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The Equivalent Conditions of the Optimal Hilbert-Type Multiple Series Inequality with Quasi-Homogeneous Kernel

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Abstract: By introducing several independent parameters, according to the structural symmetry of quasi-homogeneous kernels and the Hilbert-type inequality, and using the weight function method, the parameter conditions of the optimal Hilbert-type n-multiple series inequality with quasi-homogeneous kernels are discussed, and several equivalent conditions and the expression formula of the best constant factor are obtained. As applications, some special symmetric inequalities are given.

Keywords: quasi-homogeneous kernel; Hilbert-type multiple series inequality; the best constant factor; equivalent condition

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1. Introduction

Suppose that $\tilde{a} = \{a_m\} \in l_2$, $\tilde{b} = \{b_n\} \in l_2$. In 1908, the literature [1] stated the well-known Hilbert series inequality with symmetric and -1 order homogeneous kernel $\frac{1}{m+n}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+n}a_m b_n \le \pi\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2} = \pi||\tilde{a}||_2||\tilde{b}||_2, \tag{1}$$

where the constant factor π is the best. In 1925, by introducing a pair of conjugate parameters (p,q) $(\frac{1}{p}+\frac{1}{q}=1,\ p>1)$, Hardy [2] generalized (1) as follows: If $\tilde{a}=\{a_m\}\in l_p$, $\tilde{b}=\{b_n\}\in l_q$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m+n}a_m b_n &\le \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}|a_m|^p\right)^{1/p}\left(\sum_{n=1}^{\infty}|b_n|^q\right)^{1/q}\\ &= \frac{\pi}{\sin(\pi/p)}||\tilde{a}||_p||\tilde{b}||_q,\end{aligned} \tag{2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best.

Define the series operator T by

$$T(\tilde{a})_n = \sum_{m=1}^{\infty}\frac{1}{m+n}a_m,\ \tilde{a}=\{a_m\}\in l_p, n=1,2,\cdots$$

then, we can prove that (2) has the following dual operator representation:

$$||T(\tilde{a})||_p \leq \frac{\pi}{\sin(\pi/p)}||\tilde{a}||_p.$$

It follows that T is a bounded operator on l_p, and the operator norm of T is $||T|| = \pi/\sin(\pi/p)$. Therefore, it has important applications to study (2).

By introducing an independent parameter λ, (2) has been extended to more general forms [3–5]. In 2020, ref. [6] considered the symmetric homogeneous kernel $(\min\{m,n\})^\lambda$ of λ-order, and obtained the Hilbert-type series inequality of the following form: If $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, $a_m \geq 0, b_m \geq 0$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\min\{m,n\})^\lambda a_m b_n \leq M\left(\sum_{m=1}^{\infty} m^{p(1+\widetilde{\lambda}_1)-1}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{q(1+\widetilde{\lambda}_2)-1}b_n^q\right)^{1/q},$$

and the equivalent conditions and expression formula for the best constant factor are discussed. In particular, when $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = 1$, the inequality with the best constant factor 2 is given:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\min\{m,n\})^2 a_m b_n \leq 2\left(\sum_{m=1}^{\infty} m^{2p-1}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{2q-1}b_n^q\right)^{1/q}.$$

The obvious feature of the above results is that their kernels are symmetric and homogeneous, and the structures of the inequalities also have some symmetry. For further study, we first generalize the space l_p to a weighted sequence space:

$$l_p^\alpha := \left\{\tilde{a} = \{a_m\} : ||\tilde{a}||_{p,\alpha} = \left(\sum_{m=1}^{\infty} m^\alpha |a_m|^p\right)^{1/p} < +\infty\right\}.$$

Next, on the basis of a symmetric homogeneous kernel, we will consider the quasi-homogeneous kernel $K(m,\ n) = G(m^{\lambda_1}, n^{\lambda_2})$, where G is a homogeneous function of λ-order. For $\tilde{a} = \{a_m\} \in l_p^\alpha$, $\tilde{b} = \{b_n\} \in l_q^\beta$, $K(m,n) \geq 0$, we call

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)a_m b_n \leq M||\tilde{a}||_{p,\alpha}||\tilde{b}||_{q,\beta}$$

the Hilbert-type series inequality. For $n \geq 2$, $\sum\limits_{i=1}^{n}\frac{1}{p_i} = 1 (p_i > 1)$, $K(m_1, m_2, \cdots, m_n) \geq 0$, $\tilde{a}(i) = \{a_{m_i}(i)\} \in l_{p_i}^{\alpha_i}$ $(i = 1, 2, \cdots, n)$, we say that

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} K(m_1, m_2, \cdots, m_n)\prod_{i=1}^{n} a_{m_i}(i) \leq M\prod_{i=1}^{n}||\tilde{a}(i)||_{p_i,\alpha_i} \quad (3)$$

is the Hilbert-type n-multiple series inequality.

Several results have been obtained on the n-multiple Hilbert-type inequality [7–11]. However, the equivalent conditions and the expression formula of the best constant factor for quasi-homogeneous kernel have not yet been seen in the literature. The purpose of this paper is to solve this problem.

By introducing matching parameters $a_1, a_2, \cdots, a_n$, and using the Hölder's inequality and weight coefficient method, we can obtain the following Hilbert-type multiple series inequality

$$\begin{aligned}&\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}K(m_1,m_2,\cdots,m_n)\prod_{i=1}^{n}a_{m_i}(i)\\ \leq\ & M(p_1,\cdots,p_n,a_1,\cdots,a_n)\prod_{i=1}^{n}||\tilde{a}(i)||_{p_i,\alpha_i(p_1,\cdots,p_n,a_1,\cdots,a_n)},\end{aligned} \tag{4}$$

where the constant factor M and parameters $\{\alpha_i\}_{i=1}^{n}$ all depend on $p_1, \cdots, p_n$ and $a_1, \cdots, a_n$. Generally speaking, for any matching parameters $a_1, \cdots, a_n$, the constant factor $M(p_1, \cdots, p_n, a_1, \cdots, a_n)$ in (4) is not the optimal value. To optimize the constant factor in (4), the matching parameters should satisfy certain conditions. In this paper, we discuss the law of best matching parameters in (4), obtain several equivalent conditions of the best matching parameters, and solve the theoretical problem of optimal matching parameters (see Theorem 1 in Section 3).

It is worth pointing out that the structural symmetry of the Hilbert-type inequality makes our treatment of each variable m_i of universal significance, which is very important.

2. Preliminary Lemmas

Lemma 1. *([12]) Assume that $p_i > 0$, $a_i > 0$, $\alpha_i > 0$ $(i = 1, 2, \cdots, n)$, $\psi(u)$ is measurable. Then*

$$\begin{aligned}&\int_{\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^{\alpha_i}\leq 1,x_i>0}\psi\left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^{\alpha_i}\right)\prod_{i=1}^{n}x_i^{p_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_n\\ =\ &\frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma\left(\frac{p_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\cdots\alpha_n\Gamma\left(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_0^1\psi(u)u^{\sum_{i=1}^{n}\frac{p_i}{\alpha_i}-1}\mathrm{d}u,\end{aligned}$$

where $\Gamma(t)$ is the Gamma function.

By using Lemma 1, it is not difficult to obtain the following formula.

Lemma 2. *Under the conditions of Lemma 1, we have*

$$\begin{aligned}&\int_{\mathbb{R}_+^n}\psi\left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^{\alpha_i}\right)\prod_{i=1}^{n}x_i^{p_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_n\\ =\ &\frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma\left(\frac{p_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\cdots\alpha_n\Gamma\left(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_0^{+\infty}\psi(u)u^{\sum_{i=1}^{n}\frac{p_i}{\alpha_i}-1}\mathrm{d}u,\end{aligned}$$

where $\mathbb{R}_+^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0\ (i = 1, 2, \cdots, n)\}$.

Lemma 3. *Suppose that $n \geq 2$, $\sum_{i=1}^{n}\frac{1}{p_i} = 1$ $(p_i > 1)$, $\lambda_i\lambda_j > 0$ $(i, j = 1, 2, \cdots, n)$, $K(x_1, x_2, \cdots, x_n) = G(x_1^{\lambda_1}, x_2^{\lambda_2}, \cdots, x_n^{\lambda_n})$, $G(u_1, u_2, \cdots, u_n)$ is a homogeneous non-negative measurable function of order λ, $K(x_1, \cdots, x_i, \cdots, x_n)x_i^{-a_i}$ is monotonically decreasing with respect to x_i on $(0, +\infty)$, denote*

$$W_j = \int_{\mathbb{R}_+^{n-1}}K(t_1,\cdots,t_{j-1},1,t_{j+1}\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n.$$

Then

$$\begin{aligned}\omega_j(m_j) &= \sum_{m_1=1}^{\infty}\cdots\sum_{m_{j-1}=1}^{\infty}\sum_{m_{j+1}=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}K(m_1,m_2,\cdots,m_n)\prod_{i\neq j}^{n}m_i^{-a_i}\\ &\leq m_j^{\lambda_j(\lambda-\sum_{i\neq j}^{n}\frac{a_i}{\lambda_i}+\sum_{i\neq j}^{n}\frac{1}{\lambda_i})}W_j.\end{aligned}$$

For $\sum_{i=1}^{n}\frac{a_i}{\lambda_i}=\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}$*, we have* $\frac{1}{\lambda_1}W_1=\frac{1}{\lambda_2}W_2=\cdots=\frac{1}{\lambda_n}W_n$*.*

Proof. First, notice that the quasi-homogeneous kernel $K(m_1,m_2,\cdots,m_n)$ has an obvious property: For $t>0$,

$$\begin{aligned}& K(m_1,\cdots,m_{j-1},tm_j,m_{j+1}\cdots,m_n)\\ =\ & t^{\lambda\lambda_j}K(t^{-\lambda_j/\lambda_1}m_1,\cdots,t^{-\lambda_j/\lambda_{j-1}}m_{j-1},m_j,t^{-\lambda_j/\lambda_{j+1}}m_{j+1},\cdots,t^{-\lambda_j/\lambda_n}m_n).\end{aligned}$$

Since $K(x_1,\cdots,x_i,\cdots,x_n)x_i^{-a_i}$ is monotonically decreasing with respect to x_i on $(0,+\infty)$, then

$$\begin{aligned}\omega_1(m_1) &= \sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}K(m_1,m_2,\cdots,m_n)\prod_{i=2}^{n}m_i^{-a_i}\\ &\leq \sum_{m_2=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\prod_{i=2}^{n-1}m_i^{-a_i}\int_0^{+\infty}K(m_1,m_2,\cdots,m_{n-1},t_n)t_n^{-a_n}\mathrm{d}t_n\\ &\leq \sum_{m_2=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\prod_{i=2}^{n-2}m_i^{-a_i}\int_{\mathbb{R}_+^2}K(m_1,m_2,\cdots,m_{n-2},t_{n-1},t_n)\prod_{i=n-1}^{n}t_i^{-a_i}\mathrm{d}t_{n-1}\mathrm{d}t_n\\ &\leq \cdots\cdots\\ &\leq \int_{\mathbb{R}_+^{n-1}}K(m_1,t_2,\cdots,t_n)\prod_{i=2}^{n}t_i^{-a_i}\mathrm{d}t_2\cdots\mathrm{d}t_n\\ &= m_1^{\lambda\lambda_1}\int_{\mathbb{R}_+^{n-1}}K(1,m_1^{-\lambda_1/\lambda_2}t_2,\cdots,m_1^{-\lambda_1/\lambda_n}t_n)\prod_{i=2}^{n}t_i^{-a_i}\mathrm{d}t_2\cdots\mathrm{d}t_n\\ &= m_1^{\lambda_1\left(\lambda-\sum_{i=2}^{n}\frac{a_i}{\lambda_i}+\sum_{i=2}^{n}\frac{1}{\lambda_i}\right)}\int_{\mathbb{R}_+^{n-1}}K(1,u_2,\cdots,u_n)\prod_{i=2}^{n}u_i^{-a_i}\mathrm{d}u_2\cdots\mathrm{d}u_n\\ &= m_1^{\lambda_1\left(\lambda-\sum_{i=2}^{n}\frac{a_i}{\lambda_i}+\sum_{i=2}^{n}\frac{1}{\lambda_i}\right)}W_1.\end{aligned}$$

Similarly, we can prove the cases of $j=2,3,\cdots,n$.
For $\sum_{i=1}^{n}\frac{a_i}{\lambda_i}=\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}$, $j=2,3,\cdots,n$, we compute

$$\begin{aligned}W_j &= \int_{\mathbb{R}_+^{n-1}}t^{\lambda\lambda_1}K(1,t_1^{-\lambda_1/\lambda_2}t_2,\cdots,t_1^{-\lambda_1/\lambda_{j-1}}t_{j-1},t_1^{-\lambda_1/\lambda_j},t_1^{-\lambda_1/\lambda_{j+1}}t_{j+1},\cdots,t_1^{-\lambda_1/\lambda_n}t_n)\\ &\quad\times\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\\ &= \frac{\lambda_j}{\lambda_1}\int_{\mathbb{R}_+^{n-1}}K(1,u_2,\cdots,u_n)u_j^{\lambda_j\left(\sum_{i=1}^{n}\frac{a_i}{\lambda_i}-\lambda-\sum_{i=1}^{n}\frac{1}{\lambda_i}\right)-a_j}\prod_{i\neq 1,i\neq j}^{n}u_i^{-a_i}\mathrm{d}u_2\cdots\mathrm{d}u_n\\ &= \frac{\lambda_j}{\lambda_1}\int_{\mathbb{R}_+^{n-1}}K(1,u_2,\cdots,u_n)\prod_{i=2}^{n}u_i^{-a_i}\mathrm{d}u_2\cdots\mathrm{d}u_n=\frac{\lambda_j}{\lambda_1}W_1.\end{aligned}$$

Hence $\frac{1}{\lambda_1}W_1=\frac{1}{\lambda_j}W_j$ $(j=2,3,\cdots,n)$. □

3. Main Results

In this section, we introduce the matching parameters $a_1, a_2, \cdots, a_n$, and use the weight coefficient method to establish the Hilbert inequality of n-multiple series in Theorem 1(i), and obtain two equivalent conditions of the best matching parameters in Theorem 1(ii).

Theorem 1. *Suppose that* $n \geq 2$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1 (p_i > 1)$, $a_i \in \mathbb{R}$, $\lambda_i \lambda_j > 0$ $(i, j = 1, 2, \cdots, n)$, $K(x_1, x_2, \cdots, x_n) = G(x_1^{\lambda_1}, x_2^{\lambda_2}, \cdots, x_n^{\lambda_n}) > 0$, $G(u_1, u_2, \cdots, u_n)$ *is a homogeneous non-negative measurable function of order* λ, $\sum_{i=1}^{n} \frac{a_i}{\lambda_i} - \left(\lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = c$, $K(x_1, \cdots, x_i, \cdots, x_n) x_i^{-a_i}$ *and* $K(x_1, \cdots, x_i, \cdots, x_n) x_i^{-a_i + \frac{\lambda_i c}{p_i}}$ *are monotonically decreasing with respect to* x_i *on* $(0, +\infty)$, *and for* $j = 1, 2, \cdots, n$,

$$W_j = \int_{\mathbb{R}_+^{n-1}} G(t_1^{\lambda_1}, \cdots, t_{j-1}^{\lambda_{j-1}}, 1, t_{j+1}^{\lambda_{j+1}}, \cdots, t_n^{\lambda_n}) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n$$

is convergent, then:

(i) Denote

$$\alpha_i = \lambda_i \left(\lambda + \sum_{k \neq i}^{n} \frac{1}{\lambda_k} - \sum_{k=1}^{n} \frac{a_k}{\lambda_k} \right) + a_i p_i,$$

one has

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} G(m_1^{\lambda_1}, m_2^{\lambda_2}, \cdots, m_n^{\lambda_n}) \prod_{i=1}^{n} a_{m_i}(i) \leq \left(\prod_{i=1}^{n} W_i^{1/p_i} \right) \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i, \alpha_i}, \tag{5}$$

where $\tilde{a}(i) = \{a_{m_i}(i)\} \in l_{p_i}^{\alpha_i}$ $(i = 1, 2, \cdots, n)$.

(ii) The following three conditions are equivalent:

(a) The constant factor $\prod_{i=1}^{n} W_i^{1/p_i}$ *of (5) is the best. That is,* $a_1, a_2, \cdots, a_n$ *are the best matching parameters;*

(b) $\sum_{i=1}^{n} \frac{a_i}{\lambda_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$;

(c) $\frac{1}{\lambda_1} W_1 = \frac{1}{\lambda_2} W_2 = \cdots = \frac{1}{\lambda_n} W_n$.

(iii) For $\sum_{i=1}^{n} \frac{a_i}{\lambda_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$, *(5) becomes*

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} G(m_1^{\lambda_1}, m_2^{\lambda_2}, \cdots, m_n^{\lambda_n}) \prod_{i=1}^{n} a_{m_i}(i) \leq \left(W_0 \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \right) \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i, a_i p_{i-1}}, \tag{6}$$

where $W_0 = \frac{1}{|\lambda_1|} W_1 = \frac{1}{|\lambda_2|} W_2 = \cdots = \frac{1}{|\lambda_n|} W_n$.

Proof. (i) Since

$$\prod_{j=1}^{n} \left[m_j^{a_j} \left(\prod_{i=1}^{n} m_i^{-a_i} \right)^{1/p_j} \right] = \left(\prod_{i=1}^{n} m_i^{-a_i} \right)^{\frac{1}{p_1} + \cdots + \frac{1}{p_n}} \prod_{j=1}^{n} m_j^{a_j} = \prod_{i=1}^{n} m_i^{-a_i} \prod_{j=1}^{n} m_j^{a_j} = 1,$$

it follows from Hölder's inequality and Lemma 3 that

$$\begin{aligned}
&\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}G(m_1^{\lambda_1},m_2^{\lambda_2},\cdots,m_n^{\lambda_n})\prod_{i=1}^{n}a_{m_i}(i)\\
=&\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}K(m_1,m_2,\cdots,m_n)\prod_{j=1}^{n}\left[m_j^{a_j}a_{m_j}(j)\left(\prod_{i=1}^{n}m_i^{-a_i}\right)^{1/p_j}\right]\\
\leq&\prod_{j=1}^{n}\left(\sum_{m_j=1}^{\infty}m_j^{a_jp_j-a_j}|a_{m_j}(j)|^{p_j}\omega_j(m_i)\right)^{1/p_j}\\
\leq&\prod_{j=1}^{n}W_j^{1/p_j}\prod_{j=1}^{n}\left(\sum_{m_j=1}^{\infty}m_j^{a_jp_j-a_j+\lambda_j(\lambda-\sum_{i\neq j}^{n}\frac{a_i}{\lambda_i}+\sum_{i\neq j}^{n}\frac{1}{\lambda_i})}|a_{m_j}(j)|^{p_j}\right)^{1/p_j}\\
=&\prod_{i=1}^{n}W_i^{1/p_i}\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}m_i^{\alpha_i}|a_{m_i}(i)|^{p_i}\right)^{1/p_i}=\left(\prod_{i=1}^{n}W_i^{1/p_i}\right)\prod_{i=1}^{n}||\tilde{a}(i)||_{p_i,\alpha_i}.
\end{aligned}$$

Hence, (5) holds.

(ii) (b)⇒(c) By Lemma 3 we have (b)⇒(c).

(b)⇒(a) Suppose that $\sum_{i=1}^{n}\frac{a_i}{\lambda_i}=\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}$, then $\alpha_i=a_ip_i-1$ $(i=1,2,\cdots,n)$. It follows from Lemma 3 that $\frac{1}{\lambda_1}W_1=\frac{1}{\lambda_2}W_2=\cdots=\frac{1}{\lambda_n}W_n$, and

$$\prod_{i=1}^{n}W_i^{1/p_i}=W_1^{1/p_1}\prod_{i=2}^{n}\left(\frac{\lambda_i}{\lambda_1}W_i\right)^{1/p_i}=\frac{W_1}{|\lambda_1|}\prod_{i=1}^{n}|\lambda_i|^{1/p_i}=W_0\prod_{i=1}^{n}|\lambda_i|^{1/p_i}.$$

Therefore, (5) becomes (6).

Let the best constant factor of (6) be M_0, then $M_0\leq W_0\prod_{i=1}^{n}|\lambda_i|^{1/p_i}$, and (6) still holds after replacing the constant factor of (6) with M_0.

For sufficiently small $\varepsilon>0$ and sufficiently large natural number N, take

$$a_{m_i}(i)=m_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i},i=2,3,\cdots,n,$$

$$a_{m_1}(1)=\begin{cases}0, & m_1=1,2,\cdots,N-1,\\ m_1^{(-a_1p_1-|\lambda_1|\varepsilon)/p_1}, & m_1=N,N+1,\cdots\end{cases}$$

then

$$\begin{aligned}
\prod_{i=1}^{n}||\tilde{a}(i)||_{p_i,a_ip_{i-1}}&=\left(\sum_{m_1=N}^{\infty}m_1^{-1-|\lambda_1|\varepsilon}\right)^{1/p_1}\prod_{i=2}^{n}\left(\sum_{m_i=1}^{\infty}m_i^{-1-|\lambda_i|\varepsilon}\right)^{1/p_i}\\
&\leq\left(\int_{N-1}^{+\infty}t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p_1}\prod_{i=2}^{n}\left(1+\int_{1}^{+\infty}t^{-1-|\lambda_i|\varepsilon}\mathrm{d}t\right)^{1/p_i}\\
&=\frac{1}{\varepsilon}(N-1)^{-|\lambda_1|\varepsilon/p_1}\prod_{i=1}^{n}|\lambda_i|^{-1/p_i}\prod_{i=2}^{n}(|\lambda_i|\varepsilon+1)^{1/p_i}.
\end{aligned}$$

Notice that $K(x_1,\cdots,x_i,\cdots,x_n)x_i^{-a_i}$ is monotonically decreasing with respect to x_i on $(0,+\infty)$, then

$$
\begin{aligned}
& \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} G(m_1^{\lambda_1}, m_2^{\lambda_2}, \cdots, m_n^{\lambda_n}) \prod_{i=1}^{n} a_{m_i}(i) \\
= & \sum_{m_1=N}^{\infty} m_1^{(-a_1p_1-|\lambda_1|\varepsilon)/p_1} \sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} K(m_1, m_2, \cdots, m_n) \prod_{i=2}^{n} m_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \\
\geq & \sum_{m_1=N}^{\infty} m_1^{(-a_1p_1-|\lambda_1|\varepsilon)/p_1} \sum_{m_2=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty} \prod_{i=2}^{n-1} m_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \\
& \times\left(\int_1^{+\infty} K(m_1, \cdots, m_{n-1}, u_n)\, u_n^{(-a_np_n-|\lambda_n|\varepsilon)/p_n} \mathrm{d}u_n\right) \\
\geq & \sum_{m_1=N}^{\infty} m_1^{(-a_1p_1-|\lambda_1|\varepsilon)/p_1} \sum_{m_2=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty} \prod_{i=2}^{n-2} m_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \\
& \times\left(\int_1^{+\infty}\int_1^{+\infty} K(m_1, \cdots, m_{n-2}, u_{n-1}, u_n) \prod_{i=n-1}^{n} u_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}u_{n-1}\mathrm{d}u_n\right) \\
\geq & \cdots\cdots \\
\geq & \sum_{m_1=N}^{\infty} m_1^{(-a_1p_1-|\lambda_1|\varepsilon)/p_1} \left(\int_1^{+\infty}\cdots\int_1^{+\infty} K(m_1, u_2, \cdots, u_n) \prod_{i=2}^{n} u_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}u_2\cdots\mathrm{d}u_n\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{m_1=N}^{\infty} m_1^{\lambda\lambda_1-\frac{a_1p_1+|\lambda_1|\varepsilon}{p_1}} \left(\int_1^{+\infty}\cdots\int_1^{+\infty} K(1, m_1^{-\frac{\lambda_1}{\lambda_2}}u_2, \cdots, m_1^{-\frac{\lambda_1}{\lambda_n}}u_n)\right. \\
& \left.\times\prod_{i=2}^{n} u_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}u_2\cdots\mathrm{d}u_n\right) \\
= & \sum_{m_1=N}^{\infty} m_1^{\lambda\lambda_1-\frac{a_1p_1+|\lambda_1|\varepsilon}{p_1}-\lambda\sum_{i=2}^{n}\frac{a_ip_i+|\lambda_i|\varepsilon}{\lambda_ip_i}+\lambda_1\sum_{i=2}^{n}\frac{1}{\lambda_i}} \left(\int_{m_1^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{m_1^{-\lambda_1/\lambda_n}}^{+\infty} K(1, t_2, \cdots, t_n)\right. \\
& \left.\times\prod_{i=2}^{n} t_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}t_2\cdots\mathrm{d}t_n\right) \\
\geq & \sum_{m_1=N}^{\infty} m_1^{-1-|\lambda_1|\varepsilon} \left(\int_{N^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{N^{-\lambda_1/\lambda_n}}^{+\infty} K(1, t_2, \cdots, t_n) \prod_{i=2}^{n} t_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}t_2\cdots\mathrm{d}t_n\right) \\
\geq & \int_N^{+\infty} t_1^{-1-|\lambda_1|\varepsilon}\mathrm{d}t_1 \left(\int_{N^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{N^{-\lambda_1/\lambda_n}}^{+\infty} K(1, t_2, \cdots, t_n) \prod_{i=2}^{n} t_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}t_2\cdots\mathrm{d}t_n\right) \\
= & \frac{1}{|\lambda_1|\varepsilon} N^{-|\lambda_1|\varepsilon} \left(\int_{N^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{N^{-\lambda_1/\lambda_n}}^{+\infty} K(1, t_2, \cdots, t_n) \prod_{i=2}^{n} t_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}t_2\cdots\mathrm{d}t_n\right),
\end{aligned}
$$

therefore

$$
\begin{aligned}
& \frac{1}{|\lambda_1|} N^{-|\lambda_1|\varepsilon} \left(\int_{N^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{N^{-\lambda_1/\lambda_n}}^{+\infty} K(1, t_2, \cdots, t_n) \prod_{i=2}^{n} t_i^{(-a_ip_i-|\lambda_i|\varepsilon)/p_i} \mathrm{d}t_2\cdots\mathrm{d}t_n\right) \\
\leq & M_0(N-1)^{-|\lambda_1|\varepsilon/p_1} \prod_{i=1}^{n} |\lambda_i|^{-1/p_i} \prod_{i=2}^{n} (|\lambda_i|\varepsilon+1)^{1/p_i}.
\end{aligned}
$$

Letting $\varepsilon \to 0^+$ and then $N \to +\infty$, by calculating the quadratic limit, we obtain

$$
\frac{1}{|\lambda_1|} \int_{\mathbb{R}_+^{n-1}} K(1, t_2, \cdots, t_n) \prod_{i=2}^{n} t_i^{-a_i} \mathrm{d}t_2\cdots\mathrm{d}t_n \leq M_0 \prod_{i=1}^{n} |\lambda_i|^{-1/p_i},
$$

consequently,

$$
W_0 \prod_{i=1}^{n} |\lambda_i|^{1/p_i} = \frac{W_1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \leq M_0.
$$

Thus, the best constant factor of (6) is $M_0 = W_0 \prod_{i=1}^{n} |\lambda_i|^{1/p_i}$.

(a)⇒(b) Let the constant factor $\prod_{i=1}^{n} W_i^{1/p_i}$ of (5) be the best. Put $a_i' = a_i - \frac{\lambda_i c}{p_i}$, then

$$\sum_{i=1}^{n} \frac{a_i'}{\lambda_i} - \left(\lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = \sum_{i=1}^{n} \left(\frac{a_i}{\lambda_i} - \frac{c}{p_i}\right) - \left(\lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = 0.$$

For $j \geq 2$, it is not difficult to obtain

$$W_j = \frac{\lambda_j}{\lambda_1} \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \left(\prod_{i=2}^{n} t_i^{-a_i}\right) t_j^{-\lambda_j c} \mathrm{d}t_2 \cdots \mathrm{d}t_n.$$

Additionally, since

$$\alpha_i' = \lambda_i \left(\lambda + \sum_{k \neq i}^{n} \frac{1}{\lambda_k} - \sum_{k=1}^{n} \frac{a_k'}{\lambda_k}\right) + a_i' p_i = \lambda_i \left(\lambda + \sum_{k \neq i}^{n} \frac{1}{\lambda_k} - \sum_{k=1}^{n} \frac{a_k}{\lambda_k}\right) + a_i p_i = \alpha_i,$$

then (5) can be written equivalently as

$$\begin{aligned} &\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} G(m_1^{\lambda_1}, m_2^{\lambda_2}, \cdots, m_n^{\lambda_n}) \prod_{i=1}^{n} a_{m_i}(i) \\ \leq\ & W_1^{1/p_1} \prod_{j=2}^{n} \left[\frac{\lambda_j}{\lambda_1} \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \left(\prod_{i=2}^{n} t_i^{-a_i}\right) t_j^{\lambda_j c} \mathrm{d}t_2 \cdots \mathrm{d}t_n\right]^{1/p_j} \\ &\times \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i, a_i'}. \end{aligned} \quad (7)$$

Hence, according to the assumption, the constant factor

$$\begin{aligned} & W_1^{1/p_1} \prod_{j=2}^{n} \left[\frac{\lambda_j}{\lambda_1} \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \left(\prod_{i=2}^{n} t_i^{-a_i}\right) t_j^{\lambda_j c} \mathrm{d}t_2 \cdots \mathrm{d}t_n\right]^{1/p_j} \\ =\ & \frac{1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} W_1^{1/p_1} \prod_{j=2}^{n} \left[\int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \left(\prod_{i=2}^{n} t_i^{-a_i}\right) t_j^{\lambda_j c} \mathrm{d}t_2 \cdots \mathrm{d}t_n\right]^{1/p_j} \end{aligned}$$

of (7) is the best. In view of $\sum_{i=1}^{n} \frac{a_i'}{\lambda_i} - \left(\lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = 0$, and

$$K(x_1, \cdots, x_i, \cdots, x_n) x_i^{-a_i'} = K(x_1, \cdots, x_i, \cdots, x_n) x_i^{-a_i + \frac{\lambda_i c}{p_i}}$$

is monotonically decreasing with respect to x_i on $(0, +\infty)$, according to the proof of (b)⇒(a), we can see that the best constant of (7) is

$$\begin{aligned} & \prod_{i=1}^{n} |\lambda_i|^{1/p_i} W_0' = \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \left(\frac{1}{|\lambda_1|} W_1'\right) \\ =\ & \frac{1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \prod_{i=2}^{n} t_i^{-a_i'} \mathrm{d}t_2 \cdots \mathrm{d}t_n \\ =\ & \frac{1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \left(\prod_{i=2}^{n} t_i^{-a_i}\right) \prod_{i=2}^{n} t_i^{\frac{\lambda_i c}{p_i}} \mathrm{d}t_2 \cdots \mathrm{d}t_n. \end{aligned}$$

Set $H(t_2,\cdots,t_n)=G(1,t_2^{\lambda_2},\cdots,t_n^{\lambda_n})\prod\limits_{i=2}^{n}t_i^{-a_i}$, then, in summary, it can be obtained that

$$\int_{\mathbb{R}_+^{n-1}}H(t_2,\cdots,t_n)\prod_{i=2}^{n}t_i^{\frac{\lambda_i c}{p_i}}\mathrm{d}t_2\cdots\mathrm{d}t_n=W_1^{1/p_1}\prod_{j=2}^{n}\left(\int_{\mathbb{R}_+^{n-1}}H(t_2,\cdots,t_n)t_j^{\lambda_j c}\mathrm{d}t_2\cdots\mathrm{d}t_n\right)^{1/p_j}. \tag{8}$$

For functions $1,t_2^{\lambda_2 c/p_2},\cdots,t_n^{\lambda_n c/p_n}$, it follows from Hölder's inequality that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n-1}}H(t_2,\cdots,t_n)\prod_{i=2}^{n}t_i^{\frac{\lambda_i c}{p_i}}\mathrm{d}t_2\cdots\mathrm{d}t_n\\
\le\ &\left(\int_{\mathbb{R}_+^{n-1}}1^{p_1}H(t_2,\cdots,t_n)\mathrm{d}t_2\cdots\mathrm{d}t_n\right)^{1/p_1}\prod_{j=2}^{n}\left(\int_{\mathbb{R}_+^{n-1}}H(t_2,\cdots,t_n)t_j^{\lambda_j c}\mathrm{d}t_2\cdots\mathrm{d}t_n\right)^{1/p_j}\\
=\ &W_1^{1/p_1}\prod_{j=2}^{n}\left(\int_{\mathbb{R}_+^{n-1}}H(t_2,\cdots,t_n)t_j^{\lambda_j c}\mathrm{d}t_2\cdots\mathrm{d}t_n\right)^{1/p_j}.
\end{aligned}\tag{9}$$

By (8), we know that (9) takes the equal sign. Then, $t_j^{\lambda_j c}$ =constant $(j=2,3,\cdots,n)$ is obtained from the condition of equal sign of Hölder's inequality. Thus, $c=0$, i.e., $\sum_{i=1}^{n}\frac{a_i}{\lambda_i}=\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}$.

(c)⇒(b) Assume that $\frac{1}{\lambda_1}W_1=\frac{1}{\lambda_2}W_2=\cdots=\frac{1}{\lambda_n}W_n$. If $\frac{1}{r}+\frac{1}{s}=1(0<r<1,s<0)$, then, by the inverse Hölder's inequality, we find for $j\ge 2$

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n-1}}K(t_1,\cdots,t_{j-1},1,t_{j+1},\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\\
=\ &W_j=\frac{\lambda_j}{\lambda_1}W_1=\frac{\lambda_j}{\lambda_1}\int_{\mathbb{R}_+^{n-1}}K(1,t_2,\cdots,t_n)\prod_{i=2}^{n}t_i^{-a_i}\mathrm{d}t_2\cdots\mathrm{d}t_n\\
=\ &\frac{\lambda_j}{\lambda_1}\int_{\mathbb{R}_+^{n-1}}K(t_j^{-\frac{\lambda_j}{\lambda_1}},t_j^{-\frac{\lambda_j}{\lambda_2}}t_2,\cdots,t_j^{-\frac{\lambda_j}{\lambda_{j-1}}}t_{j-1},1,t_j^{-\frac{\lambda_j}{\lambda_{j+1}}}t_{j+1},\cdots t_j^{-\frac{\lambda_j}{\lambda_n}}t_n)t_j^{\lambda\lambda_j}\prod_{i=2}^{n}t_i^{-a_i}\mathrm{d}t_2\cdots\mathrm{d}t_n\\
=\ &\int_{\mathbb{R}_+^{n-1}}K(u_1,\cdots,u_{j-1},1,u_{j+1},\cdots,u_n)u_j^{-\frac{\lambda_1}{\lambda_j}(\lambda\lambda_j-a_j)-\lambda_1\sum_{i=2}^{n}\frac{1}{\lambda_i}-1}\\
&\times\prod_{i=2(\neq j)}^{n}u_j^{\frac{\lambda_1}{\lambda_i}a_i}u_i^{-a_i}\mathrm{d}u_1\cdots\mathrm{d}u_{j-1}\mathrm{d}u_{j+1}\cdots\mathrm{d}u_n\\
=\ &\int_{\mathbb{R}_+^{n-1}}K(u_1,\cdots,u_{j-1},1,u_{j+1},\cdots,u_n)u_j^{\lambda_1(\sum_{i=2}^{n}\frac{a_i}{\lambda_i}-\lambda-\sum_{i=1}^{n}\frac{1}{\lambda_i})}\prod_{i=2(\neq j)}^{n}u_i^{-a_i}\\
&\times\mathrm{d}u_1\cdots\mathrm{d}u_{j-1}\mathrm{d}u_{j+1}\cdots\mathrm{d}u_n\\
=\ &\int_{\mathbb{R}_+^{n-1}}K(u_1,\cdots,u_{j-1},1,u_{j+1},\cdots,u_n)u_j^{\lambda_1(c-\frac{a_1}{\lambda_1})}\prod_{i=2(\neq j)}^{n}u_i^{-a_i}\mathrm{d}u_1\cdots\mathrm{d}u_{j-1}\mathrm{d}u_{j+1}\cdots\mathrm{d}u_n\\
=\ &\int_{\mathbb{R}_+^{n-1}}u_j^{\lambda_1 c}K(t_1,\cdots,t_{j-1},1,t_{j+1},\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\\
\ge\ &\left(\int_{\mathbb{R}_+^{n-1}}1^{r}K(t_1,\cdots,t_{j-1},1,t_{j+1},\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\right)^{1/r}\\
&\times\left(\int_{\mathbb{R}_+^{n-1}}t_1^{\lambda_1 cs}K(t_1,\cdots,t_{j-1},1,t_{j+1},\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\right)^{1/s}\\
=\ &W_j^{1/r}\left(\int_{\mathbb{R}_+^{n-1}}t_1^{\lambda_1 cs}K(t_1,\cdots,t_{j-1},1,t_{j+1},\cdots,t_n)\prod_{i\neq j}^{n}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\right)^{1/s}.
\end{aligned}$$

Thereupon,

$$W_j \geq \int_{\mathbb{R}_+^{n-1}} t_1^{\lambda_1 cs} K(t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_n) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n.$$

If $\lambda_1 c > 0$, then $\lambda_1 cs < 0$ and

$$\begin{aligned} W_j &\geq \int_0^{\frac{1}{2}} \int_{\mathbb{R}_+^{n-2}} t_1^{\lambda_1 cs} K(t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_n) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n \\ &\geq \left(\frac{1}{2}\right)^{\lambda_1 cs} \int_0^{\frac{1}{2}} \int_{\mathbb{R}_+^{n-2}} K(t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_n) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n. \end{aligned}$$

Letting $s \to -\infty$, then $W_j = +\infty$. According to the convergence of W_j, we have a contradiction. Hence, $\lambda_1 c > 0$ is impossible.

If $\lambda_1 c < 0$, then $\lambda_1 cs > 0$ and

$$\begin{aligned} W_j &\geq \int_2^{+\infty} \int_{\mathbb{R}_+^{n-2}} t_1^{\lambda_1 cs} K(t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_n) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n \\ &\geq 2^{\lambda_1 cs} \int_2^{+\infty} \int_{\mathbb{R}_+^{n-2}} K(t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_n) \prod_{i \neq j}^{n} t_i^{-a_i} \mathrm{d}t_1 \cdots \mathrm{d}t_{j-1} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_n. \end{aligned}$$

Letting $s \to -\infty$, then $W_j = +\infty$, which is contrary to the convergence of W_j; thus, $\lambda_1 c < 0$ cannot hold.

In conclusion, $\lambda_1 c = 0$,; therefore, $c = 0$, that is $\sum_{i=1}^{n} \frac{a_i}{\lambda_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$.

It has been proved that (b)⇒(c) and (b)⇒(a), (a)⇒(b) and (c)⇒(b), hence (a), (b) and (c) are equivalent to each other.

(iii) It can be obtained by the proof of (b)⇒(a). □

4. Applications

According to Theorem 1, the following several inequalities involving symmetry can be obtained.

Corollary 1. *If* $n \geq 2$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1 (p_i > 1)$, $0 < \lambda_i \leq 1$, $\tilde{a}(i) = \{a_{m_i}(i)\} \in l_{p_i}^{p_i - 1}$ $(i = 1, 2, \cdots, n)$, *then*

$$\begin{aligned} &\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\min\{m_1^{\lambda_1}, \cdots, m_n^{\lambda_n}\}}{\max\{m_1^{\lambda_1}, \cdots, m_n^{\lambda_n}\}} \prod_{i=1}^{n} a_{m_i}(i) \\ \leq\ & n! \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i} - 1} \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i, p_{i-1}}, \end{aligned}$$

where the constant factor $n! \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i} - 1}$ *is the best.*

Proof. Let

$$G\left(x_1^{\lambda_1}, \cdots, x_n^{\lambda_n}\right) = \frac{\min\{x_1^{\lambda_1}, x_2^{\lambda_2}, \cdots, x_n^{\lambda_n}\}}{\max\{x_1^{\lambda_1}, x_2^{\lambda_2}, \cdots, x_n^{\lambda_n}\}},$$

then $G(u_1, u_2, \cdots, u_n)$ a is a homogeneous function of order $\lambda = 0$.

Set $a_i p_i - 1 = p_i - 1$, then $a_i = 1$. Choose $a_i = 1$ $(i = 1, 2, \cdots, n)$ as the matching parameter, we have

$$c = \sum_{i=1}^{n} \frac{p_i}{\lambda_i} - \left(\lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = \sum_{i=1}^{n} \frac{1}{\lambda_i} - \left(0 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right) = 0.$$

Hence $a_i = 1$ $(i = 1, 2, \cdots, n)$ is the best matching parameter.

Since $0 < \lambda_i \leq 1$ $(i = 1, 2, \cdots, n)$, then

$$G\left(x_1^{\lambda_1}, \cdots, x_i^{\lambda_i}, \cdots, x_n^{\lambda_n}\right) x_i^{-a_i} = \frac{\min\{x_1^{\lambda_1}, \cdots, x_i^{\lambda_i}, \cdots, x_n^{\lambda_n}\}}{\max\{x_1^{\lambda_1}, \cdots, x_i^{\lambda_i}, \cdots, x_n^{\lambda_n}\}} x_i^{-1}$$

decreases monotonically on $(0, +\infty)$ with respect to x_i. According to the results of [13], it is calculated that

$$\begin{aligned} W_1 &= \int_{\mathbb{R}_+^{n-1}} G(1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}) \prod_{i=2}^{n} t_i^{-a_i} \mathrm{d}t_2 \cdots \mathrm{d}t_n \\ &= \int_{\mathbb{R}_+^{n-1}} \frac{\min\{1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}\}}{\max\{1, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}\}} \prod_{i=2}^{n} t_i^{-1} \mathrm{d}t_2 \cdots \mathrm{d}t_n \\ &= \prod_{i=2}^{n} \frac{1}{\lambda_i} \int_{\mathbb{R}_+^{n-1}} \frac{\min\{1, u_2, \cdots, u_n\}}{\max\{1, u_2, \cdots, u_n\}} \prod_{i=2}^{n} u_i^{-1} \mathrm{d}u_2 \cdots \mathrm{d}u_n \\ &= n! \prod_{i=2}^{n} \frac{1}{\lambda_i}. \end{aligned}$$

The conclusion of Corollary 1 can be obtained according to Theorem 1. □

Corollary 2. *If* $n \geq 2$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1 (p_i > 1)$, $1 \leq k < n$, $\lambda > 0$, $\lambda_i > 0$, $0 < a_i < 1$, $0 \leq \sigma \leq \min\{\frac{a_1}{\lambda 1}, \cdots, \frac{a_k}{\lambda k}\}$, $\tilde{a}(i) = \{a_{m_i}(i)\} \in l_{p_i}^{a_i p_i - 1}$ $(i = 1, 2, \cdots, n)$, $\sum_{i=1}^{n} \frac{a_i}{\lambda_i} = \sigma - \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$, *then*

$$\begin{aligned} &\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\left(m_1^{\lambda_1} + \cdots + m_k^{\lambda_k}\right)^{\sigma}}{\left(m_1^{\lambda_1} + \cdots + m_k^{\lambda_k} + \cdots + m_n^{\lambda_n}\right)^{\lambda}} \prod_{i=1}^{n} a_{m_i}(i) \\ \leq\ & M_0 \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i, a_i p_{i-1}}, \end{aligned}$$

where the constant factor

$$M_0 = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i} - 1} \frac{\Gamma\left(\sigma + \sum_{i=1}^{k} \frac{1-a_i}{\lambda_i}\right)}{\Gamma\left(\sum_{i=1}^{k} \frac{1-a_i}{\lambda_i}\right)} \prod_{i=1}^{n} \Gamma\left(\frac{1-a_i}{\lambda_i}\right)$$

is the best.

Proof. Let
$$G\left(x_1^{\lambda_1},\cdots,x_n^{\lambda_n}\right)=\frac{\left(x_1^{\lambda_1}+\cdots+x_k^{\lambda_k}\right)^{\sigma}}{\left(x_1^{\lambda_1}+\cdots+x_k^{\lambda_k}+\cdots+x_n^{\lambda_n}\right)^{\lambda}},$$
then $G(u_1,\cdots,u_n)$ is a homogeneous function of order $\sigma-\lambda$.

For $i>k$, notice that $\lambda>0,\lambda_i>0,a_i>0$, then
$$G\left(x_1^{\lambda_1},\cdots,x_n^{\lambda_n}\right)x_i^{-a_i}=\frac{\left(x_1^{\lambda_1}+\cdots+x_k^{\lambda_k}\right)^{\sigma}}{\left(x_1^{\lambda_1}+\cdots+x_k^{\lambda_k}+\cdots+x_n^{\lambda_n}\right)^{\lambda}}x_i^{-a_i}$$
is monotonically decreasing on $(0,+\infty)$ with respect to x_i.

For $1\le i\le k$, since $\lambda>0,\lambda_i>0,a_i>0,\ 0\le\sigma\le\min\{\frac{a_1}{\lambda 1},\cdots,\frac{a_k}{\lambda k}\}$, then
$$G\left(x_1^{\lambda_1},\cdots,x_n^{\lambda_n}\right)x_i^{-a_i}=\frac{\left(x_i^{-a_i/\sigma}x_1^{\lambda_1}+\cdots x_i^{\lambda_i-a_i/\sigma}+\cdots+x_i^{-a_i/\sigma}x_k^{\lambda_k}\right)^{\sigma}}{\left(x_1^{\lambda_1}+\cdots+x_i^{\lambda_i}+\cdots+x_n^{\lambda_n}\right)^{\lambda}}$$
is also monotonically decreasing on $(0,+\infty)$ with respect to x_i.

According to Lemma 2, we calculate
$$\begin{aligned}
W_n &= \int_{\mathbb{R}_+^{n-1}} G(t_1^{\lambda_1},\cdots,t_{n-1}^{\lambda_{n-1}},1)\prod_{i=1}^{n-1}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\\
&= \int_{\mathbb{R}_+^{n-1}} \frac{\left(t_1^{\lambda_1}+\cdots+t_k^{\lambda_k}\right)^{\sigma}}{\left(t_1^{\lambda_1}+\cdots+t_k^{\lambda_k}+t_{k+1}^{\lambda_{k+1}}+\cdots+t_{n-1}^{\lambda_{n-1}}+1\right)^{\lambda}}\prod_{i=1}^{n-1}t_i^{-a_i}\mathrm{d}t_2\cdots\mathrm{d}t_{n-1}\\
&= \int_{\mathbb{R}_+^{n-k-1}}\prod_{i=k+1}^{n-1}t_i^{-a_i}\left(\int_{\mathbb{R}_+^{k}}\frac{\left(t_1^{\lambda_1}+\cdots+t_k^{\lambda_k}\right)^{\sigma}}{\left[\left(t_1^{\lambda_1}+\cdots+t_k^{\lambda_k}\right)+\left(t_{k+1}^{\lambda_{k+1}}+\cdots+t_{n-1}^{\lambda_{n-1}}\right)+1\right]^{\lambda}}\right.\\
&\quad\left.\times\prod_{i=1}^{k}t_i^{-a_i}\mathrm{d}t_1\cdots\mathrm{d}t_k\right)\mathrm{d}t_{k+1}\cdots\mathrm{d}t_{n-1}\\
&= \frac{\prod_{i=1}^{k}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{k}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}\int_{\mathbb{R}_+^{n-k-1}}\prod_{i=k+1}^{n-1}t_i^{-a_i}\left(\int_0^{+\infty}\frac{u^{\sigma}}{\left[u+\left(t_{k+1}^{\lambda_{k+1}}+\cdots+t_{n-1}^{\lambda_{n-1}}\right)+1\right]^{\lambda}}\right.\\
&\quad\left.\times u^{\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}-1}\mathrm{d}u\right)\mathrm{d}t_{k+1}\cdots\mathrm{d}t_{n-1}
\end{aligned}$$

$$= \frac{\prod_{i=1}^{k}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{k}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}\int_0^{+\infty}u^{\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}-1}\left(\int_{\mathbb{R}_+^{n-k-1}}\frac{1}{\left[1+u+\left(t_{k+1}^{\lambda_{k+1}}+\cdots+t_{n-1}^{\lambda_{n-1}}\right)\right]^{\lambda}}\right.$$

$$\left.\times\prod_{i=k+1}^{n-1}t_i^{-a_i}\mathrm{d}t_{k+1}\cdots\mathrm{d}t_{n-1}\right)\mathrm{d}u$$

$$= \frac{\prod_{i=1}^{k}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{k}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}\frac{\prod_{i=1}^{n-1}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=k+1}^{n-1}\lambda_i\Gamma\left(\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}\int_0^{+\infty}u^{\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}-1}$$

$$\times\left(\int_0^{+\infty}\frac{1}{(1+u+v)^{\lambda}}v^{\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}-1}\mathrm{d}v\right)\mathrm{d}u$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{n-1}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}\int_{\mathbb{R}_+^2}\frac{1}{(1+u+v)^{\lambda}}u^{\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}-1}v^{\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}-1}$$

$$\times\mathrm{d}u\mathrm{d}v$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{n-1}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}\frac{\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}{\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}+\sum_{i=k+1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}$$

$$\times\int_0^{+\infty}\frac{1}{(1+t)^{\lambda}}t^{\sigma+\sum_{i=1}^{n-1}\frac{1-a_i}{\lambda_i}-1}\mathrm{d}t$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}{\prod_{i=1}^{n-1}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sigma+\sum_{i=1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}\frac{\Gamma\left(\sigma+\sum_{i=1}^{n-1}\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\lambda-\sigma-\sum_{i=1}^{n-1}\frac{1-a_i}{\lambda_i}\right)}{\Gamma(\lambda)}$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}{\Gamma(\lambda)\prod_{i=1}^{n-1}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}\Gamma\left(\frac{1-a_n}{\lambda_n}\right)=\frac{\prod_{i=1}^{n}\Gamma\left(\frac{1-a_i}{\lambda_i}\right)\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}{\Gamma(\lambda)\prod_{i=1}^{n-1}\lambda_i\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}.$$

Thereupon,

$$W_0=\frac{1}{\lambda_n}W_n=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\lambda_i^{-1}\frac{\Gamma\left(\sigma+\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}{\Gamma\left(\sum_{i=1}^{k}\frac{1-a_i}{\lambda_i}\right)}\prod_{i=1}^{n}\Gamma\left(\frac{1-a_i}{\lambda_i}\right).$$

It follows from Theorem 1 that Corollary 2 holds. □

In Corollary 2, take $\sigma=0$; the following corollary can be obtained.

Corollary 3. *Suppose that* $n\geq 2$, $\sum_{i=1}^{n}\frac{1}{p_i}=1(p_i>1)$, $\lambda>0,\lambda_i>0,0<a_i<1$, $\sum_{i=1}^{n}\frac{a_i}{\lambda_i}=-\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}$, $\tilde{a}(i)=\{a_{m_i}(i)\}\in l_{p_i}^{a_ip_i-1}$ $(i=1,2,\cdots,n)$. *Then*

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\left(m_1^{\lambda_1}+\cdots+m_n^{\lambda_n}\right)^{\lambda}}\prod_{i=1}^{n}a_{m_i}(i)\leq M_0\prod_{i=1}^{n}||\tilde{a}(i)||_{p_i,a_ip_i-1},$$

where the constant factor

$$M_0 = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i}-1} \prod_{i=1}^{n} \Gamma\left(\frac{1-a_i}{\lambda_i}\right)$$

is the best.

Take $a_i = \frac{\lambda_i}{p_i}(\sigma - \lambda) - 1$ $(i = 1, 2, \cdots, n)$ in Corollary 2, we can obtain the following result.

Corollary 4. *If* $n \geq 2$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1 (p_i > 1)$, $1 \leq k < n$, $\lambda > 0$, $\lambda_i > 0$, $0 < \lambda - \sigma < \frac{p_i}{\lambda i}$, $\alpha_i = \lambda_i(\sigma - \lambda) + p_i - 1$, $0 < \sigma < \left(1 - \frac{1}{p_i}\right)^{-1}\left(\frac{1}{\lambda i} - \frac{\lambda}{p_i}\right)$, $\tilde{a}(i) = \{a_{m_i}(i)\} \in l_{p_i}^{\alpha_i}$ $(i = 1, 2, \cdots, n)$, *then*

$$\begin{aligned} &\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\left(m_1^{\lambda_1} + \cdots + m_k^{\lambda_k}\right)^{\sigma}}{\left(m_1^{\lambda_1} + \cdots + m_k^{\lambda_k} + \cdots + m_n^{\lambda_n}\right)^{\lambda}} \prod_{i=1}^{n} a_{m_i}(i) \\ \leq \quad & M_0 \prod_{i=1}^{n} ||\tilde{a}(i)||_{p_i,\alpha_i}, \end{aligned}$$

where the constant factor

$$M_0 = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i}-1} \frac{\Gamma\left(\sigma + \sum_{i=1}^{k} \frac{\lambda-\sigma}{p_i}\right)}{\Gamma\left(\sum_{i=1}^{k} \frac{\lambda-\sigma}{\lambda_i}\right)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda-\sigma}{\lambda_i}\right)$$

is the best.

5. Conclusions

In this paper, based on some symmetric homogeneous kernels, the concept of the quasi-homogeneous kernel is proposed and extended to the high-dimensional case. Then, using the symmetry of the Hilbert-type inequality for each variable and the weight function method, the matching parameters $a_1, \cdots, a_n$ are introduced to obtain the Hilbert-type inequality of an n-multiple series. In Theorem 1, the equivalent parameter conditions of the best constant factor of the n-multiple Hilbert-type inequality is established, and the parameter problem of constructing a Hilbert-type series inequality with the best constant factor is solved. As applications, some special cases are given in Corollaries 1–4, and many Hilbert series inequalities with symmetric quasi-homogeneous kernels and best constant factors are obtained.

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