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Coincidence Points for Mappings in Metric Spaces Satisfying Weak Commuting Conditions

Salvatore Sessa [1,*] and Mohamed Akkouchi [2]

[1] Dipartimento di Architettura, Universita degli Studi di Napoli Federico II, Via Toledo 402, 80134 Napoli, Italy
[2] Department of Mathematics, Faculy of Sciences-Semlalia, University Cadi Ayyad, Av. Prince My. Abdellah, BP 2390, Marrakesh (40.000-Marrakech), Morocco; akkm555@yahoo.fr
* Correspondence: sessa@unina.it

Abstract: In this note, we prove some results of elementary fixed point theory for mappings defined in metric spaces satisfying conditions of weak commutativity. Suitable examples are proven as well.

Keywords: weak commutativity; elementary fixed point theory

1. Introduction

Throughout the paper, we always consider selfmaps f, g of the interval $[0,1]$ in its natural topology. We denote by $C(f,g)$ the set of coincidence points, i.e., the set given by

$$C(f,g) := \{x \in [0,1] : fx = gx\}.$$

The purpose of this paper is to investigate conditions ensuring the existence of coincidence points.

Suppose that f, g are satisfying the following condition:

$$f(0) < g(0), \quad \text{and} \quad f(1) > g(1), \tag{$*$}$$

without being not necessarily commuting. It is well known that a simple application of Bolzano's theorem assures that the set $\{x \in [0,1] : fx = gx\}$ is not empty.

By omitting the condition $(*)$ but assuming that f and g commute, Jungck proved in 1966 [1] that they have coincidence points, i.e., that the set $C(f,g)$ is not empty.

Sessa [2] introduced, in the Euclidean metric, the concept of weakly commuting maps f, g as a generalization of commuting maps in the following way:

$\mid fg(x) - gf(x) \mid \leq \mid f(x) - g(x) \mid$, for any $x \in [0,1]$.

The literature is full of examples of selfmaps of $[0,1]$ (more in general, in the context of metric spaces and related generalizations, e.g., [3,4]) of selfmaps, not necessarily continuous, which are weakly commuting, but not commuting.

An extension of another famous theorem of Jungck [5] to weakly commuting selfmaps of a complete metric space is a well-known result, widely generalized to weak compatibility selfmaps (there exist various definitions of compatible selfmaps (e.g., see [3]) which here not recalled and compared).

To extend the above theorem of Jungck to the weak commuting selfmaps f, g of $[0,1]$ which are continuous is a fallacious operation, as shown by the following trivial constant selfmaps of $[0,1]$ defined via $fx = a$ and $gx = b$, $a \neq b$, $0 \leq a, b \leq 1$, for every $x \in [0,1]$. Clearly, f and g are not commuting, but they are weakly commuting, for which the set $C(f,g)$ is empty. However, we are able to show that $C(f,g)$ is not empty if we impose some suitable conditions (in our opinion, easy to verify in many examples).

For brevity, from now on, we put $|x - y| = d(x,y)$ for any $x, y \in [0,1]$.

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2. Results

We start with the following theorem:

Theorem 1. *Let f,g be two continuous selfmaps of $[0,1]$ such that*

(i) $d(fgx, gfx) \leq k \max\{d(fx, gfx), d(fx, gx), d(gx, fgx)\}$ for any $x \in [0,1]$, where k is a constant with $0 < k < 1$.

(ii) $gfx \geq gx$ implies $gfx \geq fgx$ for any $x \in [0,1]$.

Then, the set $C(f,g)$ is not empty.

Proof. Following the proof of Jungck [1], assume that the set $C(f,g) = \{x \in [0,1] : fx = gx\}$ is empty, and given the continuity of f and g, without loss of generality, we can consider that $fx > gx$ for any $x \in [0,1]$.

Certainly, the set $B(g) := \{x \in [0,1] : gx \geq x\}$ is not empty because $g0 \geq 0$. As $B(g)$ is closed (hence compact), there exists a maximum point $c \in B(g)$, such that $gc = c$, and so $fc > gc = c$.

By (i), we have

$$d(fc, gfc) \leq k \max\{d(fc, gfc), d(fc, c)\},$$

which implies $d(fc, gfc) \leq k\, d(fc, c)$.

Now, if $gfc \geq fc$, then fc should be in $B(g)$, and we should have $c < fc \leq c$, a contradiction. So $fc > gfc$, and by condition (i), we get

$$fc - gfc \leq k(fc - c) < fc - c,$$

so $gfc > c = gc$. In virtue of (ii) applied for $x = c$, then $gfc \geq fgc = fc$, and hence, we should deduce again that fc should be in $B(g)$, again a contradiction. Then, $C(f,g)$ is not empty. This ends the proof $\square$

Remark 1. *Clearly, for reasons of symmetry, a similar theorem holds if one assumes (i) and the following condition:*

(ii)′ $fgx \geq fx$ implies $fgx \geq gfx$, for any $x \in [0,1]$.

Remark 2. *Obviously, the above theorem of Jungck is generalized from Theorem 2.1 being $d(fgx, gfx) = 0$ for any $x \in [0,1]$, and trivially, the condition (ii) or (ii) is satisfied. Indeed, the following example shows that Theorem 2.1 holds, but not the above Jungck's theorem.*

Example 1. *Let $gx = \frac{x}{x+2}$ and $fx = \frac{x}{2}$ for any $x \in [0,1]$.*
(a) Clearly, f and g do not commute because

$$gf0 = 0 = fg0 \quad \text{and} \quad gfx = \frac{x}{x+4} > fgx = \frac{x}{4+2x}, \quad \text{for all} \quad x \in (0,1],$$

(b) f and g verify the inequality (i) with the constant $k := \frac{1}{4}$.

Indeed, we have $d(fgx, gfx) = \frac{x^2}{(4+x)(4+2x)}$ and $d(fx, gfx) = \frac{x}{2} - \frac{x}{x+4} = \frac{x^2+2x}{4+x}$.

Therefore, for any $x \in [0,1]$, we obtain

$$\begin{aligned} d(fgx, gfx) &\leq \frac{x^2+2x}{(4+2x)(4+x)} \\ &\leq \frac{1}{4}\, d(fx, gfx) \\ &\leq \frac{1}{4} \max\{d(fx, gfx), d(fx, gx), d(gx, fgx)\}. \end{aligned}$$

(c) The property (ii) is satisfied for the pair of functions $\{f, g\}$, since the inequality $gfx = x/(x+4) \geq x/(x+2) = gx$ holds only for $x = 0$, and at this point, we have $gf(0) = 0 \geq 0 = fg(0)$.

Then, all the assumptions of Theorem 2.1 are verified for the pair $\{f, g\}$, and indeed, we have that $C(f, g) = \{0\}$.

(d) We note that in this example, the condition (ii) is also satisfied by the pair of functions $\{f, g\}$. Indeed, the inequality $fgx = x/(4 + 2x) \geq fx = \frac{x}{2}$ holds only for $x = 0$, and at this point, we have $fg(0) = 0 \geq 0 = gf(0)$.

We recall that in 1982, Sessa introduced the concept of weak commutativity relaxing the commutativity condition of mappings.

Definition 1 (S. Sessa [2])**.** *Two selfmappings f and g of a metric space (X, d) are called weakly commuting iff $d(fgx, gfx) \leq d(fx, gx)$ for all x in X.*

There are many kinds of generalizations of the above concept. The reader is invited to consult the references for more information on them, and to see many comparison results between these generalizations.

By returning to weakly commuting mappings in a metric space, now we have the following result, for not necessarily continuous selfmaps, which is inspired from a theorem given in ([6], p. 41):

Theorem 2. *Let f, g be two selfmaps of a metric space (X, d) and h be a surjective isometry of X into the metric space (Y, d').*

Then, the following assertions are equivalent.

(a) The maps f and g are weakly commuting in X and they have a common fixed point in X.

(b) The maps hfh^{-1} and hgh^{-1} are weakly commuting in Y (with respect to the metric d') and they have a common fixed point in Y.

Proof. (i) Let z in X be such that $fz = gz = z$ and hence $hz = y$ for some unique y in Y and thus $z = fz = fh^{-1}y = gz = gh^{-1}y$, which implies

$$y = hz = hfz = hfh^{-1}y = hgz = hgh^{-1}y,$$

that is y is a common fixed point of the selfmaps hfh^{-1} and hgh^{-1} in Y.

Viceversa, let y in Y be such that $y = hfh^{-1}y = hgh^{-1}y$, which implies, by setting $z = h^{-1}y$ for some unique z in X, that

$$z = h^{-1}y = h^{-1}hfh^{-1}y = h^{-1}hgh^{-1}y = fh^{-1}y = fz = gh^{-1}y = gz.$$

Hence, z is a common fixed point in X for the selfmaps f, g on X.

Now, we suppose that f and g are weakly commuting in X. We have for any x in X the following

$$\begin{aligned} d'(hfh^{-1}(hgh^{-1}x), hgh^{-1}(hfh^{-1}x)) &= d'(hf(h^{-1}h)gh - 1x, hg(h^{-1}h)fh^{-1}x) \\ &= d'(hfgh^{-1}x, hgfh^{-1}x) \\ &= d(fgh^{-1}x, gfh^{-1}x) \\ &\leq d(fh^{-1}x, gh^{-1}x) = d'(hfh^{-1}x, hgh^{-1}x). \end{aligned}$$

Thus, the selfmaps hfh^{-1} and hgh^{-1} are weakly commuting in Y with respect to d'.

Viceversa, let hfh^{-1} and hgh^{-1} be weakly commuting in Y. Then, by setting $x = h^{-1}y$, we have

$$
\begin{aligned}
d(fx,gx) &= d(fh^{-1}y, gh^{-1}y) \\
&= d'(hfh^{-1}y, hgh^{-1}y) \\
&\geq d'(hfh^{-1}(hgh^{-1}y), hgh^{-1}(hfh^{-1}y)) \\
&= d'(hf(h^{-1}h)gh^{-1}x, hg(h^{-1}h)fh^{-1}x) \\
&= d'(hfgh^{-1}y, hgfh^{-1}y) \\
&= d(fgh^{-1}y, gfh^{-1}y) = d(fgx, gfx),
\end{aligned}
$$

which means that f and g are weakly commuting in X with respect to d. □

The following example is borrowed from [4]:

Example 2. *Let $(X,d) = (Y,d') = ([0,1],d)$ with Euclidean metric $d = d'$ and $fx = \frac{x+2}{3}$ and $gx = x^2$ for any x in X. Then we have that*

$$
\begin{aligned}
d(fgx, gfx) &= |(x+2)^2/9 - (x^2+2)/3| \\
&= |(x^2+4x+4-3x^2-6)/9| = |-2x^2+4x-2|/9 \\
&= 2|x-1|^2/9 \\
&\leq |x-1||3x+2|/3 = d(fx,gx),
\end{aligned}
$$

for any x in X, so f and g are weakly commuting. Assume that $hx = 1-x$ for any x in X, and then $h = h^{-1}$ is a surjective isometry. This implies that $hfh^{-1}x = \frac{x}{3}$ and $hgh^{-1}x = -x^2+2x$ for any x in X.

We observe that $h(1) = 0$, $f1 = g1 = 1$ and $hfh^{-1}(0) = 0 = hgh^{-1}(0)$. and $f1 = g1 = 1$. So, this example supports all the statements of the above theorem.

It is instructive also to give the following variant of Example 2.

Example 3. *Let $(X,d) = (Y,d') = (\mathbb{R},d)$ with Euclidean metric $d = d'$. Let h be the surjective isometry defined as $hx = x+k$ for any x in X, where k is a positive number, $fx = (x+2)/3$ and $gx = x^2$, for any real number x. As in Example 2, we have f, g are weakly commuting in the set $\mathbb{R}$ of all reals. We have $hfh^{-1}x = (x+2k+2)/3$ and $hgh^{-1}x = (x-k)^2+k$, for any real number x, thus $hfh^{-1}(1+k) = hgh^{-1}(1+k) = 1+k$.*

Let (X,d) be a metric space and (X^*, d^*, h) be the (see, e.g., T. B. Singh [7]) completion of Cauchy (up to isomorphisms) of (X,d). For each point x in X, we denote by x^* the set of all Cauchy sequences $\{x_n\}$ in X converging to x. We recall that the map $h : X \to X^*$ is defined for any $x \in X$ by $h(x) = x^*$.

The map h is an isometry of X into X^*, satisfying:

$$
d(x,y) = d^*(x^*,y^*) = \lim_{n\to\infty} d(x_n,y_n),
$$

for all points x, y in X, where as above, x^* (resp. y^*) is the set of all Cauchy sequences $\{x_n\}$ (resp. $\{y_n\}$) in X converging to x (resp. y).

The map h is surjective if, and only if, X is complete.

Before stating our last results, we need to recall some concepts.

In 2000, Sastri and Krishna Murthy [8] introduced the following notion:

Definition 2 ([8])**.** *Let (X,d) be a metric space and f and g be two self-mappings.*

A point $t \in X$ is said to be tangent to the pair (f,g), if there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$.

The pair (f,g) is called tangential if there exists a point t in X which is tangent to (f,g).

If the pair (f,g) is tangential, we shall denote by $\mathcal{T}(f,g)$ the set of tangent points to the pair (f,g).

In 2002 (two years later), Aamri and Moutawakil [9] rediscovered this notion and called it property (E.A).

Definition 3 ([9])**.** *Let (X,d) be a metric space and f and g be two self-mappings. The pair (f,g) satisfies the property (E.A), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.*

In 2011, M. Akkouchi [10] introduced the following concept.

Definition 4 ([10])**.** *Let (X,d) be a metric space and $f,g : X,\to X$ be two self-mappings. f and g are said to be weakly tangential mappings if there exists a sequence $\{x_n\}$ of points in X, such that $\lim_{n\to\infty} d(fx_n, gx_n) = 0$.*

After the recalls and notations above, now we formulate the following theorem:

Theorem 3. *Let (X,d) be a complete metric space and f,g be two continuous selfmaps of X, which are weakly commuting in X. We suppose that the pair (f,g) is tangential. Then,*

$\mathcal{T}(f,g) = C(f,g) = \{x \in X : fx = gx\}$. In particular, $C(f,g)$ is not empty.

Furthermore, for any $z \in X$, we have $z \in C(f,g)$ if, and only if, $z = h(z)$ is a common fixed point of hfh^{-1} and hg^{-1} in X^* (up to isomorphisms).*

Proof. (i) It is obvious that $C(f,g) \subset \mathcal{T}(f,g)$.

Conversely, let $t \in \mathcal{T}(f,g)$ and let $\{x_n\}$ be a sequence of points in X, such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$. By virtue of the continuity and weak commutativity of f,g, we have

$$d(ft, gt) = \lim_{n\to\infty} d(fg(x_n), gf(x_n)) \le \lim_{n\to\infty} d(f(x_n), g(x_n)) = 0.$$

Hence, $t \in C(f,g)$.

(ii) The second part comes from Theorem 2. □

For compact case, we have the following result.

Theorem 4. *Let (X,d) be a compact metric space and f,g be two continuous selfmaps of X which are weakly commuting in X. Then, the following assertions are equivalent:*

(i) The set $C(f,g)$ is not empty.

(ii) The pair (f,g) is tangential.

(iii) The maps f and g are weakly tangential.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

Suppose that (iii) holds true and let $\{x_n\}$ be a sequence of points in X, such that $\lim_{n\to\infty} d(fx_n, gx_n) = 0$. By the Bolzano–Weirstrass theorem, we can find a subsequence $\{x_{n_k}\}$ which converges to a point t in X. By using the continuity and weak commutativity of f,g, we have

$$d(ft, gt) = \lim_{k\to\infty} d(fg(x_{n_k}), gf(x_{n_k})) \le \lim_{n\to\infty} d(f(x_{n_k}), g(x_{n_k})) = 0.$$

Hence, $t \in C(f,g)$. This completes the proof. □

Example 4. *Let (X,d) be a subspace of a complete metric space (S,d) and $(\overline{X},d)$ be the closure of (X,d). Then, $(\overline{X},d)$ is complete because closed in S, and let $i : x \to i(x) = x$ be the canonical embedding of $\overline{X}$ into S. It is easily seen that the completion $(\overline{X},d,i)$ of (X,d) is isomorphic to the completion (X^*,d^*,h) of Cauchy of (X,d).*

Let f, g be two continuous selfmaps of X, which are weakly commuting on X. We suppose that f (resp. g) has a continuous extension denoted by $\overline{f}$ (resp. $\overline{g}$) to $\overline{X}$. For any $x \in \overline{X}$, it is well known that there exists x_n of points of X converging to x in (S, d). Then, by the definition of $\overline{f}$ and $\overline{g}$, we have $\overline{f}(x) = \lim_{n\to\infty} f(x_n)$ and $\overline{g}(x) = \lim_{n\to\infty} g(x_n)$.

Hence, the functions $\overline{f}, \overline{g}$ are continuous selfmaps of the complete metric space $(\overline{X}, d)$ which are weakly commuting in $(\overline{X}, d)$, since we have for any $x \in \overline{X}$:

$$\begin{aligned} d(\overline{f}\overline{g}x, \overline{g}\overline{f}x) &= d(\overline{f}\overline{g}(\lim_{n\to\infty} x_n), \overline{g}\overline{f}(\lim_{n\to\infty} x_n)) \\ &= \lim_{n\to\infty} d(\overline{f}\overline{g}(x_n), \overline{g}\overline{f}(x_n)) \\ &= \lim_{n\to\infty} d(fg(x_n), gf(x_n)) \\ &\leq \lim_{n\to\infty} d(g(x_n), f(x_n)) \\ &= d(\overline{g}x, \overline{f}x), \end{aligned}$$

because of the definition and continuity of $\overline{f}, \overline{g}$ and the continuity of d.

For instance, let $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open circle of radius 1 endowed with the Euclidean metric d, then $\overline{X} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. The metric space $(\overline{X}, d)$ is complete because it is a closed subspace of $(\mathbb{R}^2, d)$, which is complete. Let f and g be the selfmaps on X defined for all $(x, y) \in X$ by setting $f(x, y) = (0, x)$ and $g(x, y) = (0, x^2)$. Clearly, f and g are continuous on X and they are commuting. Their extensions $\overline{f}$ and $\overline{g}$ have the same property on $\overline{X}$. Of course, $(0, 0)$ and $(0, 1)$ are their two common fixed points of f and g (here $ifi^{-1} = f$ and $igi^{-1} = g$), hence $h((0, 0))$, which is equal to the set of all the Cauchy (or convergent) sequences of $\overline{X}$ with limits equal to $(0, 0)$ and $h((0, 1))$ which are equal to the set of all the Cauchy (or convergent) sequences of $\overline{X}$ with a limit equal to $(0, 1)$ are fixed points of the mappings $h\overline{f}h^{-1}$ and $h\overline{g}h^{-1}$, where $h : \overline{X} \to \overline{X}^*$ is the map defined for any $x \in \overline{X}$ by $h(x) = x^*$.

3. Conclusions

Our scope was to extend results already known for commutative selfmaps of the [0, 1] interval to a weakly commutative case also in abstract metric spaces, so enlarging the study in this setting. We point out that the conditions of weak commutativity used are symmetric, in accordance to the intents of this Special Issue. This paper is in this direction; for further information, we refer to the book [6], which, to the best of our knowledge, gives the idea of the actual art of the elementary fixed point theory. Such a theory has not yet received the necessary attention from the worldwide fixed point theorists community, although it has been going since for the last 60 and 70 years of the last century. Finally, we recommend reading the book [6], which has inspired us deeply, before conducting any further research.

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