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Trapezium-like Inequalities Involving k -th Order Differentiable $\mathcal{R}_\gamma$ -Convex Functions and Applications

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Abstract: We introduce the class of $\mathcal{R}_\gamma$ -convex functions and discuss that it relates to some other classes of convexity. We study the class of $\mathcal{R}_\gamma$ -convex functions in the perspective of trapezium-like inequalities, for which we also derive a new integral identity involving a k -th order differentiable function. In order to show the significance of our results, we also discuss several special cases and offer some interesting applications.

Keywords: convex; $\mathcal{R}_\gamma$ -convex functions; k -times differentiable; trapezium inequalities; means

MSC: 26A51; 26D15

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1. Introduction and Preliminaries

The theory of convexity has great significance in modern analysis and also plays an important role in different branches of pure and applied sciences through its numerous applications. For example, convexity plays a significant role in optimization, mathematical economics, and operations research. The concepts of convexity and symmetry also have a close relation. There are many important properties of symmetric convex sets. A significance of the relation between between convexity and symmetry is that we work on one and apply it to the other. In recent decades the classical concepts of convex sets and convex functions have been extended and generalized in different directions according to the need of the problems. In 1978, Brckner [1] introduced the notion of s_1 -convex functions and noticed that we can recapture the classical convexity from s_1 -convexity by taking $s = 1$. In 1995, Dragomir et al. [2] introduced and studied the class of P -convex functions. Godunova and Levin [3] introduced the notion of Godunova–Levin type convex functions. Varosanec [4] introduced and studied the class of h -convex functions. It is worth mentioning here that the class of h -convex functions unifies all of these discussed classes and also enjoys some favorable properties of classical convex functions. In 2015, Dragomir [5] introduced the notion of s_2 -Godunova–Levin type convex functions and observed that this class is also contained in the class of h -convex functions. In 2013, Dragomir [6] introduced another generalization of classical convexity that is called φ -convex functions. Taking inspiration from this, Awan et al. [7] introduced another significant generalization of classical convexity that is called γ -preinvex functions [8]. Very recently, Cortez et al. [9] introduced the class of $\mathcal{R}$ -convex sets as:

Definition 1 ([9]). Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A non-empty set $\mathcal{L}$ is said to be generalized convex, if:

$$v_1 + \tau \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1) \in \mathcal{L}, \quad \forall v_1, v_2 \in \mathcal{L}, \tau \in [0, 1].$$

Here $\mathcal{R}_{\rho, \lambda, \sigma}(\cdot)$ is Raina's function and is defined as follows:

$$\mathcal{R}_{\rho, \lambda, \sigma}(z) = \mathcal{R}_{\rho, \lambda}^{\sigma(0), \sigma(1), \dots}(z) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k,$$

where $\rho, \lambda > 0$, $|z| < R$ and $\sigma = \{\sigma(0), \sigma(1), \dots, \sigma(k), \dots\}$ is a bounded sequence of positive real numbers. For details, see [10].

Using $\mathcal{R}$ -convex sets as a domain, Cortez et al. [9] also defined the class of $\mathcal{R}$ -convex functions as:

Definition 2 ([9]). Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be generalized convex, if:

$$\Psi(v_1 + \tau \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq (1 - \tau)\Psi(v_1) + \tau\Psi(v_2), \quad \forall v_1, v_2 \in \mathcal{L}, \tau \in [0, 1].$$

For some more details regarding generalizations, extensions and applications of classical convexity, see [4,7,11–16].

As a matter of fact, the theory of convex functions has a close relationship with the theory of inequalities. We can easily obtain a huge number of inequalities by just using the convexity property of the functions and their generalizations. Hermite–Hadamard's inequality is one of the most important results that can be viewed as an equivalent property of the convex functions. This inequality is also known as a trapezium inequality. In recent years, this result has been extended and generalized in different directions using novel and innovative ideas.

Cortez et al. [9] derived a new version of Hermite–Hadamard's inequality using the class of generalized convex functions. This result reads as follows:

Theorem 1. Let $\Psi : \mathcal{L} = [c_1, c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)] \rightarrow \mathbb{R}$ be a generalized convex function; then:

$$\Psi\left(\frac{2c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)}{2}\right) \leq \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Psi(x) dx \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}.$$

Note that if we take $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) = c_2 - c_1$, then we can recapture the classical Hermite–Hadamard's inequality from the above inequality, which reads as:

Theorem 2. Let $\Psi : \mathcal{I} = [c_1, c_2] \rightarrow \mathbb{R}$ be a convex function; then:

$$\Psi\left(\frac{c_1 + c_2}{2}\right) \leq \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \Psi(x) dx \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}.$$

For more details regarding Hermite–Hadamard's inequality and its applications, see [14,17,18].

The main objective of this paper is to generalize the notion of $\mathcal{R}$ -convex functions and introduce the class of $\mathcal{R}_\gamma$ -convex functions. We show that the class of $\mathcal{R}_\gamma$ -convex functions includes some other classes of classical convexity. We study the class of $\mathcal{R}_\gamma$ -convex functions in the perspective of trapezium-like inequalities. To obtain our main results,

we derive a new integral identity involving a k -th order differentiable function. Using this identity as an auxiliary result, we then derive associated trapezium-like inequalities essentially using the class of k -th order $\mathcal{R}_\gamma$ -convex functions. We also discuss several special cases that can be deduced from the main results of our paper. In order to show the significance of our results, we also discuss several special cases and offer some interesting applications. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Main Results

In this section, we discuss our main results.

2.1. $\mathcal{R}_\gamma$ -Convex Functions

We now define the class of $\mathcal{R}_\gamma$ -convex functions.

Definition 3. Let $\gamma : (0, 1) \rightarrow (0, \infty)$ be a real function. Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be $\mathcal{R}_\gamma$ -convex, if:

$$\Psi(v_1 + \mu \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq (1 - \mu)\gamma(1 - \mu)\Psi(v_1) + \mu\gamma(\mu)\Psi(v_2), \quad \forall v_1, v_2 \in K, \mu \in [0, 1].$$

We now discuss some special cases of Definition 3.

I. If we take $\gamma(\mu) = 1$ in Definition 3, then we have the class of generalized convex functions [9].

II. If we take $\gamma(\mu) = \mu^{-1}$ in Definition 3, then we have the definition of $\mathcal{R}_p$ -convex functions.

Definition 4. Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be $\mathcal{R}_p$ -convex, if:

$$\Psi(v_1 + \mu \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq \Psi(v_1) + \Psi(v_2), \quad \forall v_1, v_2 \in K, \mu \in [0, 1].$$

III. If we take $\gamma(\mu) = \mu^{s-1}$ in Definition 3, where $s \in (0, 1)$, then we have the class of $\mathcal{R}_{s_1}$ -convex functions of Breckner type.

Definition 5. Let $s \in (0, 1]$. Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be $\mathcal{R}_{s_1}$ -convex, if:

$$\Psi(v_1 + \mu \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq (1 - \mu)^s \Psi(v_1) + \mu^s \Psi(v_2), \quad \forall v_1, v_2 \in K, \mu \in [0, 1].$$

IV. If we take $\gamma(\mu) = \mu^{-s-1}$, then Definition 3 reduces to the definition of $\mathcal{R}_{s_2}$ -Godunova–Levin convex function.

Definition 6. Let $s \in [0, 1]$. Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be $\mathcal{R}_{s_2}$ -convex, if:

$$\Psi(v_1 + \mu \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq (1 - \mu)^{-s} \Psi(v_1) + \mu^{-s} \Psi(v_2), \quad \forall v_1, v_2 \in K, \mu \in (0, 1).$$

V. If we take $\gamma(\mu) = 1 - \mu$ in Definition 3, then we have the definition of $\mathcal{R}_{tgs}$ -convex function.

Definition 7. Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ is said to be $\mathcal{R}_{tgs}$ -convex, if:

$$\Psi(v_1 + \mu \mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1)) \leq \mu(1 - \mu)[\Psi(v_1) + \Psi(v_2)], \quad \forall v_1, v_2 \in K, \mu \in [0, 1].$$

VI. If we take $\mathcal{R}_{\rho, \lambda, \sigma}(v_2 - v_1) = v_2 - v_1$ in Definition 3, then we have the definition of γ -convex function, see [6].

Our next result depends upon condition M, which was introduced by Noor and Noor [19].

Condition M. Assume that the function $\mathcal{R}(\cdot)$ satisfies the following condition:

$$\mathcal{R}(\theta\mathcal{R}(v-u)) = \theta\mathcal{R}(v-u), \quad \theta \in [0, 1].$$

We now give a new version of Hermite–Hadamard’s inequality essentially using the class of $\mathcal{R}_\gamma$ -convex functions.

Theorem 3. Let $\Psi : \mathcal{L} = [c_1, \Delta] \rightarrow \mathbb{R}$ be a $\mathcal{R}_\gamma$ -convex function and $\mathcal{R}_{\rho, \lambda, \sigma}(\cdot)$ satisfy condition M; then for $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) > 0$ and $\gamma(\frac{1}{2}) \neq 0$, we have:

$$\frac{1}{\gamma(\frac{1}{2})} \Psi\left(\frac{2c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)}{2}\right) \leq \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Psi(x) dx \leq (\Psi(c_1) + \Psi(c_2)) \int_0^1 \mu \gamma(\mu) d\mu.$$

Proof. It is given that $\mathcal{R}_{\rho, \lambda, \sigma}(\cdot)$ satisfies condition M and since Ψ is a $\mathcal{R}_\gamma$ -convex function, by taking $x = c_1 + \mu\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)$ and $y = c_1 + (1 - \mu)\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)$, we have:

$$\begin{aligned} \Psi\left(\frac{2c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)}{2}\right) &= \Psi\left(\frac{2x + \mathcal{R}_{\rho, \lambda, \sigma}(y - x)}{2}\right) \\ &\leq \gamma\left(\frac{1}{2}\right) \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Psi(x) dx. \end{aligned}$$

This implies:

$$\frac{1}{\gamma(\frac{1}{2})} \Psi\left(\frac{2c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)}{2}\right) \leq \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Psi(x) dx. \quad (1)$$

Additionally, using the fact that Ψ is a $\mathcal{R}_\gamma$ -convex function, we have:

$$\frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Psi(x) dx \leq (\Psi(c_1) + \Psi(c_2)) \int_0^1 \mu \gamma(\mu) d\mu. \quad (2)$$

Combining (1) and (2) completes the proof. $\square$

Remark 1. Note that for different suitable choices of the function $\gamma(\cdot)$ in Theorem 3, we obtain other versions of Hermite–Hadamard’s inequality. For instance, if we take $\gamma(\mu) = 1, \mu^{-1}, \mu^{s-1}, \mu^{-s-1}$ and $\gamma(\mu) = 1 - \mu$, then we obtain Hermite–Hadamard’s inequality for generalized convex functions, $\mathcal{R}_P$ -convex functions, $\mathcal{R}_{s_1}$ -convex functions, $\mathcal{R}_{s_2}$ -convex functions and for $\mathcal{R}_{tgs}$ -convex functions, respectively.

2.2. Auxiliary Result

In this section, we derive a new integral identity involving k-times differentiable functions. For the sake of simplicity, we now consider $\Omega_\mu := c_1 + \mu\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)$, $\Omega_{1-\mu} := c_1 + (1 - \mu)\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)$, $\Delta := c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)$ and $\Lambda(c_1, \mu) := c_1 k \mu^k + \dots + c_{11} \mu + c_{10}$.

Lemma 1. Let $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ be a k -times differentiable function on $\mathcal{L}^\circ$ with $c_1, c_2 \in \mathcal{L}$ and $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) > 0$ where k is an even number. If $\Psi^k \in L[c_1, c_1 + \mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)]$, then

$$\mathcal{M}(c_1, c_2; k) = \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!c_{1k}} \int_0^1 (\Lambda(c_1, \mu)) [\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})] d\mu.$$

where:

$$\begin{aligned} \mathcal{M}(c_1, c_2; k) = & \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_{c_1}^{\Delta} \Psi(x) dx - \frac{\Psi(c_1) + \Psi(\Delta)}{2} + \dots \\ & - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})\phi^{k-4}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-4}(\Delta) + \Psi^{k-4}(c_1)] \\ & + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})\phi^{k-3}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-3}(\Delta) - \Psi^{k-3}(c_1)] \\ & - \frac{(kc_{1k} + \dots + 2c_{12})\phi^{k-2}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-2}(\Delta) + \Psi^{k-2}(c_1)] \\ & + \frac{(c_{1k} + \dots + c_{11} + 2c_{10})\phi^{k-1}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-1}(\Delta) - \Psi^{k-1}(c_1)]. \end{aligned}$$

Proof. Consider the following integral and integrating by parts repeatedly, we have:

$$\begin{aligned} \mathcal{K}_1 = & \int_0^1 (\Lambda(c_1, \mu)) \Psi^k(\Omega_{1-\mu}) d\mu \\ = & -(\Lambda(c_1, \mu)) \frac{\Psi^{k-1}(\Omega_{1-\mu})}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \Big|_0^1 \\ & + \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_0^1 (kc_{1k}\mu^{k-1} + \dots + c_{11}) \Psi^{k-1}(\Omega_{1-\mu}) d\mu \\ = & \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} [c_{10} \Psi^{k-1}(\Delta) - (c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(c_1)] \\ & + \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} \int_0^1 (kc_{1k}\mu^{k-1} + \dots + c_{11}) \Psi^{k-1}(\Omega_{1-\mu}) d\mu \\ = & \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1)} [c_{10} \Psi^{k-1}(\Delta) - (c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(c_1)] \\ & - (kc_{1k}\mu^{k-1} + \dots + 2c_{12}\mu + c_{11}) \frac{\Psi^{k-2}(\Omega_{1-\mu})}{\mathcal{R}_{\rho, \lambda, \sigma}^2(c_2 - c_1)} \Big|_0^1 \\ & - \frac{1}{\mathcal{R}_{\rho, \lambda, \sigma}^2(c_2 - c_1)} \int_0^1 (k(k-1)c_{1k}\mu^{k-2} + \dots + 3.2c_{13}\mu + 2c_{12}) \Psi^{k-2}(\Omega_{1-\mu}) d\mu \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[c_{10} \Psi^{k-1}(\Delta) - (c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[(kc_{1k} + \dots + 2c_{12} + c_{11}) \Psi^{k-2}(c_1) - c_{11} \Psi^{k-2}(\Delta) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \int_0^1 (k(k-1)c_{1k}\mu^{k-2} + \dots + 3.2c_{13}\mu + 2c_{12}) \Psi^{k-2}(\Omega_{1-\mu}) d\mu \\
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[c_{10} \Psi^{k-1}(\Delta) - (c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[(kc_{1k} + \dots + 2c_{12} + c_{11}) \Psi^{k-2}(c_1) - c_{11} \Psi^{k-2}(\Delta) \right] \\
&\quad + \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^3(c_2 - c_1)} \left[2c_{12} \Psi^{k-3}(\Delta) - (k(k-1)c_{1k} + \dots + 3.2c_{13} + 2c_{12}) \Psi^{k-3}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^4(c_2 - c_1)} \left[(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14} + 3.2c_{13}) \Psi^{k-4}(c_1) \right. \\
&\quad \left. - 3.2c_{13} \Psi^{k-4}(\Delta) \right] \\
&\quad + \dots - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} \left[(k!c_{1k} + (k-1)!c_{1k-1}) \Psi(c_1) - (k-1)!c_{1k-1} \Psi(\Delta) \right] \\
&\quad + \frac{k!c_{1k}}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} \int_0^1 \Psi(\Omega_{1-\mu}) d\mu. \tag{3}
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathcal{K}_2 &= \int_0^1 (\Lambda(c_1, \mu)) \Psi^k(\Omega_\mu) d\mu \\
&= (\Lambda(c_1, \mu)) \frac{\Psi^{k-1}(\Omega_\mu)}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \Big|_0^1 \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_0^1 (kc_{1k}\mu^{k-1} + \dots + c_{11}) \Psi^{k-1}(\Omega_\mu) d\mu \\
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[(c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(\Delta) - c_{10} \Psi^{k-1}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_0^1 (kc_{1k}\mu^{k-1} + \dots + c_{11}) \Psi^{k-1}(\Omega_\mu) d\mu \\
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[(c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(\Delta) - c_{10} \Psi^{k-1}(c_1) \right] \\
&\quad - (kc_{1k}\mu^{k-1} + \dots + 2c_{12}\mu + c_{11}) \frac{\Psi^{k-2}(\Omega_\mu)}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \Big|_0^1 \\
&\quad + \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \int_0^1 (k(k-1)c_{1k}\mu^{k-2} + \dots + 3.2c_{13}\mu + 2c_{12}) \Psi^{k-2}(\Omega_\mu) d\mu
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[(c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(\Delta) - c_{10} \Psi^{k-1}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[(kc_{1k} + \dots + 2c_{12} + c_{11}) \Psi^{k-2}(\Delta) - c_{11} \Psi^{k-2}(c_1) \right] \\
&\quad + \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \int_0^1 (k(k-1)c_{1k}\mu^{k-2} + \dots + 3.2c_{13}\mu + 2c_{12}) \Psi^{k-2}(\Omega_\mu) d\mu \\
&= \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[(c_{1k} + \dots + c_{11} + c_{10}) \Psi^{k-1}(\Delta) - c_{10} \Psi^{k-1}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[(kc_{1k} + \dots + 2c_{12} + c_{11}) \Psi^{k-2}(\Delta) - c_{11} \Psi^{k-2}(c_1) \right] \\
&\quad + \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^3(c_2 - c_1)} \left[(k(k-1)c_{1k} + \dots + 3.2c_{13} + 2c_{12}) \Psi^{k-3}(\Delta) - 2c_{12} \Psi^{k-3}(c_1) \right] \\
&\quad - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^4(c_2 - c_1)} \left[(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14} + 3.2c_{13}) \Psi^{k-4}(\Delta) - 3.2c_{13} \Psi^{k-4}(c_1) \right] \\
&\quad + \dots - \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} \left[(k!c_{1k} + (k-1)!c_{1k-1}) \Psi(\Delta) - (k-1)!c_{1k-1} \Psi(c_1) \right] \\
&\quad + \frac{k!c_{1k}}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} \int_0^1 \Psi(\Omega_\mu) d\mu.
\end{aligned} \tag{4}$$

Summing up Equations (3) and (4), we have:

$$\begin{aligned}
\mathcal{K}_1 + \mathcal{K}_2 &= \int_0^1 (\Lambda(c_1, \mu)) [\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})] d\mu \\
&= \frac{c_{10}}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[\Psi^{k-1}(\Delta) - \Psi^{k-1}(c_1) \right] \\
&\quad + \frac{(c_{1k} + \dots + c_{11} + c_{10})}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \left[\Psi^{k-1}(\Delta) - \Psi^{k-1}(c_1) \right] \\
&\quad + \frac{c_{11}}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[\Psi^{k-2}(\Delta) + \Psi^{k-2}(c_1) \right] \\
&\quad - \frac{(kc_{1k} + \dots + 2c_{12} + c_{11})}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} \left[\Psi^{k-2}(\Delta) + \Psi^{k-2}(c_1) \right] \\
&\quad + \frac{2c_{12}}{\mathcal{R}_{\rho,\lambda,\sigma}^3(c_2 - c_1)} \left[\Psi^{k-3}(\Delta) - \Psi^{k-3}(c_1) \right] \\
&\quad + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13} + 2c_{12})}{\mathcal{R}_{\rho,\lambda,\sigma}^3(c_2 - c_1)} \left[\Psi^{k-3}(\Delta) - \Psi^{k-3}(c_1) \right] \\
&\quad + \frac{3.2c_{13}}{\mathcal{R}_{\rho,\lambda,\sigma}^4(c_2 - c_1)} \left[\Psi^{k-4}(\Delta) + \Psi^{k-4}(c_1) \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14} + 3.2c_{13})}{\mathcal{R}_{\rho,\lambda,\sigma}^4(c_2 - c_1)} [\Psi^{k-4}(\Delta) - \Psi^{k-4}(c_1)] \\
& + \dots - \frac{(k!c_{1k})}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} [\Psi(c_1) + \Psi(\Delta)] + \frac{2k!c_{1k}}{\mathcal{R}_{\rho,\lambda,\sigma}^{k+1}(c_2 - c_1)} \int_{c_1}^{\Delta} \Psi(x) dx \\
& = \frac{(c_{1k} + \dots + c_{11} + 2c_{10})}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} [\Psi^{k-1}(\Delta) - \Psi^{k-1}(c_1)] \\
& - \frac{(kc_{1k} + \dots + 2c_{12})}{\mathcal{R}_{\rho,\lambda,\sigma}^2(c_2 - c_1)} [\Psi^{k-2}(\Delta) + \Psi^{k-2}(c_1)] \\
& + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})}{\mathcal{R}_{\rho,\lambda,\sigma}^3(c_2 - c_1)} [\Psi^{k-3}(\Delta) - \Psi^{k-3}(c_1)] \\
& - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})}{\mathcal{R}_{\rho,\lambda,\sigma}^4(c_2 - c_1)} [\Psi^{k-4}(\Delta) - \Psi^{k-4}(c_1)] \\
& + \dots - \frac{k!c_{1k}}{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)} [\Psi(c_1) + \Psi(\Delta)] + \frac{2k!c_{1k}}{\mathcal{R}_{\rho,\lambda,\sigma}^{k+1}(c_2 - c_1)} \int_{c_1}^{\Delta} \Psi(x) dx.
\end{aligned}$$

Multiplying the above equality by $\frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!c_{1k}}$, we obtain:

$$\begin{aligned}
& \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1)} \int_{c_1}^{\Delta} \Psi(x) dx - \frac{\Psi(c_1) + \Psi(\Delta)}{2} + \dots \\
& - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})\phi^{k-4}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-4}(\Delta) + \Psi^{k-4}(c_1)] \\
& + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})\phi^{k-3}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-3}(\Delta) - \Psi^{k-3}(c_1)] \\
& - \frac{(kc_{1k} + \dots + 2c_{12})\phi^{k-2}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-2}(\Delta) + \Psi^{k-2}(c_1)] \\
& + \frac{(c_{1k} + \dots + c_{11} + 2c_{10})\phi^{k-1}(c_2, c_1)}{2k!c_{1k}} [\Psi^{k-1}(\Delta) - \Psi^{k-1}(c_1)] \\
& = \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!c_{1k}} \int_0^1 (\Lambda(c_1, \mu)) [\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})] d\mu.
\end{aligned}$$

This completes the proof. $\square$

2.3. Trapezium-like Inequalities

Now using Lemma 1, we derive new trapezium-like inequalities.

Theorem 4. Let $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ be a function such that $\Psi^{(k)}$ exists on $\mathcal{L}^\circ$ and $\Psi^{(k)}$ is integrable on $\mathcal{L}$, where $c_1, c_2 \in \mathcal{L}$ with $\mathcal{R}_{\rho,\lambda,\sigma}(c_2 - c_1) > 0$ and k is an even number. If $|\Psi^{(k)}|$ is the $\mathcal{R}_\gamma$ -convex function on $\mathcal{L}$, then:

$$\begin{aligned}
|\mathcal{M}(c_1, c_2; k)| & \leq \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} [|\Psi^k(c_1)| + |\Psi^k(c_2)|] \\
& \times \int_0^1 [c_{1k}\mu^k + \dots + c_{11}\mu + c_{10}] [\mu\gamma(\mu) + (1-\mu)\gamma(1-\mu)] d\mu.
\end{aligned}$$

Proof. Using Lemma 1 and the γ -preinvexity of $|\Psi^{(k)}|$, we have:

$$\begin{aligned}
& |\mathcal{M}(c_1, c_2; k)| \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| |\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})| d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})| \right] d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu)| + |\Psi^k(\Omega_{1-\mu})| \right] d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \\
& \quad \times \left[(1-\mu)\gamma(1-\mu)|\Psi^k(c_1)| + \mu\gamma(\mu)|\Psi^k(c_2)| + \mu\gamma(\mu)|\Psi^k(c_1)| + (1-\mu)\gamma(1-\mu)|\Psi^k(c_2)| \right] d\mu \\
& = \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \\
& \quad \times \int_0^1 |\Lambda(c_1, \mu)| [\mu\gamma(\mu) + (1-\mu)\gamma(1-\mu)] d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \\
& \quad \times \int_0^1 \left[|c_{1k}\mu^k| + \dots + |c_{11}\mu| + |c_{10}| \right] [\mu\gamma(\mu) + (1-\mu)\gamma(1-\mu)] d\mu.
\end{aligned}$$

This completes the proof. $\square$

Now we will discuss some special cases of Theorem 4.

I. If $\gamma(\mu) = 1$, then Theorem 4 reduces to the following result in the class of generalized convex functions.

Corollary 1. Under the assumptions of Theorem 4 if $|\Psi^{(k)}|$ is generalized convex function on $\mathcal{L}$, then

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \sum_{j=0}^k \frac{|c_{1j}|}{j+1}.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 4 reduces to the following result in the class of $\mathcal{R}_P$ -convex function.

Corollary 2. Under the assumptions of Theorem 4 if $|\Psi^{(k)}|$ is an $\mathcal{R}_P$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \sum_{j=0}^k \frac{|c_{1j}|}{j+1}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 4 reduces to the following result in the class of $\mathcal{R}_{s_1}$ -convex functions.

Corollary 3. Under the assumptions of Theorem 4 if $|\Psi^{(k)}|$ is an $\mathcal{R}_{s_1}$ -convex function on $\mathcal{L}$, then

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \sum_{j=0}^k |c_{1j}| \left(\frac{1}{j+s+1} - B(j+1, s+1) \right),$$

where $B(\cdot, \cdot)$ is the beta function and is defined as:

$$B(x, y) = \int_0^1 v^{x-1} (1-v)^{y-1} dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 4 reduces to the following result in the class of $\mathcal{R}_{s_2}$ -convex functions.

Corollary 4. Under the assumptions of Theorem 4, if $|\Psi^{(k)}|$ is an $\mathcal{R}_{s_2}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \sum_{j=0}^k |c_{1j}| \left(\frac{1}{j-s+1} - B(j+1, 1-s) \right).$$

V. If $\gamma(\mu) = 1 - \mu$, then Theorem 4 reduces to the following result in the class of $\mathcal{R}_{tgs}$ -convex functions.

Corollary 5. Under the assumptions of Theorem 4, if $|\Psi^{(k)}|$ is an $\mathcal{R}_{tgs}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left[|\Psi^k(c_1)| + |\Psi^k(c_2)| \right] \sum_{j=0}^k \frac{|c_{1j}|}{(j+2)(j+3)}.$$

Theorem 5. Let $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ be a function such that $\Psi^{(k)}$ exists on $\mathcal{L}^\circ$ and $\Psi^{(k)}$ is integrable on $\mathcal{L}$, where $c_1, c_2 \in \mathcal{L}$ with $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) > 0$ and k is an even number. If $|\Psi^{(k)}|^q$ is the $\mathcal{R}_\gamma$ -convex function on $\mathcal{L}$, then for $p^{-1} + q^{-1} = 1$, we have:

$$\begin{aligned} |\mathcal{M}(c_1, c_2; k)| &\leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left[\left(|\Psi^k(c_1)|^q \int_0^1 (1-\mu)\gamma(1-\mu) d\mu + |\Psi^k(c_2)|^q \int_0^1 \mu\gamma(\mu) d\mu \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(|\Psi^k(c_1)|^q \int_0^1 \mu\gamma(\mu) d\mu + |\Psi^k(c_2)|^q (1-\mu)\gamma(1-\mu) d\mu \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Using Lemma 1, Hölder's inequality, Minkowski's integral inequality, and the γ -preinvexity of $|\Psi^{(k)}|^q$, we have:

$$\begin{aligned} |\mathcal{M}(c_1, c_2; k)| &\leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| |\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})| d\mu \\ &\leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})| \right] d\mu \\ &\leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu)| + |\Psi^k(\Omega_{1-\mu})| \right] d\mu \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 |\Psi^k(\Omega_\mu)|^q d\mu \right)^{\frac{1}{q}} \\
&\quad + \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 |\Psi^k(\Omega_{1-\mu})|^q d\mu \right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left[(1-\mu)\gamma(1-\mu)|\Psi^k(c_1)| + \mu\gamma(\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \\
&\quad + \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left[\mu\gamma(\mu)|\Psi^k(c_1)|^q + (1-\mu)\gamma(1-\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left[\left(\int_0^1 \left[(1-\mu)\gamma(1-\mu)|\Psi^k(c_1)| + \mu\gamma(\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 \left[\mu\gamma(\mu)|\Psi^k(c_1)|^q + (1-\mu)\gamma(1-\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \right] \\
&= \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left[\left(|\Psi^k(c_1)|^q \int_0^1 (1-\mu)\gamma(1-\mu) d\mu + |\Psi^k(c_2)|^q \int_0^1 \mu\gamma(\mu) d\mu \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(|\Psi^k(c_1)|^q \int_0^1 \mu\gamma(\mu) d\mu + |\Psi^k(c_2)|^q \int_0^1 (1-\mu)\gamma(1-\mu) d\mu \right)^{\frac{1}{q}} \right].
\end{aligned}$$

This completes the proof. $\square$

Now we will discuss some special cases of Theorem 5.

I. If $\gamma(\mu) = 1$, then Theorem 5 reduces to the following result in the class of generalized convex functions.

Corollary 6. Under the assumptions of Theorem 5, if $|\Psi^{(k)}|^q$ is a generalized convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho,\lambda,\sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(\frac{|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q}{2} \right)^{\frac{1}{q}}.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 5 reduces to the following result in the class of $\mathcal{R}_P$ -convex functions.

Corollary 7. Under the assumptions of Theorem 5, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_p$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q \right)^{\frac{1}{q}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 5 reduces to the following result in the class of $\mathcal{R}_{s_1}$ -convex functions.

Corollary 8. Under the assumptions of Theorem 5, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{s_1}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(\frac{|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q}{1+s} \right)^{\frac{1}{q}}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 5 reduces to the following result in the class of $\mathcal{R}_{s_2}$ -convex functions.

Corollary 9. Under the assumptions of Theorem 5, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{s_2}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(\frac{|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q}{1-s} \right)^{\frac{1}{q}}.$$

V. If $\gamma(\mu) = 1 - \mu$, then Theorem 5 reduces to the following result in the class of $\mathcal{R}_{tgs}$ -convex functions.

Corollary 10. Under the assumptions of Theorem 5, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{tgs}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(\frac{|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q}{6} \right)^{\frac{1}{q}}.$$

Theorem 6. Let $\Psi : \mathcal{L} \rightarrow \mathbb{R}$ be a function such that $\Psi^{(k)}$ exists on $\mathcal{L}^\circ$ and $\Psi^{(k)}$ is integrable on $\mathcal{L}$, where $c_1, c_2 \in \mathcal{L}$ with $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) > 0$ and k is an even number. If $|\Psi^{(k)}|^q$ is the $\mathcal{R}_\gamma$ -convex function on $\mathcal{L}$ for $q \geq 1$, then:

$$\begin{aligned} |\mathcal{M}(c_1, c_2; k)| &\leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\ &\times \left[\left(|\Psi^k(c_1)|^q \int_0^1 (1-\mu)\gamma(1-\mu)|\Lambda(c_1, \mu)|d\mu + |\Psi^k(c_2)|^q \int_0^1 \mu\gamma(\mu)|\Lambda(c_1, \mu)|d\mu \right)^{\frac{1}{q}} \right. \\ &\left. + \left(|\Psi^k(c_1)|^q \int_0^1 \mu\gamma(\mu)|\Lambda(c_1, \mu)|d\mu + |\Psi^k(c_2)|^q \int_0^1 (1-\mu)\gamma(1-\mu)|\Lambda(c_1, \mu)|d\mu \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Using Lemma 1, the power mean integral inequality, and the γ -preinvexity of $|\Psi^{(k)}|^q$, we have:

$$\begin{aligned}
& |\mathcal{M}(c_1, c_2; k)| \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| |\Psi^k(\Omega_\mu) + \Psi^k(\Omega_{1-\mu})| d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu)| + |\Psi^k(\Omega_{1-\mu})| \right] d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \int_0^1 |\Lambda(c_1, \mu)| \left[|\Psi^k(\Omega_\mu)| + |\Psi^k(\Omega_{1-\mu})| \right] d\mu \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)| d\mu \right)^{1-\frac{1}{q}} \left(\int_0^1 |\Lambda(c_1, \mu)| |\Psi^k(\Omega_\mu)|^q d\mu \right)^{\frac{1}{q}} \\
& \quad + \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\int_0^1 |\Lambda(c_1, \mu)| d\mu \right)^{1-\frac{1}{q}} \left(\int_0^1 |\Lambda(c_1, \mu)| |\Psi^k(\Omega_{1-\mu})|^q d\mu \right)^{\frac{1}{q}} \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{j+1} \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 |\Lambda(c_1, \mu)| \left[(1-\mu)\gamma(1-\mu)|\Psi^k(c_1)| + \mu\gamma(\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \\
& \quad + \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{j+1} \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 |\Lambda(c_1, \mu)| \left[\mu\gamma(\mu)|\Psi^k(c_1)|^q + (1-\mu)\gamma(1-\mu)|\Psi^k(c_2)|^q \right] d\mu \right)^{\frac{1}{q}} \\
& \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!c_{1k}} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\left(|\Psi^k(c_1)|^q \int_0^1 (1-\mu)\gamma(1-\mu)|\Lambda(c_1, \mu)| d\mu + |\Psi^k(c_2)|^q \int_0^1 \mu\gamma(\mu)|\Lambda(c_1, \mu)| d\mu \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(|\Psi^k(c_1)|^q \int_0^1 \mu\gamma(\mu)|\Lambda(c_1, \mu)| d\mu + |\Psi^k(c_2)|^q \int_0^1 (1-\mu)\gamma(1-\mu)|\Lambda(c_1, \mu)| d\mu \right)^{\frac{1}{q}} \right].
\end{aligned}$$

This completes the proof. $\square$

Now we will discuss some special cases of Theorem 6.

I. If $\gamma(\mu) = 1$, then Theorem 6 reduces to the following result in the class of generalized convex functions.

Corollary 11. Under the assumptions of Theorem 6, if $|\Psi^{(k)}|^q$ is a generalized convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\ \times \left[\left(|\Psi^k(c_1)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+1)(j+2)} + |\Psi^k(c_2)|^q \sum_{j=0}^k \frac{|c_{1j}|}{j+2} \right)^{\frac{1}{q}} \right. \\ \left. + \left(|\Psi^k(c_1)|^q \sum_{j=0}^k \frac{|c_{1j}|}{j+2} + |\Psi^k(c_2)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+1)(j+2)} \right)^{\frac{1}{q}} \right].$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 6 reduces to the following result in the class of $\mathcal{R}_P$ -convex functions.

Corollary 12. Under the assumptions of Theorem 6, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_P$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right) \left(|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q \right)^{\frac{1}{q}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 6 reduces to the following result in the class of $\mathcal{R}_{s_1}$ -convex functions.

Corollary 13. Under the assumptions of Theorem 6, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{s_1}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\ \times \left[\left(|\Psi^k(c_1)|^q \sum_{j=0}^k B(j+1, s+1) + |\Psi^k(c_2)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+s+1)} \right)^{\frac{1}{q}} \right. \\ \left. + \left(|\Psi^k(c_1)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+s+1)} + |\Psi^k(c_2)|^q \sum_{j=0}^k B(j+1, s+1) \right)^{\frac{1}{q}} \right].$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 6 reduces to the following result in the class of $\mathcal{R}_{s_2}$ -convex functions.

Corollary 14. Under the assumptions of Theorem 6, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{s_2}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2k!|c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\ \times \left[\left(|\Psi^k(c_1)|^q \sum_{j=0}^k B(j+1, 1-s) + |\Psi^k(c_2)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j-s+1)} \right)^{\frac{1}{q}} \right. \\ \left. + \left(|\Psi^k(c_1)|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j-s+1)} + |\Psi^k(c_2)|^q \sum_{j=0}^k B(j+1, 1-s) \right)^{\frac{1}{q}} \right].$$

V. If $\gamma(\mu) = 1 - \mu$, then Theorem 6 reduces to the following result in the class of $\mathcal{R}_{tgs}$ -convex functions.

Corollary 15. Under the assumptions of Theorem 6, if $|\Psi^{(k)}|^q$ is an $\mathcal{R}_{tgs}$ -convex function on $\mathcal{L}$, then:

$$|\mathcal{M}(c_1, c_2; k)| \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{k! |c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(j+2)(j+3)} \right)^{\frac{1}{q}} \left(|\Psi^k(c_1)|^q + |\Psi^k(c_2)|^q \right)^{\frac{1}{q}}.$$

2.4. Applications

We now consider the following special means for different positive real numbers c_1 and c_2 , where $c_1 < c_2$:

- Arithmetic mean: $A(c_1, c_2) = \frac{c_1 + c_2}{2}$;
- k -logarithmic mean: $L_k(c_1, c_2) = \left(\frac{c_2^{k+1} - c_1^{k+1}}{(k+1)(c_2 - c_1)} \right)^{\frac{1}{k}}, \quad k \in \mathbb{R} \setminus \{0, -1\}$.

Proposition 1. Let $c_1, c_2 \in \mathbb{R}, 0 < c_1 < c_2$, where k is an even number; then:

$$\begin{aligned} & \left| L_{k+1}^{k+1}(c_1, c_2) - A(c_1^k, c_2^k) + \dots \right. \\ & \quad - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})(c_2 - c_1)^{k-4}}{2.4!c_{1k}} [c_2^4 + c_1^4] \\ & \quad + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})(c_2 - c_1)^{k-3}}{2.3!c_{1k}} [c_2^3 - c_1^3] \\ & \quad - \frac{(kc_{1k} + \dots + 2c_{12})(c_2 - c_1)^{k-2}}{2.2!c_{1k}} [c_2^2 + c_1^2] \\ & \quad \left. + \frac{(c_{1k} + \dots + c_{11} + 2c_{10})(c_2 - c_1)^{k-1}}{2c_{1k}} (c_2 - c_1) \right| \\ & \leq \frac{(c_2 - c_1)^k}{2|c_{1k}|} [|c_1^k| + |c_2^k|] \sum_{j=0}^k |c_{1j}| \left(\frac{1}{j+s+1} - B(j+1, s+1) \right). \end{aligned}$$

Proof. The proof directly follows from Corollary 3 by taking $\Psi : [0, 1] \rightarrow [0, 1], \Psi(x) = x^k$ and $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) = c_2 - c_1$. $\square$

Proposition 2. Let $c_1, c_2 \in \mathbb{R}, 0 < c_1 < c_2$, where k is an even number; then:

$$\begin{aligned} & \left| L_{k+1}^{k+1}(c_1, c_2) - A(c_1^k, c_2^k) + \dots \right. \\ & \quad - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})(c_2 - c_1)^{k-4}}{2.4!c_{1k}} [c_2^4 + c_1^4] \\ & \quad + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})(c_2 - c_1)^{k-3}}{2.3!c_{1k}} [c_2^3 - c_1^3] \\ & \quad - \frac{(kc_{1k} + \dots + 2c_{12})(c_2 - c_1)^{k-2}}{2.2!c_{1k}} [c_2^2 + c_1^2] \\ & \quad \left. + \frac{(c_{1k} + \dots + c_{11} + 2c_{10})(c_2 - c_1)^{k-1}}{2c_{1k}} (c_2 - c_1) \right| \\ & \leq \frac{(c_2 - c_1)^k}{(k!)^{1-\frac{1}{q}} |c_{1k}|} \left(\sum_{j=0}^k \frac{|c_{1j}|}{(jp+1)^{\frac{1}{p}}} \right) \left(\frac{|c_1^k|^q + |c_2^k|^q}{1+s} \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof directly follows from Corollary 8 by taking $\Psi : [0, 1] \rightarrow [0, 1], \Psi(x) = x^k$ and $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) = c_2 - c_1$. $\square$

Proposition 3. Let $c_1, c_2 \in \mathbb{R}, 0 < c_1 < c_2$, where k is an even number; then:

$$\begin{aligned} & \left| L_{k+1}^{k+1}(c_1, c_2) - A(c_1^k, c_2^k) + \dots \right. \\ & \quad - \frac{(k(k-1)(k-2)c_{1k} + \dots + 4.3.2c_{14})(c_2 - c_1)^{k-4}}{2.4!c_{1k}} [c_2^4 + c_1^4] \\ & \quad + \frac{(k(k-1)c_{1k} + \dots + 3.2c_{13})(c_2 - c_1)^{k-3}}{2.3!c_{1k}} [c_2^3 - c_1^3] \\ & \quad - \frac{(kc_{1k} + \dots + 2c_{12})(c_2 - c_1)^{k-2}}{2.2!c_{1k}} [c_2^2 + c_1^2] \\ & \quad \left. + \frac{(c_{1k} + \dots + c_{11} + 2c_{10})(c_2 - c_1)^{k-1}}{2c_{1k}} (c_2 - c_1) \right| \\ & \leq \frac{\mathcal{R}_{\rho, \lambda, \sigma}^k(c_2 - c_1)}{2|c_{1k}|} \left(\frac{1}{k!} \sum_{j=0}^k \frac{|c_{1j}|}{(j+1)} \right)^{1-\frac{1}{q}} \\ & \quad \times \left[\left(|c_1^k|^q \sum_{j=0}^k B(j+1, s+1) + |c_2^k|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+s+1)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|c_1^k|^q \sum_{j=0}^k \frac{|c_{1j}|}{(j+s+1)} + |c_2^k|^q \sum_{j=0}^k B(j+1, s+1) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The proof directly follows from Corollary 13 by taking $\Psi : [0, 1] \rightarrow [0, 1], \Psi(x) = x^k$ and $\mathcal{R}_{\rho, \lambda, \sigma}(c_2 - c_1) = c_2 - c_1$. $\square$

3. Conclusions

We have introduced the class of $\mathcal{R}_\gamma$ -convex functions involving Raina's function. We have shown that by making suitable choices of the real function $\gamma(\cdot)$, we can recapture some other new classes of the classical convexity. This shows that our new class relates to several other unrelated classes of the convex functions. We then studied this class from the perspective of integral inequalities of the trapezium type. In order to establish our main results, we have derived a new integral identity involving k -th order differentiable functions. Finally, we have presented some applications to means that show the significance of our obtained results. We hope that the ideas and techniques of this paper will inspire interested readers working in this field. We would like to point out here that the results of this paper can be extended by using the class of higher-order $\mathcal{R}_\gamma$ -convex functions. This will be an interesting problem for future research.

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