psi(z) = \frac{g(z) - g^{\tau}(z)}{g(z) + g^{\tau}(z)} = \frac{5z^3 - 10z^2 + 10z - 5}{3z^3 - 2z^2 - 2z + 3}$$

Using basic algebra to expand $\Psi(s)$ *in the variable* $(z-1)/(z+1)$ *leads to the following expansion*

$$\Psi(z) = \frac{15}{11}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{121}{40}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{40}{11}\left(\frac{z-1}{z+1}\right)}}$$

This is exactly the continued fraction expansion of Ψ*as expressed in Theorem 4 with*

$$k_0 = k_1 = k_2 = 0,\ h_0 = \frac{15}{11} > 0,\ h_1 = \frac{121}{40} > 0,\ h_2 = \frac{40}{11} > 0.$$

5. Conclusions

The search for a unified approach to the study of eigenvalue distribution with respect to the left-half plane for continuous systems and with respect to the unit disc for discrete systems has been advocated by many eminent researchers in the field. In the current work, we provide a framework to be pursued to reconcile the two most important types of stability, namely Routh–Hurwitz and Schur–Cohn. The striking symmetries between these two types were highlighted. The proof of the main theorems provides further insight into the intriguing relationships that exist between the two stability criteria. The results we established are simply a contribution to various attempts to put different types of stability on common ground. However, many research efforts are still directed toward the search for a unified approach to the study of root distribution, not only for the further theoretical development of this subject, but also for the sake of obtaining simpler and more easily realizable stability criteria in practice.

The author thanks the referees for their suggestions, which certainly improved the quality of the paper.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Horn, R.; Johnson, C. *Topics in Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1991.
2. Hovstad, R. A short proof of continued fraction test for the stability of polynomials. *Proc. Am. Math. Soc.* **1989**, *105*, 76–79. [CrossRef]
3. Kharitonov, V.L. The asymptotic stability of the equilibrium state of a family of systems of linear differential equations. *Differ. Uravn.* **1978**, *14*, 2086–2088.
4. Howland, J.L. Matrix equations and the separation of matrix eigenvalues. *J. Math. Anal. Appl.* **1971**, *33*, 683–691. [CrossRef]
5. Barnett, S. *Some Modern Applications of Mathematics*; Ellis Horwood: London, UK, 1995.
6. Heinig, G.; Jungnickel, U. On the Routh-Hurwitz and Schur-Cohn Problems for Matrix Polynomials and Generalized Bezoutians. *Math. Nachr.* **1984**, *116*, 185–196. [CrossRef]
7. Krein, M.G.; Naimark, M.A. The method of symmetric and Hermitian forms in the separation of roots of algebraic equations. *Linear Multilinear Alg.* **1981**, *10*, 265–308. [CrossRef]

8. Mawhin, J. Some remarks on the Ω-stability for families of polynomials. *Arch. Math.* **1997**, *33*, 139–145.
9. Zahreddine, Z. New Versions of the Hermite Bieler Theorem in Stability Contexts. *Am. J. Appl. Math.* **2019**, *7*, 1–4. [CrossRef]
10. Zahreddine, Z. Structural Properties of a Class of Functions Arising in Stability Theory. *J. Interdiscip. Math.* **2021**, *24*, 2313–2321. [CrossRef]
11. Levinson, N.; Redheffer, R.M. *Complex Variables*; Tata McGraw-Hill Publishing Company Limited: New Delhi, India, 1980.
12. Zahreddine, Z. An Extension of the Routh Array for the Asymptotic Stability of a System of Differential Equations with Complex Coefficients. *Appl. Anal.* **1993**, *49*, 61–72. [CrossRef]
13. Schussler, H.W. A Stability Theorem for Discrete Systems. *IEEE Trans. Acoust. Speech Signal Process.* **1976**, *24*, 87–89. [CrossRef]