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Some Results in the Theory of a Cosserat Thermoelastic Body with Microtemperatures and Inner Structure

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Abstract: This study is concerned with the theory of Cosserat thermoelastic media, whose micro-particles possess microtemperatures. The mixed initial boundary value problem considered in this context is transformed in a temporally evolutionary equation on a Hilbert space. Using some results from the theory of semigroups, the existence and uniqueness of solution is proved. In the same manner, it approached the continuous dependence of the solution upon initial data and loads. From what we have studied, neither on the internet nor in the databases, we have not found qualitative issues addressed regarding the mixed problem in the context of the theory of thermoelasticity of Cosserat environments, in which the contribution of inner structure and microtemperatures are taken into account.

Keywords: cosserat thermoelastic body; microtemperatures; micro-particles; semigroup; continuous dependence

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1. Introduction

The study of elastic materials with microstructure was initiated by the French Cosserat brothers in a famous book that was published in 1909. Recently, this theory has been addressed in a large number of specialized studies. Eringen attached great importance to these environments (for instance, [1]), which improved the theory of these environments by adding a conservation law for the tensor of inertia. In this way, he laid the foundations of the theory of micromorphic continua. Important results on the Cosserat media are presented in the works [2–5]. For instance, in [2], the authors give explicit solutions for surface waves propagation in a homogeneous half space filled with an isotropic Cosserat elastic material. Additionally, in [5], in the framework of the linear theory, a uniqueness result and a solution of Galerkin type are established. In classical theories, the fact, that the reaction of a body to some external actions is influenced by the intimate structure of that body, is ignored. In the theory of Cosserat bodies, there are three extra freedom degrees for the "rotation" of the points in the media. In this way a new tensor appears—the couple stress tensor, which completes the classical stress tensor. We should give some examples of bodies with a microstructure, in order to point out how important it is to consider the micromotions. So, we have: polymers, suspensions, crystals, composites, grid and multibar systems, or blood. Another theory in this regard is the theory of microstretch materials (for instance, Eringen [6]). This is an extension of the Cosserat theory, and a particular case of the theory of micromorphic media, in which there are considered three deformable directors that only have microdeformations of the breathing type. Some results on bodies with a microstructure can be found in [7–10].

For instance, in [8], in a microstretch material, a Toupin-type measure associated with the corresponding steady-state vibration is used, and by assuming that the angular frequency of oscillations is lower than a certain critical frequency, it is shown that the amplitude of the vibrations decays exponentially with the distance to the base.

All of the above generalized theories have a common aim, namely, to eliminate the disagreements between the experiments and the classical theory of elasticity. The contribution of the microstructure on the media's overall evolutions was first noticed in the case of graphite, human bones, polymers, and granular materials with large molecules. Other discrepancies that should have disappeared are those observed between short wavelengths and elastic vibrations of high frequency.

The first studies dedicated to the theory of media having microtemperatures were published by Grot (for instance, [11]). He proposed a theory of solids with a microstructure, in which each microelement is endoved with microtemperatures. As a consequence, the inequality of the entropy production has been modified to evidentiate the presence of the microtemperatures. Additionally, together with the known energy equations of bodies with microtemperatures, the first-order moment of the equations of energy has been considered. There are many studies dedicated to the thermoelasticity of media having microtemperatures, such as Refs. [12–14].

In [14], in the context of a linear theory of thermoelasticity with microtemperatures, the basic boundary value problems of steady vibrations are investigated using the potential method. For different results regarding the inner structures of materials, we recommend the studies [15–22]. In [16], the authors show that, in the analysis with finite elements of a solid body having a general rigid motion, additional elements occur, which can change the dynamic response of the system.

In our present paper, we take into account the effect of the inner structure and microtemperatures in the deformation of a Cosserat thermoelastic solid. Our study can be considered as belonging to the generalized theories of continuous media, because the effects we consider do not prevent the propagation of thermal waves at finite speed. To get the main results for the mixed problem in the context of Cosserat thermoelastic bodies with inner structure and microtemperatures, we use a few results from the theory of semi-groups of operators. The problem in this context will be replaced by an abstract problem of the Cauchy type, attached to an evolutionary equation, defined on a specific Hilbert space, built in context. This procedure makes it much easier to get results regarding the existence of at least one solution and the uniqueness of the solution of our problem. In addition, this technique eases even the continuous dependence of solutions on both initial values and loads.

2. Basic Conditions and Equations

In what follows, we consider a bounded domain D from Euclidean space R^3, which is occupied by a Cosserat thermoelastic body. The closure of D is denoted by $\bar{D}$ and we have $\bar{D} = D \cup \partial D$, where, as usual, ∂D is the boundary of the region D, which we consider a piece-wise smooth surface, with the outward unit normal of components n_i. To designate a vector field and a tensor field, we use letters in boldface. The components of a vector field $\mathbf{w}$ are denoted by w_i. To designate the time derivative of a function h, we use the notation $\dot{h}$, that is, a superposed dot. The notation h, m is used for the partial differentiation of function h with respect to its m-th variable. The Einstein summation rule regarding the repeated subscripts is also used. If there is no likelihood of confusion, it can be omitted to write the dependence of a function on its spatial variables or time variable. In order to characterize the evolution of a Cosserat thermoelastic body, we use a displacement vector with the components v_m, a microrotation vector with the components ϕ_m, a microstretch function φ, and the temperature θ measured from the constant absolute temperature T_0 of the body in its reference state.

With the help of these internal state variables, the components of the strain tensors e_{mn}, ε_{mn} and γ_m can be defined, by means of the following usual kinematic relations:

$$e_{mn} = v_{n,m} + \epsilon_{mnk}\phi_k,\ \varepsilon_{mn} = \phi_{n,m},\ \gamma_m = \varphi_{,m}, \tag{1}$$

where ϵ_{ijk} is the Riccis's symbol.

We suppose that the stress tensor has the components τ_{mn} and the components of the tensor of the couple stress are denoted by σ_{mn}.

According to Iesan and Nappa [5], the motion equations in the theory of Cosserat thermoelastic media have the following form:

$$\begin{aligned} \tau_{ji,j} + \varrho f_i &= \varrho \ddot{v}_i, \\ \sigma_{ji,j} + \epsilon_{ijk}\tau_{jk} + \varrho g_i &= I_{ij}\ddot{\phi}_j. \end{aligned} \tag{2}$$

The moment of the first stress has the following balance [5]:

$$h_{i,i} - \lambda + \varrho l = J\ddot{\varphi}. \tag{3}$$

Here, the following notations are used: $\varrho =$ the mass density in the initial state of the body, $f = (f_m) =$ the body force, $g = (g_m) =$ the body couple, and $L =$ an external body charge. In addition, the components of inertia are noted by $I_{mn} = I_{nm}$ and J.

The variation of the temperature is denoted by θ and we have $\theta = T - T_0$, where T_0 is the temperature of the body in the undeformed state.

In addition, it is assumed that the temperature in the media is given by the following sum

$$\theta + T_m\left(X_m^{\prime} - X_m\right), \tag{4}$$

where $(X_i^{\prime})$ are the coordinates of the mass center of an arbitrary microelement in the initial state of the body and (X_i) are the elements of a generic point of the body. In (4), it is denoted by T_m microtemperatures of media.

Similar to ϑ_i, entered above, the variation of the microtemperatures ϑ_i, relative to the microtemperatures T_m^0 in the initial state of the body is considered, namely, $\vartheta_m = T_m - T_m^0$.

The deformation of a Cosserat thermoelastic solis with inner structure and microtemperatures will be evaluated using the above introduced variables v_m, ϕ_m, φ and the thermal fields χ and τ_m, having the following form:

$$\chi = \int_{t_0}^{t} \theta d\tau,\ \tau_m = \int_{t_0}^{t} \vartheta_m d\tau, \tag{5}$$

t_0 being the initial time.

Being in the context of a linear theory, it is natural to consider that the internal energy is a quadratic form relative to all its constitutive functions, that is, as follows:

$$\begin{aligned} U = \frac{1}{2}\Big[& A_{ijmn}e_{ij}e_{mn} + 2B_{ijmn}e_{ij}\varepsilon_{mn} + 2a_{ij}e_{ij}\varphi + 2D_{ijmn}e_{ij}\tau_{m,n} \\ & + C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2b_{ij}\varepsilon_{ij}\varphi + 2E_{ijmn}\varepsilon_{ij}\tau_{m,n} + A_{ij}\varphi_{,i}\varphi_{,j} \\ & + 2H_{ij}\varphi_{,i}\chi_{,j} + \zeta\phi^2 + 2F_{ij}\tau_{i,j}\varphi + K_{ij}\chi_{,i}\chi_{,j} + G_{ijmn}\tau_{m,n}\tau_{i,j} \Big]. \end{aligned} \tag{6}$$

Using a procedure analogous to that in [4], we obtain:

$$\begin{aligned} \tau_{ij} = \frac{\partial U}{\partial e_{ij}}, \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, h_i = \frac{\partial U}{\partial \varphi_{,i}}, \lambda = \frac{\partial U}{\partial \varphi}, \\ \eta = \frac{\partial U}{\partial \theta}, S_i = \frac{\partial U}{\partial \chi_{,i}}, \Lambda_{ij} = \frac{\partial U}{\partial \tau_{i,j}}, \end{aligned}$$

and so we obtain the constitutive equations for a homogeneous and anisotropic Cosserat thermoelastic media with an inner structure and microtemperatures:

$$\begin{aligned}
\tau_{ij} &= A_{ijmn}e_{mn} + B_{ijmn}\varepsilon_{mn} + a_{ij}\varphi - \alpha_{ij}\dot{\chi} + D_{ijmn}\tau_{m,n},\\
\sigma_{ij} &= B_{ijmn}\, e_{mn} + C_{ijmn}\varepsilon_{mn} + b_{ij}\varphi - \beta_{ij}\dot{\chi} + E_{ijmn}\tau_{m,n},\\
h_i &= A_{ij}\gamma_j - d_{ij}\dot{\tau}_j + H_{ij}\chi_{,j},\\
\lambda &= \frac{\partial U}{\partial \varphi} = a_{ij}e_{ij} + b_{ij}\varepsilon_{ij} + \zeta\varphi - \kappa\dot{\chi} + F_{ij}\tau_{i,j},\\
\varrho\eta &= \alpha_{ij}e_{ij} + \beta_{ij}\varepsilon_{ij} + \kappa\varphi + a\dot{\chi} + L_{ij}\tau_{i,j},\\
\varrho\eta_i &= d_{ji}\gamma_j + B_{ij}\dot{\tau}_j + C_{ij}\chi_{,j},\\
S_i &= H_{ji}\gamma_j - C_{ji}\dot{\tau}_j + K_{ij}\chi_{,j},\\
\Lambda_{ij} &= D_{ijmn}e_{mn} + E_{ijmn}\varepsilon_{ij} + F_{ji}\varphi - L_{ji}\dot{\chi} + G_{ijmn}\tau_{m,n}.
\end{aligned} \tag{7}$$

The new notations that appear in (6) have the following significations: τ_{mn} and σ_{mn} are the stress tensors in the body, h_m is the internal hypertraction function, λ is a generalized internal charge, η is the mass entropy, η_m is the first vector of the entropy moment, S_m is the vector of the entropy flux, and Λ_{mn} are the components of the tensor of the first entropy flux moment.

The constitutive thermoelastic coefficients A_{ijmn}, B_{ijmn}, …, L_{ji}, and G_{ijmn}, which also appear above, satisfy the following relations of symmetry:

$$\begin{aligned}
&A_{ijmn} = A_{mnij},\ C_{ijmn} = C_{mnij},\ A_{ij} = A_{ji},\\
&K_{ij} = K_{ji},\ D_{ijmn} = D_{jimn},\ G_{ijmn} = G_{mnij}.
\end{aligned} \tag{8}$$

By using the equation of energy, we obtain the following equation [11]:

$$\varrho\xi_m + S_m - H_m = 0, \tag{9}$$

in which ξ_m is the internal rate of entropy production, H_m are the components of vector of the mean entropy flux, and, as above, S_m are the components of the vector of the entropy flux.

The rate of supply of entropy by s and the first moment components of the rate of entropy supply by Q_m will be denoted. Then, we deduce two more relations of energy [9]:

$$\begin{aligned}
\varrho\dot{\eta} &= S_{m,m} + \varrho s,\\
\varrho\dot{\eta}_m &= \Lambda_{mn,n} + \varrho Q_m.
\end{aligned} \tag{10}$$

Now, we consider the kinematic Equation (5) and take into account the constitutive relations (7), such that from the motion Equation (1), the equation of first moment of stress (2), and the energy Equation (10), the following partial differential equations are obtained:

$$\begin{aligned}
&A_{ijmn}\left(v_{m,nj} + \epsilon_{mnk}\phi_{k,j}\right) + B_{ijmn}\phi_{n,mj} + a_{ij}\varphi_{,j} - \alpha_{ij}\dot{\chi}_{,j} + D_{ijmn}\tau_{m,nj} + \varrho F_i = \varrho\ddot{v}_i;\\
&B_{ijmn}\left(v_{m,nj} + \epsilon_{mnk}\phi_{k,j}\right) + C_{ijmn}\phi_{n,mj} + b_{ij}\varphi_{,j} - \beta_{ij}\dot{\chi}_{,j} + E_{ijmn}\tau_{m,nj}\\
&\quad + \epsilon_{ijk}\Big[A_{jkmn}(v_{m,n} + \epsilon_{mnk}\phi_k) + B_{jkmn}\phi_{n,m} + a_{jk}\varphi - \alpha_{jk}\dot{\chi} + D_{jkmn}\tau_{m,n}\Big] + \varrho G_i = I_{ij}\ddot{\phi}_j,\\
&A_{ij}\phi_{,ij} - d_{ij}\dot{\tau}_{j,i} + H_{ij}\chi_{,ij} - a_{ij}\left(v_{j,i} + \epsilon_{ijk}\phi_k\right) - b_{ij}\phi_{j,i} - \zeta\varphi - \kappa\dot{\chi} - F_{ij}\tau_{i,j} + \varrho L = J\ddot{\varphi},\\
&H_{ji}\phi_{,ij} - D_{ij}\dot{\tau}_{j,i} + K_{ij}\chi_{,ij} - \alpha_{ij}\left(\dot{v}_{j,i} + \epsilon_{ijk}\dot{\phi}_k\right) - \beta_{ij}\dot{\phi}_{j,i} - \kappa\dot{\varphi} - a\ddot{\chi} = -\varrho s,\\
&D_{ijmn}\left(v_{m,nj} + \epsilon_{mnk}\phi_{k,j}\right) + E_{ijmn}\phi_{n,mj} + F_{ji}\varphi_{,j}\\
&\qquad - D_{ji}\dot{\chi}_{,j} + G_{ijmn}\tau_{m,nj} - d_{ij}\dot{\varphi}_{,j} - B_{ij}\ddot{\tau}_j = -\varrho Q_i.
\end{aligned} \tag{11}$$

in which it is clearer what the unknown functions are, namely v_m, ϕ_m, φ, χ and τ_m.

The signification of D_{mn} is: $D_{mn} = C_{mn} + L_{mn}$.

In the case of a Dirichlet problem attached to Equation (11), the boundary relations in the following form are used:

$$v_m = \bar{v}_m,\ \phi_m = \bar{\phi}_m,\ \varphi = \bar{\varphi},\ \chi = \bar{\chi},\ \tau_m = \bar{\tau}_m,\ \text{ on } \partial D \times (0, \infty), \tag{12}$$

in which $\bar{v}_m$, $\bar{\phi}_m$, $\bar{\varphi}$, $\bar{\chi}$, $\bar{\tau}_m$ are the prescribed functions.

If we consider a Neumann type boundary value problem, associated to Equation (11), then the relations to the limit (12) are substituted by the following conditions:

$$\tau_{km} n_m = \bar{t}_m,\ \sigma_{kn} n_k = \bar{m}_n,\ \lambda_m n_m = \bar{\lambda},\ S_m n_m = \bar{S},\ \Lambda_{km} n_k = \bar{\Lambda}_m,\ on\ \partial D \times (0, \infty), \tag{13}$$

in which $\bar{t}_m$, $\bar{m}_k$, $\bar{\lambda}$, $\bar{S}$ and $\bar{\Lambda}_m$ are the known functions.

In what follows, we only take into account a boundary problem with Dirichlet conditions.

In order to complete the mixed problem for the Cosserat thermoelastic bodies with microtemperatures and inner structure, we need to associate the initial data, which we use in the following form:

$$\begin{gathered} v_m(x,0) = v_m^0(x),\ \dot{v}_m(x,0) = v_m^1(x),\ \phi_m(x,0) = \phi_m^0(x), \\ \dot{\phi}_m(x,0) = \phi_m^1(x),\ \varphi(x,0) = \varphi^0(x),\ \dot{\varphi}(x,0) = \varphi^1(x), \\ \chi(x,0) = \chi^0(x),\ \dot{\chi}(x,0) = \chi^1(x),\ \tau_m(x,0) = \tau_m^0(x),\ \dot{\tau}_m(x,0) = \tau_m^1(x), \end{gathered} \tag{14}$$

for any $x \in D$. Here, the functions v_m^0, v_m^1, ϕ_m^0, ϕ_m^1, ϕ^0, ϕ^1, χ^0, χ^1, τ_m^0, and τ_m^1 are prescribed.

Let us denote by $\mathcal{P}$ the mixed problem consisting of basic Equation (11), the boundary conditions (12), and the initial data (14).

3. Results and Discussion

We address in this section our main results, namely, we formulate and demonstrate a result regarding the existence of a solution for problem $\mathcal{P}$, which was previously formulated. We also study under what conditions the formulated mixed problem admits only one solution, that is, the uniqueness of the solution of problem $\mathcal{P}$ is proven. One last and equally important result that we approach at the end of our study, is whether the solution to our problem depends continuously on both the initial data and the loads. All three theorems will be obtained based on results from the semigroup theory of operators. In order to avoid repeating certain conditions, we impose that all the functions that appear in the conditions and the equations below are well-defined at the whole of their domain in order to perform some mathematical operations.

To obtain our first result regarding the uniqueness, we will use the preliminary result, which is proven in following proposition.

Proposition 1. *Between the internal functions that describe the evolution of a thermoelastic Cosserat solid having microtemperatures and inner structure, the next identity takes place:*

$$\begin{aligned} &\tau_{ij} e_{ij} + \sigma_{ij} \varepsilon_{ij} + \lambda_i \phi_{,i} + \sigma \varphi + \varrho \eta \dot{\chi} + \varrho \eta_i \dot{\tau}_i + S_i \chi_{,i} + \Lambda_{ij} \tau_{i,j} \\ &\quad = A_{ijmn} e_{ij} e_{mn} + 2B_{ijmn} e_{ij} \varepsilon_{mn} + 2a_{ij} e_{ij} \varphi + 2D_{ijmn} e_{ij} \tau_{m,n} \\ &\qquad + C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + 2b_{ij} \varepsilon_{ij} \varphi + 2E_{ijmn} \varepsilon_{ij} \tau_{m,n} + A_{ij} \varphi_{,i} \varphi_{,j} \\ &\qquad + 2H_{ij} \phi_{,i} \chi_{,j} + \zeta \varphi^2 + 2F_{ij} \tau_{i,j} \varphi + K_{ij} \chi_{,i} \chi_{,j} \\ &\qquad + G_{ijmn} \tau_{m,n} \tau_{i,j} + a \dot{\chi}^2 + B_{ij} \dot{\tau}_i \dot{\tau}_j. \end{aligned} \tag{15}$$

Proof. We multiply, in the following manner, $\tau_{mn}.e_{mn}$, $\sigma_{mn}.\varepsilon_{mn}$, $\lambda_m.\varphi_{,m}$, $\sigma.\varphi$, $\varrho\eta.\dot{\chi}$, $\varrho\eta_m.\dot{\tau}_m$, $S_m.\chi_{,m}$ and $\Lambda_{mn}.\tau_{m,n}$ in all equations from the constitutive relations (6).

After that, all obtained equalities are gathered member by member. Finally, if it takes into account the symmetry relations (7) we are led to the anticipated relation (14). □

Taking into account the expression of the internal energy and taking into account the identity (15), the first result of uniqueness can be obtained, regarding the solution of problem $\mathcal{P}$.

This will be obtained in the following proposition.

Proposition 2. *Suppose that the following hypotheses take the place:*

It is assumed that the following hypotheses take places:

1. *ϱ, a, I_{ij} and J are positive (strictly);*
2. *The relations of symmetry (8) are satisfied;*
3. *The internal energy U is a positive semi-definite quadratic form;*
4. *The tensor, having as components the constitutive expressions B_{ij}, is a positively defined one.*

Then, the mixed problem $\mathcal{P}$ has a unique solution.

Proof. Using the same procedure as in the proof of Proposition 1 we begin by multiplying any of the constitutive relations from (7) in the following manner: $\tau_{mn}.\dot{e}_{mn}$, $\dot{\varepsilon}_{mn}.\sigma_{mn}$, $\lambda_m.\dot{\phi}_{,m}$, $\dot{\varphi}.\sigma$, $\varrho\dot{\chi}.\dot{\eta}$, $\varrho\dot{\tau}_m\dot{\eta}_m.$, $S_m.\dot{\chi}_{,m}$ and $\Lambda_{mn}.\dot{\tau}_{m,n}$. After that, the equalities equals that result are added, member to member, so that after we take into account the symmetry relations (8) and the internal energy U having the form from (15), we are led to the following identity:

$$\begin{aligned}\tau_{ij}\dot{e}_{ij}+\sigma_{ij}\dot{\varepsilon}_{ij}+\lambda_i\dot{\varphi}_{,i}+\sigma\dot{\varphi}+\varrho\dot{\eta}\dot{\chi}+\varrho\dot{\eta}_i\dot{\tau}_i+S_i\dot{\chi}_{,i}+\Lambda_{ij}\dot{\tau}_{i,j}\\=\frac{\partial}{\partial t}\left(U+\frac{1}{2}a\dot{\chi}^2+\frac{1}{2}B_{ij}\dot{\tau}_i\dot{\tau}_j\right).\end{aligned} \tag{16}$$

Now there are considered the kinematic Equation (5), the motion Equation (1), the first stress moment Equation (2), and the equations of energy (9), and in this way the following equality is obtained:

$$\begin{aligned}\tau_{ij}\dot{e}_{ij}+\dot{\varepsilon}_{ij}\sigma_{ij}+\dot{\varphi}_{,i}\lambda_i+\sigma\dot{\varphi}+\varrho\dot{\chi}\dot{\eta}+\varrho\dot{\tau}_i\dot{\eta}_i+S_i\dot{\chi}_{,i}+\Lambda_{ij}\dot{\tau}_{i,j}\\=\left(\tau_{ij}\ \dot{u}_i+\sigma_{ij}\dot{\phi}_i+\dot{\varphi}h_j+\dot{\chi}S_j+\Lambda_{ij}\dot{\tau}_i\right)_{,j}\\+\varrho(F_i\dot{v}_i+G_i\dot{\phi}_i+L\dot{\varphi}+s\dot{\chi}+Q_i\dot{\tau}_i)-\varrho\ddot{v}_i\dot{v}_i-I_{ij}\ddot{\phi}_i\dot{\phi}_j-J\ddot{\varphi}\dot{\varphi}.\end{aligned} \tag{17}$$

Of course, by using the equalities (16) and (17), the following identity is deduced:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(2U+\varrho\dot{v}_i\dot{v}_i+I_{ij}\dot{\phi}_i\dot{\phi}_j+J\dot{\varphi}^2+a\dot{\chi}^2+B_{ij}\dot{\tau}_i\dot{\tau}_j\right)\\=\left(\tau_{ij}\dot{v}_i+\sigma_{ij}\dot{\phi}_i+\lambda_j\dot{\varphi}+S_j\dot{\chi}+\Lambda_{ij}\dot{\tau}_i\right)_{,j}+\varrho(F_i\dot{v}_i+G_i\dot{\phi}_i+L\dot{\varphi}+s\dot{\chi}+Q_i\dot{\tau}_i).\end{aligned} \tag{18}$$

Now, over the domain D the identity (18) is integrated, and then the theorem of divergence is used in order to obtain the relation:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_B\left(2U+\varrho\dot{v}_i\dot{v}_i+I_{ij}\dot{\phi}_i\dot{\phi}_j+J\dot{\varphi}^2+a\dot{\chi}^2+B_{ij}\dot{\tau}_i\dot{\tau}_j\right)dV\\=\int_{\partial B}(\tau_{ji}\dot{v}_i+\sigma_i\dot{\phi}_i+\lambda_j\dot{\varphi}+S_j\dot{\chi}+\Lambda_{ij}\dot{\tau}_i)n_j dA+\int_B\varrho(F_i\dot{v}_i+G_i\dot{\phi}_i+L\dot{\varphi}+s\dot{\chi}+Q_i\dot{\tau}_i)dV.\end{aligned} \tag{19}$$

Let us designate by "*" the difference between two arbitrary solutions of the problem $\mathcal{P}$, i.e.,

$$v^*_m=v^2_m-v^1_m,\ \phi^*_m=\phi^2_m-\phi^1_m,\ \varphi^*=\varphi^2-\varphi^1,\ \chi^*=\chi^2-\chi^1,\ \tau^*_m=\tau^2_m-\tau^1_m.$$

With the same mark "*", any other sizes relative to the previous differences are designated.

Of course, because of linearity, the above differences also satisfy the motion Equation (1), the first stress moment balance (2), and the equations of energy (9). Clearly, all these equations are satisfied in the case of null body charges. Obviously, in this situation the initial relations become homogeneous, so that for every $x \in D$, it results:

$$\begin{gathered} v_m^*(0,x) = 0,\ \dot{v}_m^*(0,x) = 0,\ \phi_m^*(0,x) = 0,\ \dot{\phi}_m^*(0,x) = 0,\ \varphi^*(0,x) = 0, \\ \dot{\phi}^*(0,x) = 0,\ \chi^*(0,x) = 0,\ \dot{\chi}^*(0,x) = 0,\ \tau_m^*(0,x) = 0,\ \dot{\tau}_m^*(0,x) = 0. \end{gathered} \tag{20}$$

In addition, in this situation, boundary values are null:

$$v_m^* = 0, \phi_m^* = 0, \phi^* = 0, \chi^* = 0,\ \tau_m^* = 0, \text{ on } \partial D \times (0,\infty), \tag{21}$$

and

$$e_{mn}^*(0,x) = 0,\ \varepsilon_{mn}^*(0,x) = 0,\ \varphi_{,m}^*(0,x) = 0,\ \chi_{,m}^*(0,x) = 0,\ \tau_{m,n}^*(0,x) = 0,\ \ x \in D. \tag{22}$$

Based on the above considerations, the relation (19) is written for the considered differences and we obtain:

$$\int_B \left(2U^* + \varrho \dot{v}_m^* \dot{v}_m^* + I_{mn} \dot{\phi}_m^* \dot{\phi}_n^* + J(\dot{\varphi}^*)^2 + a(\dot{\chi}^*)^2 + B_{mn} \dot{\tau}_m^* \dot{\tau}_n^* \right) dV = 0,\ t \geq 0. \tag{23}$$

Considering hypothesis 3 of the theorem, and taking into account (22), it is obtained that the internal energy U, which is written for the differences, becomes zero, such that from (23), we deduce:

$$\int_D \left[\varrho \dot{v}_m^* \dot{v}_m^* + I_{mn} \dot{\phi}_m^* \dot{\phi}_n^* + J(\dot{\varphi}^*)^2 + a(\dot{\chi}^*)^2 + B_{mn} \dot{\tau}_m^* \dot{\tau}_n^* \right] dV = 0. \tag{24}$$

Based on hypothesis 4 of Proposition, with respect to the tensor B_{mn} and taking into account hypothesis 1 of the Proposition with respect to the quantities ϱ, I_{mn}, J and a, from (24) we deduce that:

$$\dot{v}_m^* = 0, \dot{\phi}_m^* = 0, \dot{\varphi}^* = 0, \dot{\chi}^* = 0,\ \dot{\tau}_m^* = 0, \text{ on } D \times (0,\infty),$$

and, as a consequence, based on (20), we obtain:

$$v_m^* = 0, \phi_m^* = 0, \varphi^* = 0, \chi^* = 0,\ \tau_m^* = 0, \text{ on } D \times (0,\infty),$$

and this ends the proof of Proposition 2. □

Our main result will be regarding the existence of a solution of the mixed problem $\mathcal{P}$. For a start, it is considered that the boundary relations are homogeneous, namely,

$$v_m = \phi_m = \varphi = \chi = \tau_m = 0,\ \ on\ \ \partial D \times (0,\infty). \tag{25}$$

As we can easily see, both the conditions and the equations that define the above mixed problem $\mathcal{P}$ are very complicated. As such, a new procedure is necessary in order to demonstrate the existence of at least one solution of our problem in this situation. For this aim, our problem is associated with a problem of the Cauchy type, for an evolutionary equation on a suitable built space of Hilbert type.

Considering the known Hilbert spaces $W_0^{1,2}$ and L^2, a new Hilbert space $\mathcal{H}$ is introduced by means of the relation:

$$\mathcal{H} = \mathbf{W}_0^{1,2} \times \mathbf{L}^2 \times \mathbf{W}_0^{1,2} \times \mathbf{L}^2 \times W_0^{1,2} \times L^2 \times W_0^{1,2} \times L^2 \times \mathbf{W}_0^{1,2} \times \mathbf{L}^2,$$

in which the notation $\mathbf{W}_0^{1,2} = W_0^{1,2} \times W_0^{1,2} \times W_0^{1,2}$ were used. In short, we have: $\mathbf{W}_0^{1,2} = \left[W_0^{1,2}\right]^3$. Also, $\mathbf{L}^2 = [L^2]^3$.

We remember that $W^{k,p}(R)$, for $1 \geq p \geq \infty$ is defined as the subset of functions $f \in L^p(R)$ such that f and its weak derivatives up to order k have a finite L^p norm. For more notions regarding the Hilbert and Sobolev spaces, the basic book is recommended [23].

We can introduce a scalar product on the space $\mathcal{H}$, as follows:

$$\begin{aligned}
&\langle (v_m, U_m, \phi_m, \Psi_m, \varphi, \Phi, \chi, \mu, \tau_m, \nu_m), (v_m^*, U_m^*, \phi_m^*, \Psi_m^*, \varphi^*, \Phi^*, \chi^*, \mu^*, \tau_m^*, \nu_m^*) \rangle \\
&\quad = \frac{1}{2}\int_D (\varrho U_m U_m^* + I_{mn}\Psi_m \Psi_n^* + J\Phi\Phi^* + a\mu\mu^* + B_{mn}\nu_m \nu_n^*) dV \\
&\qquad + \frac{1}{2}\int_D \Big[A_{ijmn} e_{ij} e_{mn}^* + B_{ijmn}\left(e_{ij}\varepsilon_{mn}^* + e_{ij}^*\varepsilon_{mn}\right) + a_{ij}\left(e_{ij}\varphi^* + e_{ij}^*\varphi\right) \\
&\qquad\quad + D_{ijmn}\left(e_{ij}\tau_{m,n}^* + e_{ij}^*\tau_{m,n}\right) + C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}^* + b_{ij}\left(\varepsilon_{ij}\varphi^* + \varepsilon_{ij}^*\varphi\right) \\
&\qquad\quad + E_{ijmn}\left(\varepsilon_{ij}\tau_{m,n}^* + \varepsilon_{ij}^*\tau_{m,n}\right) + A_{ij}\varphi_{,i}\varphi_{,j}^* + H_{ij}\left(\varphi_{,i}\chi_{,j}^* + \varphi_{,i}^*\chi_{,j}\right) \\
&\qquad\quad + \zeta\varphi\varphi^* + F_{ij}\left(\tau_{i,j}\varphi^* + \tau_{i,j}^*\varphi\right) + K_{ij}\chi_{,i}\chi_{,j}^* + G_{ijmn}\tau_{m,n}\tau_{i,j}^* \Big] dV.
\end{aligned} \tag{26}$$

Of course, the product scalar (26) will induce a specific norm, and it is not difficult to prove the equivalence between this norm and the initial norm on space $\mathcal{H}$, which is also a Hilbert space.

Now, inspired by the equations from (10), we consider the operators:

$$\begin{aligned}
&A_i^1\mathbf{v} = \frac{1}{\varrho}A_{ijmn}v_{m,nj},\ A_i^2\boldsymbol{\phi} = \frac{1}{\varrho}\Big[A_{ijmn}\epsilon_{mnk}\phi_{k,j} + B_{ijmn}\phi_{n,mj}\Big],\ B_i^1\varphi = \frac{1}{\varrho}a_{ij}\varphi_{,j}, \\
&C_i^1\mu = -\frac{1}{\varrho}\alpha_{ij}\varepsilon_{,j},\ D_i^1\boldsymbol{\tau} = \frac{1}{\varrho}D_{ijmn}\tau_{m,nj},\ A_i^3\mathbf{u} = \frac{1}{I_{ij}}\left(B_{ijmn}v_{m,nj} + \epsilon_{ijk}A_{jkmn}v_{m,n}\right), \\
&A_s^4\boldsymbol{\phi} = W_{si}\left[A_{ijmn}\epsilon_{jmn}\phi_j + B_{ijmn}\epsilon_{jmn}\phi_{n,m} + C_{ijmn}\phi_{n,mj}\right], B_s^2\varphi = W_{si}\left(b_{ij}\varphi_{,j} + a_{jk}\epsilon_{ijk}\varphi\right) \\
&C_s^2\mu = -W_{si}\left(\beta_{ij}\mu_{,j} + \epsilon_{ijk}\alpha_{jk}\mu\right),\ D_s^2\boldsymbol{\tau} = W_{si}\left(E_{ijmn}\tau_{m,nj} + \epsilon_{ijk}D_{jkmn}\tau_{m,n}\right), \\
&E\varphi = \frac{1}{J}(A_{ij}\varphi_{,ij} - \zeta\varphi),\ F\boldsymbol{\nu} = -\frac{1}{J}d_{ij}\nu_{j,i},\ G\chi = \frac{1}{J}H_{ij}\chi_{,ij},\ H\mathbf{u} = -\frac{1}{J}a_{ij}v_{j,i}, \\
&K\boldsymbol{\phi} = -\frac{1}{J}\left(a_{ij}\epsilon_{ijk}\phi_k + b_{ij}\phi_{j,i}\right),\ L\mu = \frac{1}{J}\kappa\mu,\ M\boldsymbol{\tau} = -\frac{1}{J}F_{ij}\tau_{i,j},\ N\chi = \frac{1}{a}K_{ij}\chi_{,ij}, \\
&P\varphi = \frac{1}{a}H_{ij}\varphi_{,ij}, Q\boldsymbol{\nu} = -\frac{1}{a}D_{ij}\nu_{j,i}, R^1\mathbf{v} = -\frac{1}{a}\alpha_{ij}v_{i,j}, R^2\boldsymbol{\Psi} = -\frac{1}{a}\left(\alpha_{ij}\epsilon_{ijk}\Psi_k + \beta_{ij}\Psi_{j,i}\right) \\
&S\Phi = -\frac{1}{a}\kappa\Phi,\ A_s^5\mathbf{u} = \Gamma_{si}D_{ijmn}v_{m,nj},\ A_s^6\boldsymbol{\phi} = \Gamma_{si}\left(D_{ijmn}\epsilon_{mnk}\phi_{k,j} + E_{ijmn}\phi_{n,mj}\right), \\
&W_s\varphi = \Gamma_{si}F_{ij}\varphi_{,j},\ X_s\mu = -\Gamma_{si}D_{ij}\mu_{,j},\ Y_s\boldsymbol{\tau} = \Gamma_{si}G_{ijmn}\tau_{m,nj},\ Z_s\Phi = -\Gamma_{si}d_{ji}\Phi_{,j},
\end{aligned} \tag{27}$$

where the matrices Γ_{si} and W_{si} satisfy the equations :

$$\Gamma_{si}B_{ir} = \delta_{sr},\ W_{si}J_{ir} = \delta_{sr}.$$

Let us introduce the matrix operator Γ, whose elements are even the operators considered in (27). In this way, the mixed problem $\mathcal{P}$ is equivalent with an abstract problem of the Cauchy type associated to an evolution equation, namely

$$\begin{aligned}
\frac{d\mathcal{V}}{dt} &= \Gamma\mathcal{V}(t) + \mathcal{F}(t), \\
\mathcal{V}(0) &= \mathcal{V}_0.
\end{aligned} \tag{28}$$

The set $D(\Gamma)$ is the domain for the operator Γ and to facilitate the next theoretical, we take it of the form:

$$\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(W_0^{1,2}\cap W^{2,2}\right)\times W_0^{1,2}\times\left(W_0^{1,2}\cap W^{2,2}\right)\times W_0^{1,2}$$
$$\times\left(W_0^{1,2}\cap W^{2,2}\right)\times W_0^{1,2}\times\left(W_0^{1,2}\cap W^{2,2}\right)\times W_0^{1,2}\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}.$$

In addition, the matrix of unknown function $\mathcal{V}$, the initial values $\mathcal{V}_0$, and the matrix of loads $\mathcal{F}$ are introduced by means of the following relations:

$$\mathcal{V} = \left(v_m, U_m, \phi_m, \Psi_m,\ \varphi, \Phi, \chi, \mu, \tau_m, \nu_m \right),$$
$$\mathcal{V}_0 = \left(v_m^0, U_m^0, \phi_m^0, \Psi_m^0,\ \varphi^0,\ \Phi^0, \chi^0,\ \mu^0, \tau_m^0, \nu_m^0\right),$$
$$\mathcal{F} = (\mathbf{0}, F_m, \mathbf{0}, G_m,\ 0,\ L,\ 0,\ s,\ \mathbf{0}, Q_m).$$

The result, which is proved in the following theorem, highlights a quality of the matrix operator Γ that is necessary for demonstrating the theorem of the existence of a solution of the problem (28).

Theorem 1. *Suppose that the next hypotheses are satisfied:*

1. $\varrho > 0$, $I_{mn} > 0$, $J > 0$, $a > 0$;
2. *Take place the symmetry relations (7);*
3. *The internal energy U defined in (15) is a positively definite quadratic form;*
4. *B_{ij} is a positively definite tensor.*

Then, we have fulfilled the following estimate:

$$\langle \Gamma\mathcal{V}, \mathcal{V}\rangle \leq 0,\ \forall\, \mathcal{U} \in D(\Gamma). \tag{29}$$

that is, operator Γ has the property of being dissipative.

Proof. An element $\mathcal{V}$ is taken and arbitrary chosen in the domain of the definition of the operator Γ. Considering the scalar product from (26) and taking into account the definition of operators from (27), we obtain:

$$\begin{aligned}\langle \Gamma\mathcal{V}, \mathcal{V}\rangle = &-\int_{\partial D}(\tau_{ji}U_i + \sigma_{ij}\Psi_i + \lambda_j\Phi + S_j\mu + \Lambda_{ij}\nu_i)n_j dA\\ &+\int_D\Big[A_{ijmn}\epsilon_{ij}e^*_{mn} + B_{ijmn}\left(e_{ij}\varepsilon^*_{mn} + e^*_{ij}\varepsilon_{mn}\right) + a_{ij}\left(e_{ij}\varphi^* + e^*_{ij}\varphi\right)\\ &+D_{ijmn}\left(e_{ij}\tau^*_{m,n} + e^*_{ij}\tau_{m,n}\right) + C_{ijmn}\varepsilon_{ij}\varepsilon^*_{mn} + b_{ij}\left(\varepsilon_{ij}\varphi^* + \varepsilon^*_{ij}\varphi\right)\\ &+E_{ijmn}\left(\varepsilon_{ij}\tau^*_{m,n} + \varepsilon^*_{ij}\tau_{m,n}\right) + A_{ij}\varphi_{,i}\varphi^*_{,j} + H_{ij}\left(\varphi_{,i}\chi^*_{,j} + \varphi^*_{,i}\chi_{,j}\right)\\ &+\zeta\varphi\varphi^* + F_{ij}\left(\tau_{i,j}\varphi^* + \tau^*_{i,j}\varphi\right) + K_{ij}\chi_{,i}\chi^*_{,j} + G_{ijmn}\tau_{m,n}\tau^*_{i,j}\Big]dV.\end{aligned} \tag{30}$$

The integrant of the second integral of (30) is an expression in the form of a square form, with respect to to the elements w and w^* of the form $w = (v_m, \phi_m, \varphi, \chi, \tau_m)$ and $w^* = (U_m, \Psi_m, \Phi, \mu, \nu_m)$. In other words, the respective integral is of the following form:

$$\int_D W(w,^*)dV = \int_D W((v_m, \phi_m, \varphi, \chi, \tau_m), (U_m, \Psi_m, \Phi, \mu, \nu_m))dV.$$

Now, the theorem of the divergence is applied in the first integral of identity (30) so that if we take into account the previous observation, the following relation results:

$$\begin{aligned}\langle \Gamma\mathcal{V}, \mathcal{V}\rangle = &-\int_D(\tau_{nm}U_{m,n} + \sigma_{nm}\Psi_{m,n} + \lambda_n\Phi_{,n} + S_n\mu_{,n} + \Lambda_{mn}\nu_{m,n})dV\\ &+\int_D W((v_m, \phi_m, \varphi, \chi, \tau_m), (U_m, \Psi_m, \Phi, \mu, \nu_m))dV = 0,\end{aligned}$$

and this ends the proof of Theorem 1. □

In the next theorem, a new property of Γ is proven, namely that the operator Γ meets the condition of range. This property is important for evaluating the solution of the problem (28).

Theorem 2. *If all the conditions of Theorem 1 are satisfied, then the operator Γ is a subjective one.*

Proof. Any element $\mathcal{V}^*$ of the Hilbert space $\mathcal{H}$ is of the form

$$\mathcal{V}^* = (v^*_m,\ U^*_m,\ \phi^*_m,\ \Psi^*_m,\ \varphi^*,\ \Phi^*,\ \chi^*,\ \mu^*,\ \tau^*_i,\ \nu^*_m).$$

If such an element $\mathcal{V}^*$ is arbitrarily taken, then the statement of the theorem can be reformulated in this form: there is at least one solution $\mathcal{V} \in D(\Gamma)$ of the equation $\Gamma\mathcal{V} = \mathcal{V}^*$.

To this aim, taking into account the operators (27), the following vector notations are introduced:

$$\begin{gathered}\mathbf{A}^1 = (A^1_m),\ \mathbf{A}^2 = (A^2_m),\ \mathbf{A}^3 = (A^3_m),\ \mathbf{A}^4 = (A^4_s),\ \mathbf{A}^5 = (A^5_s),\ \mathbf{A}^6 = (A^6_s),\\ \mathbf{B}^1 = (B^1_m),\ \mathbf{B}^2 = (B^2_s),\ \mathbf{C}^1 = (C^1_m),\ \mathbf{C}^2 = (C^2_s),\ \mathbf{D}^1 = (D^1_m),\ \mathbf{D}^2 = (D^2_s),\\ \mathbf{W} = (W_s),\ \mathbf{X} = (X_s),\ \mathbf{Y} = (Y_s),\ \mathbf{Z} = (Z_s).\end{gathered} \tag{31}$$

By using the notations (31) and considering the operators defined in (27), the differential Equation (10) receives the following form:

$$\begin{aligned}&\mathbf{U} = \mathbf{v}^*,\\ &\qquad \mathbf{A}^1\mathbf{v} + \mathbf{A}^2\boldsymbol{\phi} + \mathbf{B}^1\varphi + \mathbf{C}^1\mu + \mathbf{D}^1\boldsymbol{\tau} = \mathbf{U}^*,\\ &\boldsymbol{\Psi} = \boldsymbol{\varphi}^*,\\ &\qquad \mathbf{A}^3\mathbf{v} + \mathbf{A}^4\boldsymbol{\phi} + \mathbf{B}^2\varphi + \mathbf{C}^2\mu + \mathbf{D}^2\boldsymbol{\tau} = \boldsymbol{\Psi}^*,\\ &\Phi = \varphi^*,\\ &\qquad H\mathbf{v} + E\varphi + G\chi + L\mu + M\boldsymbol{\tau} + F\boldsymbol{\nu} = \Phi^*,\\ &\mu = \chi^*,\\ &\qquad R\mathbf{U} + P\varphi + S\Phi + N\chi + Q\boldsymbol{\nu} = \mu^*,\\ &\boldsymbol{\nu} = \boldsymbol{\tau}^*,\\ &\qquad \mathbf{A}^5\mathbf{v} + \mathbf{A}^6\boldsymbol{\phi} + \mathbf{W}\varphi + \mathbf{Z}\Phi + \mathbf{X}\mu + \mathbf{Y}\boldsymbol{\tau} = \boldsymbol{\nu}^*.\end{aligned} \tag{32}$$

We go through another stage so that from the above system, (32) obtains new equations, whose basic constitutive functions are principal unknowns functions: $(\mathbf{v}, \boldsymbol{\varphi}, \phi, \chi, \boldsymbol{\tau})$. The other variables, which are automatically secondary, will be passed to the right-hand member with the "free term" status. Thus, the new system of equations is deduced, which has the following form:

$$\begin{aligned}&\mathbf{A}^1\mathbf{v} + \mathbf{A}^2\boldsymbol{\phi} + \mathbf{B}^1\varphi + \mathbf{D}^1\boldsymbol{\tau} = \mathbf{U}^* - \mathbf{C}^1\chi^*,\\ &\mathbf{A}^3\mathbf{v} + \mathbf{A}^4\boldsymbol{\phi} + \mathbf{B}^2\varphi + \mathbf{D}^2\boldsymbol{\tau} = \boldsymbol{\Psi}^* - \mathbf{C}^2\chi^*,\\ &\quad H\mathbf{v} + E\varphi + G\chi + M\boldsymbol{\tau} = \Phi^* - L\chi^* - F\boldsymbol{\tau}^*,\\ &\quad P\varphi + N\chi = \mu^* - R\mathbf{v}^* - S\varphi^* - Q\boldsymbol{\tau}^*,\\ &\mathbf{A}^5\mathbf{v} + \mathbf{A}^6\boldsymbol{\phi} + \mathbf{W}\varphi + \mathbf{Y}\boldsymbol{\tau} = \boldsymbol{\nu}^* - \mathbf{Z}\varphi^* - \mathbf{X}\chi^*.\end{aligned} \tag{33}$$

To simplify the writing, the new variables $\tilde{\mathbf{v}}$, $\tilde{\boldsymbol{\phi}}$, $\tilde{\varphi}$, $\tilde{\chi}$ and $\tilde{\boldsymbol{\tau}}$ are introduced, defined by:

$$
\begin{aligned}
\tilde{\mathbf{v}} &= \mathbf{A}^1\mathbf{v} + \mathbf{A}^2\boldsymbol{\phi} + \mathbf{B}^1\varphi + \mathbf{D}^1\boldsymbol{\tau},\\
\tilde{\boldsymbol{\phi}} &= \mathbf{A}^3\mathbf{v} + \mathbf{A}^4\boldsymbol{\phi} + \mathbf{B}^2\varphi + \mathbf{D}^2\boldsymbol{\tau},\\
\tilde{\varphi} &= H\mathbf{v} + E\varphi + G\chi + M\boldsymbol{\tau},\\
\tilde{\chi} &= P\varphi + N\chi,\\
\tilde{\boldsymbol{\tau}} &= \mathbf{A}^5\mathbf{v} + \mathbf{A}^6\boldsymbol{\phi} + \mathbf{W}\varphi + \mathbf{Y}\boldsymbol{\tau}.
\end{aligned} \tag{34}
$$

In this way, a new bilinear form defined on $W_0^{1,2}$ is obtained, by means of the following scalar product: such that the scalar product $\langle(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\phi}}, \tilde{\varphi}, \tilde{\chi}, \tilde{\boldsymbol{\tau}}), (\mathbf{v}, \boldsymbol{\phi}, \varphi, \chi, \boldsymbol{\tau})\rangle$. Furthermore, after simple calculations, the following scalar product is obtained:

$$
\begin{aligned}
&\langle(\mathbf{v}, \boldsymbol{\phi}, \varphi, \chi, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\phi}, \varphi, \chi, \boldsymbol{\tau})\rangle\\
&= \int_D \Big[A_{ijmn}e_{ij}e_{mn} + 2B_{ijmn}e_{ij}\varepsilon_{mn} + 2a_{ij}e_{ij}\varphi + 2D_{ijmn}e_{ij}\tau_{m,n}\\
&\quad + C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2b_{ij}\varepsilon_{ij}\varphi + 2E_{ijmn}\varepsilon_{ij}\tau_{m,n} + A_{ij}\varphi_{,i}\varphi_{,j}\\
&\quad + 2H_{ij}\varphi_{,i}\chi_{,j} + \zeta\varphi^2 + 2F_{ij}\tau_{i,j}\varphi + K_{ij}\chi_{,i}\chi_{,j} + G_{ijmn}\tau_{m,n}\tau_{i,j}\Big]dV.
\end{aligned} \tag{35}
$$

Taking into account the hypotheses of the theorem, from the above relation, the coerciveness of this expression is deduced in all points of the Sobolev space $W_0^{1,2}$.

In addition, it is not difficult to show that the functions from the right side of the system of Equation (33), i.e.,

$$
\mathbf{V}^* - \mathbf{C}^1\chi^*, \boldsymbol{\Psi}^* - \mathbf{C}^2\chi^*, \Phi^* - L\chi^* - F\boldsymbol{\tau}^*, \mu^* - R\mathbf{u}^* - S\varphi^* - Q\boldsymbol{\tau}^*, \boldsymbol{\nu}^* - \mathbf{Z}\varphi^* - \mathbf{X}\chi^*,
$$

are elements from the Sobolev space $W^{1,2}$. Based on the last observation, it results that all the conditions are satisfied in order to be able to use the theorem of Lax-Milgram, based on which the elements of vector $\mathcal{U} = (\mathbf{v}, \boldsymbol{\phi}, \varphi, \chi, \boldsymbol{\tau})$ form a solution of our above system (33). As a consequence, it is deduced that the system (32) has a solution.
This concludes the proof of Theorem 2. □

Our results regarding the operator Γ, from Theorems 1 and 2, are exactly the conditions of the corollary Lumer-Phillips's, obtained by the Hille-Yosida theorem (Pazy, [24]). Based on this, our first main result is obtained, proven in the next theorem.

Theorem 3. *Assume that all the conditions of Theorem 1 are satisfied. Then, the matrix operator Γ generates a semi-group of contraction type operators, all defined on Hilbert space $\mathcal{H}$.*

Furthermore, by using the same Lumer-Phillips corollary, the second main result of our paper is deduced, which is the one that ensures the uniqueness of the solution.

Theorem 4. *Assume that all the conditions of Theorem 1 are satisfied. Furthermore, we suppose that the loads $F_m, G_m, L, s, Q_m \in C^1([0,\infty), L^2) \cap C^0([0,\infty), W_0^{1,2})$ and the initial data $\mathcal{V}_0$ are included in the domain of matrix operator Γ.*

In these conditions, our Cauchy problem (28) has only one solution, namely $\mathcal{V}(t) \in C^1([0,\infty), \mathcal{H})$.

In the third important result of our study, it is shown that the solution of the Cauchy problem (28) depends continuously on the initial data and charges. Based on the above considerations, it is deduced that the solution of our mixed problem $\mathcal{P}$ depends continuously on the initial data and charges. As with the previous two results, we use Lumer-Phillips's corollary again.

Theorem 5. *If all the conditions of Theorem 1 are satisfied, then the matrix solution* $\mathcal{V} = (\mathbf{v}, \boldsymbol{\phi}, \varphi, \chi, \boldsymbol{\tau})$ *of the problem of Cauchy type (28) continuously depends on charges* F_m, G_m, L, s, Q_m *and the initial data* $\mathcal{V}_0$*, i.e.,*

$$|\mathcal{V}(t)| \leq \int_0^t \|(F_m,\ G_m,\ L,\ s,\ Q_m)\| ds + |\mathcal{V}_0|.$$

4. Conclusions

For a better description of the behaviour of many materials, some suggestions that have a great impact on the development of these materials have been made, e.g., the inner structure of the media and the fact that the microparticles have micro temperatures.

In classical theories, the fact that the reaction of a body to some external actions is influenced by the intimate structure of that body and microtemperatures is ignored [25]. Consequently, many more unknown functions have appeared, as the system of differential equations includes many more relations, which has increased the number of the initial values and of the boundary relations. Additionally, their complexity is much higher. That is why we turned to results from the theory of contraction semigroups and due to the elegance of this theory, the above mentioned complications failed to affect the main results regarding the mixed problem from the theory of Cosserat thermoelastic media with inner structure and microtemperatures. Thus, we could demonstrate both the existence and uniqueness results and also that the solutions continuously depend on the charges and the initial values.

It is worth noting that, from what we have studied, neither on the internet nor in the databases, that we did not find any qualitative issues addressed regarding the mixed problem in the context of the theory of the thermoelasticity of Cosserat environments, in which the contribution of inner structure and microtemperatures are taken into account.

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