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Chiral Chemical Potential and Magnetic Effects on QCD Matter in NJL Model with a Self-Consistent Method

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Abstract: The NJL model of one-flavor quark is employed to study the properties of QCD mater with finite temperature, external magnetic field, and chiral chemical potential. Through the mean-field approximation and a self-consistent method, a non-perturbative quark propagator is proposed to deduce the gap equations, and it can be proved that besides the classic vacuum condensate, there are non-zero statistical averages of a quark current and quark magnetic moment. Through a rigorous algebraic method, the quark current leads to a modified chiral magnetic effect. Through a numerical method, the quark magnetic moment is non-zero in the chiral breaking phase, and its relation with chiral chemical potential is studied.

Keywords: NJL model; magnetic field; chiral chemical potential; chiral magnetic effect

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1. Introduction

Studying the phase diagram and the phase transitions of QCD matter is important for the theories and experiments of high energy physics [1,2]. It is believed that at different temperatures and densities, the QCD matter is divided into three phases in the phase diagram [3–5]; they are the phase of quark-gluon plasma (QGP), the phase of hadrons, and the phase of color-superconductivity [6,7]. The order parameter to distinguish between QGP and hadrons is the vacuum condensate, which provides a dynamical mass to the quark. In the phase of QGP, the vacuum condensate is zero (while the chiral limit, the bare quark, is massless), while in the phase of hadrons, the condensate is non-zero; therefore, the phase of QGP is also known as the chiral symmetry restoring phase and the phase of hadrons is known as the chrial symmetry breaking phase.

The NJL model is an effective model to study the non-perturbative properties of QCD matter [8–12]. Through the mean-field approximation, this model could give a qualitative description of the vacuum condensate and the phase transitions. It also has a nice extendibility to study the QCD matter with diquark pairing and confinement [13–15], the color-superconductivity [16], and the QCD matter with external magnetic field. One of the situations to consider a magnetic field in QCD matter is the experiment of high energy particle collision: when the charged particles collide non-centrally in the accelerator, an extremely strong magnetic field is produced [17], and such a magnetic field impacts QCD matter significantly [16–25].

One of the most intriguing properties induced by a magnetic field in the QCD matter is the 'Chiral Magnetic Effect' (CME), which has been widely studied in recent decades [26–30]. This effect is highly relative to the magnetic field and chiral chemical potential. The importance of CME is that it leads to a local polarization effect in the product of high energy collisions [31–33] and produces a testable physical effect for experiments [34], which gives us the ability to verify the rationality of QCD theory and understand the evolution of QCD matter.

In our previous work [35], we proposed a self-consistent method to prove that, with external magnetic field and chemical potential, there is a non-zero axial vector current, which leads to the 'Chiral Separation Effect' and a non-zero quark magnetic moment. In this paper, we employ the same method to study the NJL model of QCD matter with an external magnetic field and chiral chemical potential and try to find out new properties of the classic CME or other effects. In Section 2, we give the basic formulae and equations, introduce the self-consistent method, and then prove how to get the proper quark propagator. In Section 3, we solve the gap equations through rigorous a algebraic method and numerical method. In Section 4, we come to the conclusion and discussions.

2. The Lagrangian and Gap Equations

2.1. Basic Formulae and the Inconsistency

Beginning with the one-flavor lagrangian of NJL model, the external magnetic field is included and treated as a classical electromagnetic field.

$$\mathcal{L} = \bar{\psi}\hat{\not{D}}\psi + \frac{G}{4N_c}(\bar{\psi}\psi)^2 + \mu_5\bar{\psi}\gamma^5\gamma^0\psi,$$
$$\hat{\not{D}} := \hat{\not{p}} - q\not{A}, \quad \hat{p}_0 := -\frac{\partial}{\partial\tau}, \quad \hat{p}_i := i\frac{\partial}{\partial x^i}, \quad (A_0, A_1, A_2, A_3) = (0, \tfrac{B}{2}x^2, -\tfrac{B}{2}x^1, 0). \tag{1}$$

In this lagrangian, the time variable is replaced by the virtual time variable $\tau \in [0, \beta]$ at finite temperature, and $\beta = \frac{1}{T}$ (T is the temperature). The factor q coupling with $\not{A}$ is an arbitrary electric charge. If the quark is u quark, then $q = \frac{2}{3}$e, and if it is d quark, then $q = -\frac{1}{3}$e. The coupling constant G comes from the approximation of the gluon propagator, and its value is determined in Section 3.2. The chiral chemical potential is μ_5; it is a bare quantity in the lagrangian and it will be renormalized in following studies.

Why do we study the 1 flavor NJL model rather than the 2-flavor or the $(2+1)$-flavor NJL model? There are two reasons. Firstly, some results depend on the sign of the electric charge of quark, so a 1-flavor NJL model with an arbitrary electric charge can clearly demonstrate the dependency; secondly, the gap equations and some results of the 1-flavor NJL model can be easily generalized to the cases of 2-flavor, because the fermion propagator of 2-flavor (u, d quarks) is the equivalent to a direct sum of u quark propagator and d quark propagator with the mean-field approximation [36]. Therefore, using the 1-flavor NJL model can keep the discussions and deductions concise. As for the $(2+1)$-flavor NJL model, it generally has a six-fermion interaction term in the lagrangian, and because of the $U_A(1)$ puzzle, the Wick-contraction approximation will turn this term into a self-energy mixed by the dynamical masses of quarks of different flavors [9]. Although the fermion propagator can still be separated into the sum of u, d, and s quark propagators, one should be aware that some of the formulae in this paper may not be easily generalized to the case of the $(2+1)$-flavor NJL model.

To deduce the gap equation of quark vacuum condensate, we employed the mean-field approximation to reorganize the four-fermion interaction term,

$$\frac{G}{4N_c}(\bar{\psi}\psi)^2 \approx -\sigma\bar{\psi}\psi - \frac{N_c}{G}\sigma^2, \qquad \sigma := -\frac{G}{2N_c}\langle\bar{\psi}\psi\rangle, \tag{2}$$

where $\langle\bar{\psi}\psi\rangle$ comes from the transformation $\bar{\psi}\psi = (\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle) + \langle\bar{\psi}\psi\rangle$, $\langle\bar{\psi}\psi\rangle$ is the mean-field, $(\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle)$ is the fluctuation, then the mean-field approximation removes $(\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle)^2$ in $(\bar{\psi}\psi)^2$ because it is a high-order correction of the fluctuation. With this approximation, we get a new lagrangian

$$\mathcal{L}' = \bar{\psi}(\hat{\not{D}} - \sigma + \mu_5\gamma^5\gamma^0)\psi - \frac{N_c}{G}\sigma^2, \tag{3}$$

and the propagator of quark is

$$\hat{S} = \frac{1}{\hat{\not{D}} - \sigma + \mu_5\gamma^5\gamma^0}. \tag{4}$$

Now we can use the new lagrangian Equation (3) and quark propagator Equation (4) to deduce the partition function and free energy,

$$\mathcal{Z} = \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \, \exp\left(\int_0^\beta \mathrm{d}\tau \int \mathrm{d}\vec{x}\,\mathcal{L}'\right) = e^{-\beta\mathcal{J}}, \qquad \mathcal{J} = \frac{N_\mathrm{c}}{G}\sigma^2 \int \mathrm{d}\vec{x} - TN_\mathrm{c}\,\mathrm{Tr}\ln\hat{S}^{-1}, \quad (5)$$

where the 'Tr' operator means taking a trace of the matrix of an operator in the Hilbert space and spinor space. The trace of color space is the factor N_c.

The gap equation of quark vacuum condensate is the equation to find the local minimums of free energy, and it is

$$\frac{\partial\mathcal{J}}{\partial\sigma} = 0 \Longrightarrow \frac{2}{G}\sigma \int \mathrm{d}\vec{x} = -T\,\mathrm{Tr}\,\hat{S}. \quad (6)$$

In order to get a more concrete form of $\mathrm{Tr}\,\hat{S}$, we introduce the eigenstate of $\hat{D}^2 = \hat{p}_0^2 - \hat{D}_\perp^2 - p_3^2$, where $\hat{D}_\perp^2 := \hat{D}_1^2 + \hat{D}_2^2$. The eigenstates is defined as $|\omega_n\rangle \otimes |m,a\rangle \otimes |p_3\rangle \equiv |n;m,a;p_3\rangle$. It has the properties as below:

$$\begin{aligned} \hat{p}_0|\omega_n\rangle &= i\omega_n|\omega_n\rangle, \quad \omega_n = (2n+1)\pi T, \quad n \in \mathbb{Z}; \\ \hat{p}_3|p_3\rangle = -p_3|p_3\rangle; &\qquad \hat{D}_\perp^2|m,a\rangle = (2m+1)|q|B|m,a\rangle, \quad m = 0,1,2,\ldots. \end{aligned} \quad (7)$$

More specific properties of the eigenstate are demonstrated in the Appendix A.

Notice that σ represents the statistical average of $\bar{\psi}\psi$. We can generalize this equation to the case of $\langle\bar{\psi}\Gamma\psi\rangle$, where $\Gamma \in \{I, \gamma^\mu, \gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu}|\mu,\nu = 0,1,2,3\}$,

$$\langle\bar{\psi}\Gamma\psi\rangle \int \mathrm{d}\vec{x} = TN_\mathrm{c}\,\mathrm{Tr}(\Gamma\hat{S}). \quad (8)$$

2.2. The Minimal Consistency Ansatz

Through the definition of the eigenstate Equation (7) and the statistical average Equation (8), one can verify that the averages, such as $\langle\bar{\psi}\gamma^3\psi\rangle$, $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$, and $\langle\bar{\psi}\sigma^{12}\psi\rangle$, are not constantly zero, and this causes an inconsistency, since the σ, which is proportional to $\langle\bar{\psi}\psi\rangle$, is included in the propagator $\hat{S}$, while other averages are not in $\hat{S}$. One can argue that these averages do not need to be in $\hat{S}$ like σ does; they are simply dynamical properties. Nut this is not true because, in the lagrangian of Equation (3), the four-fermion interaction term has the Fierz identity,

$$\mathscr{F}[(\bar{\psi}\psi)^2] = \frac{1}{4N_\mathrm{c}}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma^\mu\psi)^2 - (i\bar{\psi}\gamma^5\psi)^2 - (\bar{\psi}\gamma^5\gamma^\mu\psi)^2 + \frac{1}{2}(\bar{\psi}\sigma^{\mu\nu}\psi)^2 + \cdots]. \quad (9)$$

The identity suggests that $(\bar{\psi}\psi)^2$ and $\mathscr{F}[(\bar{\psi}\psi)^2]$ are dynamically equivalent; therefore, after introducing the mean-field approximation, the terms such as $\langle\bar{\psi}\Gamma\psi\rangle$ should also be parts of the self-energy of the non-perturbative quark propagator. For this reason, we should start the lagrangian in the form as below:

$$\mathcal{L} = \bar{\psi}\hat{\not{D}}\psi + \frac{G}{4N_\mathrm{c}}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma^\mu\psi)^2 - (i\bar{\psi}\gamma^5\psi)^2 - (\bar{\psi}\gamma^5\gamma^\mu\psi)^2 + \frac{1}{2}(\bar{\psi}\sigma^{\mu\nu}\psi)^2 + \cdots] + \mu_5\bar{\psi}\gamma^5\gamma^0\psi. \quad (10)$$

The mean-field approximation to this new lagrangian will then produce terms such as $\langle\bar{\psi}\Gamma\otimes\lambda_i\psi\rangle(\bar{\psi}\Gamma\otimes\lambda_i\psi)$, where λ_i stands for the Gell-Mann matrices of color space, and the non-perturbative quark propagator will take $\langle\bar{\psi}\Gamma\otimes\lambda_i\psi\rangle$ as parts of quark self-energy. However, an unwieldy problem is that this kind of propagator can hardly carry forward through algebraic calculus, let alone the numerical calculation. Of course one can try to prove that some of these averages in the non-perturbative propagator are zero and simplify the propagator, but it also needs to employ the algebraic method to transform $\mathrm{Tr}(\Gamma\hat{S})$ at first, which falls into an endless loop.

To break the loop, we propose an ansatz, the minimal consistency assumption: that the self-energy of the proper non-perturbative propagator of quark only has the least terms that will not cause inconsistency. In the case here, although Equation (4) is not the proper propagator, it reveals that the terms $\langle\bar{\psi}\gamma^3\psi\rangle$, $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$, and $\langle\bar{\psi}\sigma^{12}\psi\rangle$ are not zero, and we can next assume that they are parts of the self-energy besides σ. In order to bring in these averages, we need to apply the mean-field approximation to the terms such as $(\bar{\psi}\Gamma\psi)^2$. Firstly,

$$g(\bar{\psi}\Gamma\psi)^2 \approx -\xi\bar{\psi}\Gamma\psi - \frac{\xi^2}{4g}, \qquad \xi := -2g\langle\bar{\psi}\Gamma\psi\rangle. \tag{11}$$

The detailed deduction of this approximation can also be found in [35]. After the mean-field approximation, we assume only $\bar{\psi}\psi$, $\bar{\psi}\gamma^3\psi$, $\bar{\psi}\gamma^5\gamma^0\psi$, and $\bar{\psi}\sigma^{12}\psi$ have non-zero statistical averages in the lagrangian Equation (10), then the lagrangian of Equation (10) becomes

$$\begin{gathered}\mathcal{L}' = \bar{\psi}(\hat{\not{D}} - \Sigma + \mu_5\gamma^5\gamma^0)\psi + \mathcal{L}_{\mathrm{M}}, \quad \Sigma = \sigma + a\gamma^3 + b\gamma^5\gamma^0 + c\sigma^{12}, \quad \mathcal{L}_{\mathrm{M}} = -\frac{N_c}{G}(\sigma^2 + a^2 - b^2 + c^2), \\ \sigma := -\frac{G}{2N_c}\langle\bar{\psi}\psi\rangle, \quad a := -\frac{G}{2N_c}\langle\bar{\psi}\gamma^3\psi\rangle, \quad b := \frac{G}{2N_c}\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle, \quad c := -\frac{G}{2N_c}\langle\bar{\psi}\sigma^{12}\psi\rangle.\end{gathered} \tag{12}$$

Consequently, the gap equations become

$$\frac{\partial\mathcal{J}}{\partial\xi} = 0, \qquad \xi = \sigma, a, b, c. \tag{13}$$

One can easily prove that Equation (8) leads to the same gap equations like (13) does. According to the assumption above, the quark propagator is

$$\hat{S} = \frac{1}{\hat{\not{D}} - \Sigma + \mu_5\gamma^5\gamma^0}. \tag{14}$$

In order to calculate $\mathrm{Tr}(\Gamma\hat{S})$ in the gap equations, we wish to use the eigenstate of $\hat{D}^2$ to implement the deduction, so $\hat{S}$ should be transformed into

$$\begin{aligned}\hat{S} =& \{[\hat{\not{D}} + \sigma - a\gamma^3 + (b-\mu_5)\gamma^5\gamma^0 - c\sigma^{12}](\hat{\not{D}} - \Sigma + \mu_5\gamma^5\gamma^0)\}^{-1} \\ &\times [\hat{\not{D}} + \sigma - a\gamma^3 + (b-\mu_5)\gamma^5\gamma^0 - c\sigma^{12}] \\ =& \{\hat{p}_0^2 - \hat{D}_\perp^2 - (\hat{p}_3 - a)^2 - \sigma^2 + (b - \mu 5)^2 + c^2 + qB\sigma^{12} + 2\hat{p}_0(b-\mu_5)\gamma^5 \\ &+ 2[c(\hat{p}_3 - a) - \sigma(b-\mu_5)]\gamma^5\gamma^0 + 2c\hat{p}_0\gamma^5\gamma^3\}^{-1}[\hat{\not{D}} + \sigma - a\gamma^3 + (b-\mu_5)\gamma^5\gamma^0 - c\sigma^{12}],\end{aligned} \tag{15}$$

then $\mathrm{Tr}(\Gamma\hat{S})$ is rewritten as

$$\mathrm{Tr}(\Gamma\hat{S}) = \mathrm{tr}\left(\Gamma\sum_{n=-\infty}^{+\infty}\sum_{m=0}^{+\infty}\int\langle n;m,a;p_3|\hat{S}|n;m,a;p_3\rangle\,\mathrm{d}a\mathrm{d}p_3\right), \tag{16}$$

where 'tr' stands for the trace of spinor space.

With the discussions above and Equations (A14) and (A16) (Scheme II and Scheme IV), one can algebraically prove that except $\langle\bar{\psi}\psi\rangle$, $\langle\bar{\psi}\gamma^3\psi\rangle$, $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$, and $\langle\bar{\psi}\sigma^{12}\psi\rangle$, the other $\langle\bar{\psi}\Gamma\psi\rangle$ are zero. Therefore, we can claim that the assumption at the beginning is rational and the propagator Equation (14) is the proper propagator. That ends the demonstration of the minimal consistency ansatz.

3. Solve the Gap Equations

Even with the minimal consistency ansatz and some simplification schemes, the final gap equations are still too complicated to handle. However, QCD matter has two phases in the NJL model that are studied in this paper; the chiral symmetry breaking phase and the chiral symmetry restoring phase. Next, we will study the gap equations in different phases.

Notice that, in the proper propagator, μ_5 and b always combine in the form $(\mu_5 - b)$. Defining $\mu_5' := \mu_5 - b$, clearly μ_5' is the renormalized chiral chemical potential. In the following discussions, we treat μ_5' as a free parameter instead of μ_5; therefore, the gap equation of b can be ignored.

Similarly, in the proper propagator Equation (14), $\hat{p}_3$ and a are always combined in the form $(\hat{p}_3 - a)$, so for the convenience of following discussions, we define $p_3' := p_3 + a$. Beware that it is $(p_3 + a)$ rather than $(p_3 - a)$, because $(\hat{p}_3 - a)|p_3\rangle = -(p_3 + a)|p_3\rangle$.

3.1. The Chiral Symmetry Restoring Phase

The chiral symmetry restoring phase is characterized by $\sigma = 0$. With Equation (A17), the gap equation of σ is

$$-\frac{4\pi^2}{G}\sigma = qBTc\sum_n \int \frac{1}{(i\omega_n)^2 - [p_3' + \mu_5' \operatorname{sgn}(q)]^2 - c^2}\, \mathrm{d}p_3. \tag{17}$$

Because $\sigma = 0$, this equation implies that c should also be zero. For preciseness, we have to verify that $c = 0$ is a solution of the gap equation of c.

Ignoring unimportant factors, the gap equation of c has the form of $c = \mathrm{Tr}(\gamma^3\hat{S})$, plus $\mathrm{Tr}(\gamma^3\hat{S}) \propto c$, then $c = 0$ is a valid solution of this equation.

Now, because $\sigma = 0$ and $c = 0$, only the gap equation of a is left, and with Equation (A13) it is

$$-\frac{4\pi^2}{G}a = |q|BT\sum_n \int \mathrm{d}p_3 \left\{ \frac{p_3' + \mu_5' \operatorname{sgn}(q)}{(i\omega_n)^2 - [p_3' + \mu_5' \operatorname{sgn}(q)]^2} + 2\sum_m \frac{p_3'}{(i\omega_n + \mu_5')^2 - 2m|q|B - p_3'^2} \right\}. \tag{18}$$

Through Residue Theorem, the above equation becomes

$$-\frac{4\pi^2}{G}a = -|q|B\int \left[\frac{1}{2}\operatorname{sgn}(p_3' + \mu_5 \operatorname{sgn}(q)) + \sum_{m=1}^{+\infty} \frac{p_3'}{\sqrt{p_3'^2 + 2m|q|B}} \right] \mathrm{d}p_3. \tag{19}$$

The integral of p_3 in the right-hand side of this equation is not convergent, and Equation (A17) can help reduce the equation to

$$-\frac{4\pi^2}{G}a = -qB\mu_5' - |q|Ba - \sum_{m=1}^{+\infty} 2|q|Ba. \tag{20}$$

Tn the result, the infinite series of m (Landau levels) is also divergent. One has to make a cut-off to the sum of m, assuming that the cut-off is M, then there is

$$-\frac{4\pi^2}{G}a = -qB\mu_5' - |q|Ba - \sum_{m=1}^{M} 2|q|Ba = -qB\mu_5' - (2M+1)|q|Ba, \tag{21}$$

and it results in

$$a = \frac{qB\mu_5'}{\frac{4\pi^2}{G} - (2M+1)|q|B}. \tag{22}$$

Considering the definition of a that $a = -\frac{G}{2N_c}\langle\bar{\psi}\gamma^3\psi\rangle$, the statistical average of the quark current is

$$\langle\bar{\psi}\gamma^3\psi\rangle = -\frac{2N_c qB\mu_5'}{4\pi^2 - (2M+1)|q|BG}. \tag{23}$$

This is the modified CME because of the quark current feedback to the self-energy. The cut-off M to the sum of Landau levels is unexpected, and we will discuss the physical meaning/effect of this cut-off in the last section.

3.2. The Chiral Symmetry Breaking Phase

The chiral symmetry breaking phase is characterized by $\sigma > 0$. Because $\sigma \neq 0$ in this phase, $c \neq 0$ either. The consequence is that the proper propagator Equation (14) leads to complicated gap equations, such that one can hardly deduce the gap equations to the forms suiting numerical calculation. To avoid this barrier, we propose an approximation in this subsection to carry out the study.

Notice, in a NJL model with chiral chemical potential but no external magnetic field, the statistical averages of quark current $\bar{\psi}\gamma^3\psi$ and quark magnetic moments $\bar{\psi}\sigma^{12}\psi$ are zero, which implies that a relatively weak external magnetic field will induce weak $\langle\bar{\psi}\gamma^3\psi\rangle$ and $\langle\bar{\psi}\sigma^{12}\psi\rangle$, so an approximation is proposed so that we can treat $\langle\bar{\psi}\gamma^3\psi\rangle = 0$ and $\langle\bar{\psi}\sigma^{12}\psi\rangle = 0$ in the proper propagator if the magnetic field is relatively weak. Then, one can use the propagator Equation (4) instead of Equation (14) to calculate $\langle\bar{\psi}\gamma^3\psi\rangle$ and $\langle\bar{\psi}\sigma^{12}\psi\rangle$.

Define $\lambda_{m\pm} = [(\sqrt{p_3^2 + 2m|q|B} \pm \mu_5)^2 + \sigma^2]^{\frac{1}{2}}$, Equations (4), (8) and (16) with all the schemes in the Appendix B and the Residue Theorem derive the gap equations of σ, a, and c as

$$\begin{aligned} -\frac{4\pi^2}{G}\sigma = &-\sigma\frac{|q|B}{2\pi}\sum_m \frac{2-\delta_{0m}}{2}\int_0^{+\infty} \mathrm{d}s \int \left[e^{-(p_0^2+\lambda_{m+}^2)s} + e^{-(p_0^2+\lambda_{m-}^2)s}\right] \mathrm{d}p_0 \mathrm{d}p_3 \\ &+ \sigma|q|B\sum_m \frac{2-\delta_{0m}}{2}\int \left(\frac{1}{\lambda_{m+}}\frac{1}{e^{\beta\lambda_{m+}}+1} + \frac{1}{\lambda_{m-}}\frac{1}{e^{\beta\lambda_{m-}}+1}\right)\mathrm{d}p_3, \end{aligned} \tag{24}$$

$$-\frac{4\pi^2}{G}a = -qB\mu_5, \tag{25}$$

$$-\frac{4\pi^2}{G}c = -\sigma\frac{qB}{2}\int_0^{+\infty}\frac{e^{-\sigma^2 s}}{s}\mathrm{d}s + \sigma qB\int \frac{1}{\sqrt{p_3^2+\sigma^2}}\frac{1}{e^{\beta\sqrt{p_3^2+\sigma^2}}+1}\mathrm{d}p_3. \tag{26}$$

Equation (25) leads to the classic CME. It seems even in the chiral symmetry breaking phase, the CME is irrelevant to temperature and vacuum condensate, but this result is not rigorous. We will come back to it in Section 3.3. In Equations (24) and (26), the integrals of the proper time s is divergent, and it needs a cut-off to the lower boundary of its integral, which is

$$\int_0^{+\infty}\mathrm{d}s \to \int_{1/\Lambda^2}^{+\infty}\mathrm{d}s. \tag{27}$$

According to [35], the cut-off parameter Λ and the coupling constant G are set as

$$G \approx 76.89312\ \mathrm{GeV}^{-2}, \qquad \Lambda \approx 1.08631\ \mathrm{GeV}. \tag{28}$$

We can then use the gap equations of σ Equation (24) to get the solutions of σ at a specific temperature, chiral chemical potential, and magnetic field through a numerical method and subsequently substitute the solutions into Equation (26) to get the values of c.

In Figure 1, the relations of σ-μ_5 and c-μ_5 with fixing temperature is drawn. Compared with σ, the values of c are much smaller, and indeed, a weaker magnetic field induces smaller c, so it verifies the rationality of the approximation that we employ in this subsection.

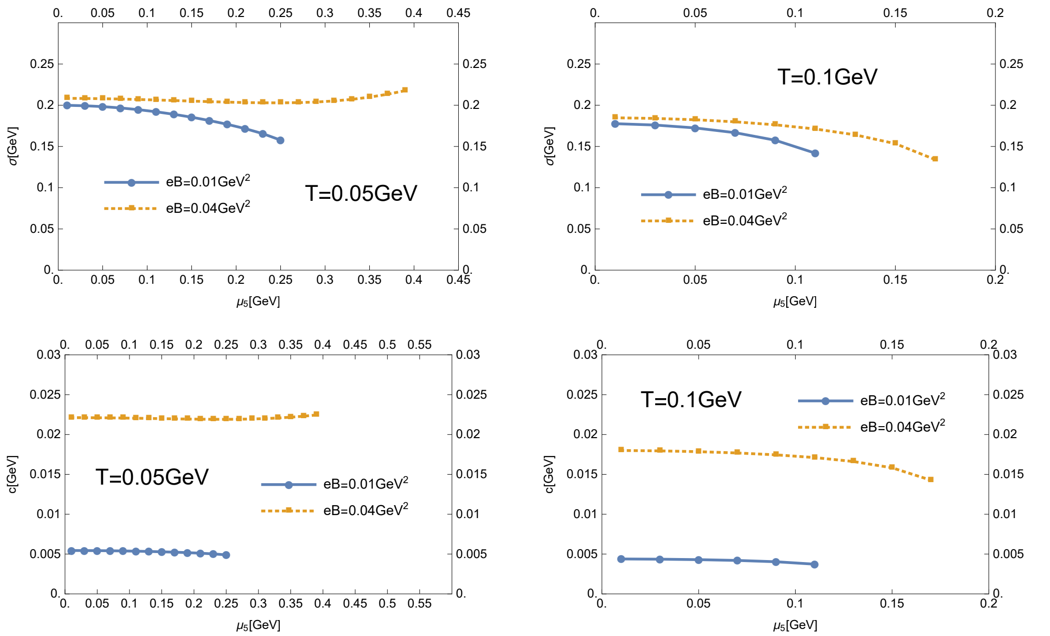

Figure 1. These figures show the σ-μ_5 and c–μ_5 relations with temperature fixed at 0.05 GeV and 0.1 GeV. In each figure, the results from different magnetic fields are compared.

Figure 2 shows the relations of σ-μ_5 and c-μ_5 when fixing a magnetic field. The detailed discussion of these results will be discussed in the last section.

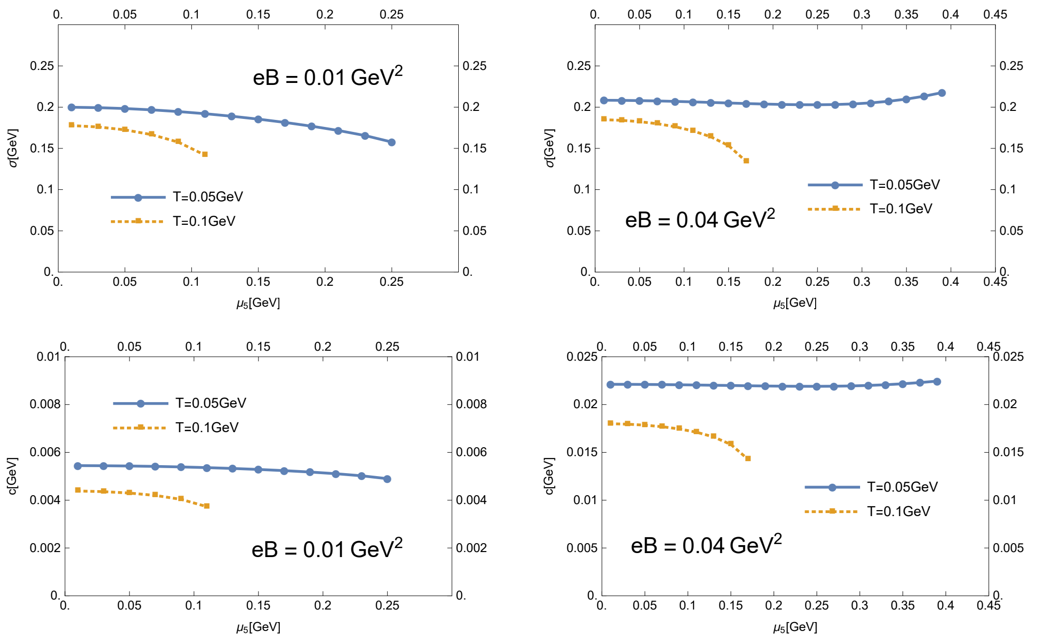

Figure 2. These figures show the σ-μ_5 and c–μ_5 relations with a magnetic field fixed at 0.01 GeV^2 and 0.04 GeV^2. In each figure, the results from different temperatures are compared.

3.3. *The Modified CME in Chiral Symmetry Breaking Phase—A Rigorous Proof*

In Section 3.2, it has been shown that using Equation (4) as the approximated non-perturbative quark propagator leads to the classic CME in the chiral symmetry breaking phase. In this subsection, we rigorously prove that, in this phase, the CME should also be modified into Equation (22) or Equation (23).

Define

$$\lambda := \sqrt{[p_3' + \mu_5' \operatorname{sgn}(q)]^2 + [\sigma + c \operatorname{sgn}(q)]^2},$$
$$\kappa_{m\pm} := \sqrt{2m|q|B + p_3'^2 + \sigma^2 + \mu_5'^2 + c^2 \pm \left[2m|q|B(\mu_5'^2 + c^2) + (p_3'\mu_5' + \sigma c)^2\right]^{\frac{1}{2}}}. \tag{29}$$

With the Residue Theorem, the gap equation of a then becomes

$$-\frac{4\pi^2}{G}a = -\frac{|q|B}{2}\int \mathrm{d}p_3 \left[\frac{p_3' + \mu_5' \operatorname{sgn}(q)}{\lambda} + \sum_{m=1}^{+\infty}\sum_{\pm}\frac{\partial \kappa_{m\pm}}{\partial p_3'}\right]$$
$$+ |q|B \int \mathrm{d}p_3 \left[\frac{p_3' + \mu_5' \operatorname{sgn}(q)}{\lambda}\frac{1}{e^{\beta\lambda}+1} + \sum_{m=1}^{+\infty}\sum_{\pm}\frac{\partial \kappa_{m\pm}}{\partial p_3'}\frac{1}{e^{\beta\kappa_{m\pm}}+1}\right]. \tag{30}$$

In the above equation, the integrals of p_3 with $\frac{1}{e^{\beta\lambda}+1}$ or $\frac{1}{e^{\beta\kappa_{m\pm}}+1}$ in the integrands are convergent and equal 0, which leaves the vacuum terms in the equation as,

$$-\frac{4\pi^2}{G}a = -\frac{|q|B}{2}\int \mathrm{d}p_3 \left[\frac{p_3' + \mu_5' \operatorname{sgn}(q)}{\lambda} + \sum_{m=1}^{+\infty}\sum_{\pm}\frac{\partial \kappa_{m\pm}}{\partial p_3'}\right]. \tag{31}$$

Similar to the discussion of Equation (19) in Section 3.1, by employing Equation (A17) in Scheme V to calculate Equation (31), it exactly equals Equation (20), then it leads to the same modified CME Equation (22) after introducing the cut-off M to the sum of Landau levels. That ends the proof about the modified CME in the chiral symmetry breaking phase.

4. Discussion and Conclusions

In this paper, we studied the one-flavor NJL model of the quark with finite temperature, chiral chemical potential, and external magnetic field. The reason why we study the 1-flavor NJL model instead of the 2-flavor or (2 + 1)-flavor NJL is for the convenience of generalizing the results to models of more flavors. The reader should be aware that the one-flavor NJL model is different from one-flavor QCD dynamically, because the anomaly in one-flavor QCD will prevent the chiral symmetry [37].

One of the goals of this paper is to determine the proper quark propagator. With the presence of a chiral chemical potential and magnetic field, there could be non-zero statistical averages of $\bar{\psi}\gamma^3\psi$, $\bar{\psi}\gamma^5\gamma^0\psi$, and $\bar{\psi}\sigma^{12}\psi$ alongside $\bar{\psi}\psi$, and all these averages contribute to the self-energy of the non-perturbative quark propagator when the mean-field approximation is applied. A rigorous way to write down the non-perturbative quark propagator is to include all possible statistical averages in the self-energy and then to prove that some of them are constantly zero. This is a top-down method and it is a great challenge for now. Instead, we used a bottom-up method to find out a non-perturbative propagator—the proper quark propagator. In our method, we propose the minimal consistency ansatz and that the proper propagator should have the least statistical averages in the self-energy, such that other averages are zero by using this propagator in Equation (8). Through this ansatz, we prove that the proper propagator is Equation (14) and the only statistical averages which could be non-zero are $\langle\bar{\psi}\psi\rangle$, $\langle\bar{\psi}\gamma^3\psi\rangle$, $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$, and $\langle\bar{\psi}\sigma^{12}\psi\rangle$.

Four statistical averages means four gap equations, but one does not need to calculate all of them. The physical effect of $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$ is to renormalize the chiral chemical potential, and the renormalized chiral chemical potential can be treated as a free parameter; therefore, the gap equation of $\langle\bar{\psi}\gamma^5\gamma^0\psi\rangle$ or 'b' is not needed. There are still three gap equations left, and the next step is to study these equations by considering different phases of QCD matter.

In the chiral symmetry restoring phase, it can be proved that $\langle\bar{\psi}\psi\rangle$ and $\langle\bar{\psi}\sigma^{12}\psi\rangle$ are simultaneously zero. This reduces the three gap equations to one equation—the gap equation Equation (21) of $\langle\bar{\psi}\gamma^3\psi\rangle$ or 'a'. This equation needs a cut-off M to the sum of Landau levels, which is unexpected but reasonable. The NJL model is an effective field theory after all, even with the mean-field approximation, and in obtaining the quantities involving self-energy, a cut-off is normally needed. Generally, at finite temperature, the cut-off happens at the upper limit of $|\vec{p}|$ if no magnetic field presents, while in this paper, $|\vec{p}|$ becomes $\sqrt{2m|q|B + p_3^2}$ because of the existence of the magnetic field; therefore, a finite result sometimes requires the cut-off to the Landau levels. Equation (21) leads to a modified CME Equation (22), which is partly different from the classic CME deduced from Equation (25). The classic CME is independent of the coupling constant 'G' and proportional to the strength of the magnetic field, while the modified CME is not. The main cause of these differences is that the classic CME does not consider the magnetic effect to the gluon propagator. We can actually define a modified coupling constant G' that satisfies

$$\frac{4\pi^2}{G'} = \frac{4\pi^2}{G} - (2M+1)|q|B, \tag{32}$$

and gives

$$G' = \frac{G}{1 - (2M+1)|q|B\frac{G}{4\pi^2}}. \tag{33}$$

This relation can shed a light on how the magnetic field affects the coupling constant or how we should approximate the gluon propagator when an external magnetic field is considered. Equation (33) also enlightens us as to how to determine the cut-off 'M' to the Landau levels; that is, if one can acquire the data of the gluon propagator under the influence of an external magnetic field, then the cut-off can be determined appropriately.

In the chiral symmetry breaking phase, the approximation of using the original propagator Equation (4) for the gap equations leads to the classic CME, but it is not correct. A more rigorous study shows that the quark current satisfies a modified CME in the breaking phase, whcih is the same as in the restoring phase. As for the gap equations of $\langle\bar{\psi}\psi\rangle$ and $\langle\bar{\psi}\sigma^{12}\psi\rangle$, we use a numerical method to find out the solutions of σ and c. iIt turns out that, much like the chemical potential, the increasing chiral chemical potential can also cause a phase transition from the breaking phase to the restoring phase. In our previous work [35], we proved that the existence of an external magnetic field and chemical potential induces non-zero $\langle\bar{\psi}\sigma^{12}\psi\rangle$, which is explained as the statistical averages of the quark magnetic moment. In this paper, the magnetic field and chiral chemical potential together also induce non-zero $\langle\bar{\psi}\sigma^{12}\psi\rangle$ in the breaking phase. By comparing the σ-μ_5 and c-μ_5 relations, the values of c are indeed much smaller than σ at the same temperature, magnetic field, and chiral chemical potential, which verifies that the approximation of using Equation (4) is appropriate.

For a brief conclusion, the presence of chiral chemical potential and external magnetic field induce non-zero quark current $\langle\bar{\psi}\gamma^3\psi\rangle$ and non-zero quark magnetic moment $\langle\bar{\psi}\sigma^{12}\psi\rangle$, which is a new effect. Additionally, because of the quark current feedback to the quark self-energy, the quark current satisfies a modified CME, which is different form the classic CME. This paper has also studies the μ_5-dependence of quark magnetic moment or 'c', and it seems 'c' has the same behavior as σ does when the phase transition happens. Nevertheless, its magnitude is normally much smaller than σ in the chiral symmetry breaking phase; therefore, it may not be a good order parameter for identifying the phases of QCD matter.

There are still more studies to be done. In the breaking phase, new techniques are needed to handle the proper propagator for studying the gap equations more rigorously. In the restoring phase, the cut-off to the Landau levels needs support from other researches, such as the experiments or QCD theory, to verify its rationality and determine its values. Alongside these, we tend to include the chemical potential, chiral chemical potential, and magnetic field in one NJL model and study their effects on the condensate and CME.

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Abbreviations

The following abbreviations are used in this manuscript:

QCD	Quantum Chromodynamics
QGP	Quark-Gluon Plasma
NJL	Nambu-Jona-Lasinio
CME	Chiral Magnetic Effect

Appendix A. The Properties of $|\omega_n; n, a; p_3\rangle$

The basic properties of $|\omega_n\rangle \otimes |m, a\rangle \otimes |p_3\rangle \equiv |n; m, a; p_3\rangle$ listed in the main body are

$$\begin{aligned} \hat{p}_0|\omega_n\rangle = i\omega_n|\omega_n\rangle, \quad \omega_n = (2n+1)\pi T, \quad n \in \mathbb{Z}; \\ \hat{p}_3|p_3\rangle = -p_3|p_3\rangle; \qquad \hat{D}_\perp^2|m, a\rangle = (2m+1)|q|B|m, a\rangle, \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{A1}$$

$|\omega_n\rangle$ is the eigenstate of $\hat{p}_0$ (noticing at finite temperature, $\hat{p}_0$ is not a Hermitian operator). Alongside Equation (A1), the orthonormalization of this eigenstate is

$$\langle \omega_n | \omega_{n'} \rangle = \delta_{nn'}. \tag{A2}$$

$|p_3\rangle$ is simply the momentum eigenstate of $\hat{p}_3$, while its orthonormalization and completeness are

$$\langle p_3' | p_3 \rangle = \delta(p_3' - p_3), \qquad \int |p_3\rangle\langle p_3| \, \mathrm{d}p_3 = 1. \tag{A3}$$

For canceling the $\int \mathrm{d}x_3$ in the left hand side of the gap equations Equation (8), we need the relation of $|p_3\rangle$ and $\int \mathrm{d}x_3$, for which there is

$$\langle p_3 | p_3 \rangle = \frac{1}{2\pi} \int \mathrm{d}x_3. \tag{A4}$$

$|m, a\rangle$ is the eigenstate of $\hat{D}_\perp^2$, which is a state in the Hilbert space spanned by $|x_1\rangle \otimes |x_2\rangle$. In order to find out the physical meaning of 'a' and its orthonormalization, we defined an orthonormalized eigenstate of $\hat{D}_1$,

$$|D_1, p\rangle := \sqrt{\frac{2}{|q|eB}}|p\rangle \otimes |X\rangle, \qquad X := \frac{2}{qeB}(D_1 - p). \tag{A5}$$

It has the properties as below:

$$\hat{D}_1|D_1, p\rangle = -D_1|D_1, p\rangle, \qquad \hat{p}_1|p\rangle = -p|p\rangle, \qquad \hat{x}_2|X\rangle = -X|X\rangle, \tag{A6}$$

$$\langle D_1, p | D_1', p' \rangle = \delta(D_1 - D_1')\delta(p - p'), \qquad \int \mathrm{d}D_1 \mathrm{d}p \, |D_1, p\rangle\langle D_1, p| = I. \tag{A7}$$

In the $|D_1, p\rangle$ representation, $|m, a\rangle$ is expressed as

$$\langle D_1, p|m, a\rangle = c_m e^{ia(\frac{D_1}{2}-p)} h_m(\sqrt{\frac{2}{|q|eB}}D_1), \qquad c_m := \left(\frac{1}{m!2\pi\sqrt{|q|eB\pi}}\right)^{\frac{1}{2}}, \tag{A8}$$

where $h_m(z)$ is an even-function of z, which is the solution of the Weber differential equation. Its orthogonality is

$$\int h_m(z)h_{m'}(z)\,\mathrm{d}z = m!\sqrt{2\pi}\delta_{mm'}, \qquad \sum_{m=0}^{+\infty}\frac{1}{m!\sqrt{2\pi}}h_m(x)h_m(y) = \delta(x-y). \tag{A9}$$

One can then employ the properties of $h_m(z)$ to demonstrate the orthonormalization and completeness of $|m, a\rangle$,

$$\langle m, a|m', a'\rangle = \int \mathrm{d}D_1\mathrm{d}p\,\langle m, a|D_1, p\rangle\langle D_1, p|m', a'\rangle = \delta_{mm'}\delta(a-a'), \tag{A10}$$

$$\sum_{m=0}^{+\infty}\int \mathrm{d}a\,|m, a\rangle\langle m, a| = \sum_{m=0}^{+\infty}\int \mathrm{d}a\mathrm{d}D_1\mathrm{d}p\mathrm{d}D_1'\mathrm{d}p'\,|D_1, p\rangle\langle D_1, p|m, a\rangle\langle m, a|D_1', p'\rangle\langle D_1', p'| = I. \tag{A11}$$

One can verify that 'a' is the eigenvalue of $(\hat{p}_2 - \frac{qeB}{2}\hat{x}^1)$ now.

Once more, in order to cancel the $\int \mathrm{d}x_1\mathrm{d}x_2$ in the left-hand side of the gap equations in Equation (8), there is the equation

$$\int \langle m, a|m, a\rangle\,\mathrm{d}a = \frac{|q|B}{2\pi}\int \mathrm{d}x_1\mathrm{d}x_2. \tag{A12}$$

Appendix B. Several Schemes for Deducing the Gap Equations

We introduce several useful schemes to assist the deductions in the main body.

Scheme I: For any function $f(\omega_n)$ of ω_n, there is

$$\sum_n f(\omega_n) = \sum_n f(-\omega_n), \tag{A13}$$

whether the infinite series is convergent or not.

Scheme II: If there is an odd function $f(\omega_n)$ of ω_n, there is

$$\sum_n f(\omega_n) = \frac{1}{2}\sum_n[f(\omega_n) + f(-\omega_n)] = 0, \tag{A14}$$

whether the infinite series is convergent or not.

Scheme III: If there is an odd function $f(x)$ of x, there is

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \lim_{\Lambda\to+\infty}\int_{-\Lambda}^{\Lambda} f(x)\,\mathrm{d}x = 0, \tag{A15}$$

whether the integral is convergent or not. In quantum field theory, this scheme is in fact the symmetric cut-off regularization scheme. According to this scheme, one can prove that $\langle x_3|\hat{p}_3|x_3\rangle = 0$. Similarly, for $\hat{D}_{1,2}$, there is another scheme shown below.

Scheme IV:

$$\langle m, a|\hat{D}_{1,2}|m, a\rangle = 0. \tag{A16}$$

Scheme V: If there is an odd function $f(x)$ of x, there is

$$\int_{-\infty}^{+\infty} f(x+\Delta)\,\mathrm{d}x = \lim_{\Lambda\to+\infty}\int_{-\Lambda}^{\Lambda} f(x+\Delta)\,\mathrm{d}x = \lim_{\Lambda\to+\infty}\int_{\Lambda-\Delta}^{\Lambda+\Delta} f(x)\,\mathrm{d}x. \tag{A17}$$

The core idea of this scheme is that the integral variable transformation must be carried out after the symmetric cut-off regularization. Of course, if $f(x)$ is integral convergent, we have $\int f(x+\Delta)\,\mathrm{d}x = 0$.

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