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Important Study on the ∇ Dynamic Hardy–Hilbert-Type Inequalities on Time Scales with Applications

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Abstract: The main objective of the present article is to prove some new ∇ dynamic inequalities of Hardy–Hilbert-type on time scales. We present and prove very important generalized results with the help of the Fenchel–Legendre transform, submultiplicative functions, and Hölder's and Jensen's inequality on time scales. We obtain some well-known time scale inequalities due to Hardy–Hilbert inequalities. For some specific time scales, we further show some relevant inequalities as special cases: integral inequalities and discrete inequalities. Symmetry plays an essential role in determining the correct methods for solutions to dynamic inequalities

Keywords: Hardy–Hilbert's inequality; Hölder's and Jensen's inequality; time scale

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1. Background and Introduction to ∇-Time Scales Calculus

In this section, we give several foundational definitions and pieces of notation for basic calculus of time scales. Stefan Hilger initiated the theory of time scales in his PhD thesis [1] in order to unify discrete and continuous analysis (see [2]). Since then, this theory has received a lot of attention. The basic notion is to establish a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary closed subset of the reals $\mathbb{R}$; see [3,4]. The three most common examples of calculus on time scales are continuous calculus, discrete calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The book by Bohner and Peterson [5] on the subject of time scales briefs and organizes much of time scale calculus.

We begin with the definition of a time scale.

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set of all real numbers $\mathbb{R}$.*

Now, we define two operators playing a central role in the analysis on time scales.

Definition 2. *If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In the previous two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\emptyset$ is the empty set.

If $\mathbb{T} \in \{[a,b],[a,\infty),(-\infty,a],\mathbb{R}\}$, then $\sigma(t) = \rho(t) = t$. We note that $\sigma(t)$ and $\rho(t)$ in $\mathbb{T}$ when $t \in \mathbb{T}$ because $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$.

Next, we define the graininess functions as follows:

Definition 3. (i) *The forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by*

$$\mu(t) = \sigma(t) - t.$$

(ii) *The backward graininess function $\nu : \mathbb{T} \to [0,\infty)$ is defined by*

$$\nu(t) = t - \rho(t).$$

With the operators defined above, we can begin to classify the points of any time scale depending on the proximities of their neighboring points in the following manner.

Definition 4. *Let $\mathbb{T}$ be a time scale. A point $t \in \mathbb{T}$ is said to be:*

(1) *Right-scattered if $\sigma(t) > t$;*
(2) *Left-scattered if $\rho(t) < t$;*
(3) *Isolated if $\rho(t) < t < \sigma(t)$;*
(4) *Right-dense if $\sigma(t) = t$;*
(5) *Left-dense if $\rho(t) = t$;*
(6) *Dense if $\rho(t) = t = \sigma(t)$.*

The closed interval on a time scale is defined by

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}.$$

Open intervals and half-open intervals are defined similarly.

Two sets we need to consider are $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$, which are defined as follows: $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has M as a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has m as a right-scattered minimum, and $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. In fact, we can write

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \big(\rho(\sup \mathbb{T}), \sup \mathbb{T}\big], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty, \end{cases}$$

and

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \setminus \big[\inf \mathbb{T}, \sigma(\inf \mathbb{T})\big), & \text{if} \quad \inf \mathbb{T} > -\infty, \\ \mathbb{T}, & \text{if} \quad \inf \mathbb{T} = -\infty. \end{cases}$$

Definition 5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function defined on a time scale $\mathbb{T}$. Then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by*

$$f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t)), \qquad t \in \mathbb{T},$$

and the function $f^\rho : \mathbb{T} \to \mathbb{R}$ by

$$f^\rho(t) = (f \circ \rho)(t) = f(\rho(t)), \qquad t \in \mathbb{T}.$$

We introduce the nabla derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}_\kappa$ as follows:

Definition 6. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_\kappa$. We define $f^\nabla(t)$ as the real number (provided it exists) with the property that for any $\epsilon > 0$, there exists a neighborhood N of t (i.e., $N = (t-\delta, t+\delta)_{\mathbb{T}}$ for some $\delta > 0$) such that*

$$\big|[f^\rho(t) - f(s)] - f^\nabla(t)[\rho(t) - s]\big| \le \epsilon|\rho(t) - s| \quad for\ every \quad s \in N.$$

We say that $f^\nabla(t)$ is the nabla derivative of f at t.

Theorem 1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $t \in \mathbb{T}_\kappa$. Then:*

(i) *f being nabla differentiable at t implies f is continuous at t.*

(ii) *f being continuous at left-scattered t implies f is nabla differentiable at t with*

$$f^\nabla(t) = \frac{f(t) - f^\rho(t)}{\nu(t)}.$$

(iii) *If t is left-dense, then f is nabla differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In such a case,

$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *$f^\rho(t) = f(t) - \nu(t) f^\nabla(t)$ whenever f is nabla differentiable at t.*

Example 1. *(i)* *Let $\mathbb{T} = \mathbb{R}$. Then*

$$f^\nabla(t) = f'(t).$$

(ii) *Let $\mathbb{T} = \mathbb{Z}$. Then*

$$f^\nabla(t) = \nabla f(t) = f(t) - f(t-1),$$

where ∇ is the backward difference operator.

Theorem 2. *Let f and $g : \mathbb{T} \to \mathbb{R}$ be functions that are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*

$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$

(ii) *If $\alpha \in \mathbb{R}$ is a constant, then the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*

$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t, and we get the product rule*

$$(fg)^\nabla(t) = f^\nabla(t) g(t) + f^\rho(t) g^\nabla(t) = f(t) g^\nabla(t) + f^\nabla(t) g^\rho(t).$$

(iv) *The function $\dfrac{1}{f} : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*

$$\left(\frac{1}{f}\right)^\nabla(t) = -\frac{f^\nabla(t)}{f(t) f^\rho(t)}, \qquad f(t) f^\rho(t) \neq 0.$$

(v) *The quotient $\dfrac{f}{g} : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t, and we get the quotient rule*

$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t) g(t) - f(t) g^\nabla(t)}{g(t) g^\rho(t)}, \qquad g(t) g^\rho(t) \neq 0.$$

Definition 7. *We say that a function $F : \mathbb{T} \to \mathbb{R}$ is a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. In this case, the nabla integral of f is defined by*

$$\int_a^t f(t) \nabla \tau = F(t) - F(a) \qquad for\ all \quad t \in \mathbb{T}_\kappa.$$

Now, we introduce the set of all ld-continuous functions in order to find a class of functions that have nabla antiderivatives.

Definition 8 (Ld-Continuous Function)**.** *We say that the function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points of $\mathbb{T}$.*

Theorem 3 (Existence of Nabla Antiderivatives)**.** *Every ld-continuous function possess a nabla antiderivative.*

Theorem 4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a ld-continuous function, and let $t \in \mathbb{T}_\kappa$. Then*

$$\int_{\rho(t)}^{t} f(\tau)\nabla\tau = \nu(t)f(t).$$

Theorem 5. *If $f^\nabla(t) \geq 0$ (respectively, $f^\nabla(t) \leq 0$), then f is nondecreasing (respectively, nonincreasing).*

Theorem 6. *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{ld}$, then*

(i) $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t;$
(ii) $\int_a^b \alpha f(t)\nabla t = \alpha \int_a^b f(t)\nabla t;$
(iii) $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t;$
(iv) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t;$
(v) $\int_a^a f(t)\nabla t = 0;$
(vi) *if* $f(t) \geq g(t)$ *on* $[a, b)_\mathbb{T}$, *then* $\int_a^b f(t)\nabla t \geq \int_a^b g(t)\nabla t;$
(vii) *if* $f(t) \geq 0$ *on* $\in [a, b)_\mathbb{T}$, *then* $\int_a^b f(t)\nabla t \geq 0.$

Theorem 7. *Let $f : \mathbb{T} \to \mathbb{R}$ be a ld-continuous function, and $a, b \in \mathbb{T}$.*

(i) *In the case that $\mathbb{T} = \mathbb{R}$, we have*

$$\int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

where the integral on the right-hand side is the Riemann integral from calculus.

(ii) *In the case that $[a, b]_\mathbb{T}$ consists of only isolated points, we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{t\in(a,b]_\mathbb{T}} \nu(t)f(t), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum\limits_{t\in(b,a]_\mathbb{T}} \nu(t)f(t), & \text{if} \quad a > b. \end{cases}$$

(iii) *In the case that $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{k=\frac{a}{h}+1}^{\frac{b}{h}} hf(hk), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum\limits_{k=\frac{b}{h}+1}^{\frac{a}{h}} hf(hk), & \text{if} \quad a > b. \end{cases}$$

(iv) *In the case that $\mathbb{T} = \mathbb{Z}$, we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum_{t=a+1}^{b} f(t), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum_{t=b+1}^{a} f(t), & \text{if} \quad a > b. \end{cases}$$

The formula for nabla integration by parts is as follows:

$$\int_a^b f(t)g^{\nabla}(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)g^{\rho}(t)\nabla t.$$

The following theorem gives a relationship between the delta and nabla derivative.

Theorem 8. *(i) Let $f : \mathbb{T} \to \mathbb{R}$ be delta differentiable on $\mathbb{T}^{\kappa}$. Then f is nabla differentiable at t and $f^{\nabla}(t) = f^{\Delta}(\rho(t))$ for any $t \in \mathbb{T}_{\kappa}$ that satisfies $\sigma(\rho(t)) = t$. If, in addition, f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is nabla differentiable at t, and $f^{\nabla}(t) = f^{\Delta}(\rho(t))$ for each $t \in \mathbb{T}_{\kappa}$.*
(ii) Let $f : \mathbb{T} \to \mathbb{R}$ be nabla differentiable on $\mathbb{T}_{\kappa}$. Then f is delta differentiable at t and $f^{\Delta}(t) = f^{\nabla}(\sigma(t))$ for any $t \in \mathbb{T}^{\kappa}$ that satisfies $\rho(\sigma(t)) = t$. If, in addition, f^{∇} is continuous on $\mathbb{T}_{\kappa}$, then f is delta differentiable at t, and $f^{\Delta}(t) = f^{\nabla}(\sigma(t))$ for each $t \in \mathbb{T}^{\kappa}$.

We will use the following relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Note that:

(*i*) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered} \rho(t) = t, \quad \nu(t) = 0, \quad f^{\nabla}(t) = f'(t), \\ \int_a^b f(t)\nabla t = \int_a^b f(t)dt. \end{gathered} \tag{1}$$

(*ii*) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered} \rho(t) = t - 1, \quad \nu(t) = 1, \\ f^{\nabla}(t) = \nabla f(t), \\ \sum_{t=a}^{b-1} f(t), \quad \int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t), \end{gathered} \tag{2}$$

where ∇ are the forward difference operators.

Now, we present the Fenchel–Legendre transform that will be needed in the proof of our results. We refer to example to [6–8] for more details.

Definition 9. *The function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is called coercive iff*

$$f(x) \longrightarrow \infty \ \ as \ \ \|x\| \longrightarrow \infty.$$

Definition 10. *Suppose $\psi : \mathbb{R}^i \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function: $\psi \neq +\infty$; i.e.,* $\mathrm{Dom}(\psi) = \{\tilde{w} \in \mathbb{R}^i, |\psi(\tilde{w}) < \infty\} \neq \emptyset$*. Then the Fenchel–Legendre transform is defined as:*

$$\psi^* : \mathbb{R}^i \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad z \longrightarrow \psi^*(\tilde{z}) = \sup\{< \tilde{z}, \tilde{w} > -\psi(\tilde{w}), \tilde{w} \in \mathrm{Dom}(\psi)\} \tag{3}$$

The scalar product is denoted by $< ., . >$ on $\mathbb{R}^i$, and $\psi \longrightarrow \psi^*$ is said to be the conjugate operation.

The domain of ψ^* is the set of slopes of all affine functions minorizing the function h over $\mathbb{R}^n$. An equivalent formula for (3) is obtained in the next corollary:

Corollary 1. *Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a strictly convex, differentiable and 1-coercive function. Then*

$$\psi^*(y) =< y, (\nabla\psi)^{-1}(y) > -\psi((\nabla\psi)^{-1}(y)), \tag{4}$$

for all $y \in \mathrm{Dom}(\psi^*)$, where $< ., . >$ denotes the inner product in $\mathbb{R}^n$.

Lemma 1 (Fenchel–Young inequality [6])**.** *Suppose a function ψ and suppose ψ^* Fenchel–Legendre are transforms of ψ, we get*

$$< \tilde{w}, \tilde{z} >\leqslant \psi(\tilde{w}) + \psi^*(\tilde{z}), \tag{5}$$

for all $\tilde{w} \in \mathrm{Dom}(\psi)$, and $\tilde{z} \in \mathrm{Dom}(\psi^*)$.

Definition 11. *We said Ω is submultiplicative function on $[0, \infty)$ if*

$$\Omega(\tilde{w}\tilde{z}) \leqslant \Omega(\tilde{w})\Omega(\tilde{z}), \ \ \forall \ \tilde{w}, \tilde{z} \geqslant 0. \tag{6}$$

The celebrated Hardy–Hilbert integral inequality [9] is

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y \leq \frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^\infty f^p(x)\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)\mathrm{d}y\right]^{\frac{1}{q}}, \tag{7}$$

where $p > 1$, $q = \frac{p}{p-1}$ and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is best possible. As special case, if $p = q = 2$, the inequality (8) is reduced to the classical Hilbert integral inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y \leq \pi\left[\int_0^\infty f^2(x)\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_0^\infty g^2(y)\mathrm{d}y\right]^{\frac{1}{2}}, \tag{8}$$

where the coefficient π is the best possible.

In [10], Pachappte established a discrete Hilbert-type inequality and its integral version, as in the following two theorems:

Theorem 9. *Let $\{a_m\}$, $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, \ldots, k$, and $n = 1, \ldots, r$ with $a_0 = b_0 = 0$; and let $\{p_m\}$, $\{q_n\}$, be two positive sequences of real numbers defined for $m = 1, \ldots, k$. and $n = 1, \ldots, r$ where k, r are natural numbers. Define $P_m = \sum_{s=1}^m p_s$ and $Q_n = \sum_{t=1}^n q_t$. Let Φ and Ψ be two real-valued nonnegative, convex, and submultiplicative functions defined on $[0, \infty)$. Then*

$$\begin{aligned}\sum_{m=1}^k\sum_{n=1}^r \frac{\Phi(a_m)\Psi(b_n)}{m+n} \leqslant M(k,r)&\left(\sum_{m=1}^k (k-m+1)\left(p_m\Phi\left(\frac{\nabla a_m}{p_m}\right)\right)^2\right)^{\frac{1}{2}} \\ &\times\left(\sum_{n=1}^r (r-n+1)\left(q_n\Psi\left(\frac{\nabla b_n}{q_n}\right)\right)^2\right)^{\frac{1}{2}},\end{aligned} \tag{9}$$

where

$$M(k,r) = \frac{1}{2}\left(\sum_{m=1}^k\left(\frac{\Phi(P_m)}{p_m}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^r\left(\frac{\Psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}}$$

and $\nabla a_m = a_m - a_{m-1}$, $\nabla b_n = b_n - b_{n-1}$.

Theorem 10. *Let $f \in C^1[[0,x], \mathbb{R}^+]$, $g \in C^1[[0,y], \mathbb{R}^+]$ with $f(0) = g(0) = 0$, and let $p(\xi)$, $q(\tau)$ be two positive functions defined for $\xi \in [0,x)$ and $\tau \in [0,y)$. Let $P(s) = \int_0^s p(\xi)d\xi$ and*

$Q(t) = \int_0^t p(\tau)d\tau$ for $s \in [0, x)$ and $t \in [0, y)$ where x, y are positive real numbers. Let Φ, and Ψ be as in Theorem 9. Then

$$\int_0^x \int_0^y \frac{\Phi(f(s))\Psi(g(t))}{s+t} dsdt \leqslant L(x,y)\left(\int_0^x (x-s)\left(p(s)\Phi\left(\frac{f'(s)}{p(s)}\right)^2 ds\right)^{\frac{1}{2}}\right. \\ \times \left(\int_0^y (y-t)\left(q(t)\Psi\left(\frac{g'(t)}{q(t)}\right)^2 dt\right)^{\frac{1}{2}} \tag{10}$$

where

$$L(x,y) = \frac{1}{2}\left(\int_0^x \left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}} \left(\int_0^y \left(\frac{\Psi(Q(t))}{Q(t)}\right)^2 dt\right)^{\frac{1}{2}}.$$

In [11], Handley et al. gave general versions of inequalities (9) and (10) in the following two theorems:

Theorem 11. *Let $\{a_{i,m_i}\}$ $(i = 1, 2, \ldots, n)$ be n sequences of nonnegative real numbers defined for $m_i = 1, \ldots, k_i$ with $a_{1,0} = a_{2,0} \ldots a_{n,0} = 0$, and let $\{p_{i,m_i}\}$ be n sequences of positive real numbers defined for $m_i = 1, \ldots, k_i$, where k_i are natural numbers. Set $P_{i,m_i} = \sum_{s_i}^{m_i} p_{i,s_i}$. Let Φ_i $(i = 1, 2, \ldots, n)$ be n real valued nonnegative convex and supmultiplicative functions defined on $(0, \infty)$. Let $\alpha_i \in (0, 1)$, and set $\alpha'_i = 1 - \alpha_i$, $(i = 1, 2, \ldots, n)$, $\alpha = \sum_{i=1}^n \alpha_i$, and $\alpha'_i = \sum_{i=1}^n \alpha'_i, = n - \alpha$. Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \Phi_i(a_{i,m_i})}{\left(\sum_{i=1}^n \alpha'_i m_i\right)^{\alpha'}} \leqslant M(k_1, \ldots, k_n) \prod_{i=1}^n \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1)\left(p_{i,m_i}\Phi_i\left(\frac{\nabla a_{i,m_i}}{p_{i,m_i}}\right)^{\frac{1}{\alpha_i}}\right)^{\alpha_i}\right.$$

where

$$M(k_1, \ldots, k_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left(\sum_{m_i=1}^{k_i} \left(\frac{\Phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\frac{1}{\alpha'_i}}\right)^{\alpha'_i}.$$

Theorem 12. *Let $f_i \in C^1([0, k_i], \mathbb{R}_+])$ $i = 1, \ldots, n$, with $f_i(0) = 0$; let $p_i(\xi_i)$ be n positive functions defined for $\xi_i \in [0, x_i]$ $(i = 1, \ldots, n)$. Set $P_i(s_i) = \int_0^{s_i} p_i(\xi_i)d\xi_i$ for $s_i \in [0, x_i]$, where x_i are positive real numbers. Let Φ_i, α_i, α'_i, α, and α' be as in Theorem 11. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \Phi_i(f(s_i))}{\left(\sum_{i=1}^n \alpha'_i s_i\right)^{\alpha'}} ds_1 \ldots ds_n \\ \leqslant L(x_1, \ldots, x_n) \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)\left(p_i(s_i)\Phi_i\left(\frac{f'(s_i)}{p(s_i)}\right)^{\frac{1}{\alpha_i}} ds_i\right)^{\alpha_i}\right.,$$

where

$$L(x_1, \ldots, x_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left(\int_0^{x_i} \left(\frac{\Phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\frac{1}{\alpha'_i}} ds_i\right)^{\alpha'_i}.$$

Hamiaz et al. [12] discussed the inequalities:

Theorem 13. *Let $r_2, r_1 \geqslant 1$, $\alpha \geqslant \beta \geqslant \frac{1}{2}$ and $(\xi_j)_{1\leq j\leq r}$, $(\delta_i)_{1\leq i\leq k}$ be sequences of non-negative real numbers where $k, r \in \mathbb{N}$. Define $\theta_i = \sum_{s=1}^{i} \delta_s$, $\phi_j = \sum_{t=1}^{j} \xi_t$. Then*

$$\sum_{i=1}^{k}\sum_{j=1}^{r} \frac{\theta_i^{2r_1}\phi_j^{2r_2}}{\psi(i)+\psi^*(j)} \leqslant C_1^*(r_1,r_2)\left(\sum_{i=1}^{k}(k-i+1)(\delta_i\theta_i^{r_1-1})^2\right) \times \left(\sum_{j=1}^{r}(r-j+1)(\xi_j\phi_j^{r_2-1})^2\right)$$

and

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{r} \frac{\theta_i^{r_1}\phi_j^{r_2}}{\left(|\psi(i)|^{\frac{1}{2\beta}}+|\psi^*(j)|^{\frac{1}{2\beta}}\right)^{\alpha}} &\leqslant \sum_{i=1}^{k}\sum_{j=1}^{r} \frac{\theta_i^{r_1}\phi_j^{r_2}}{\sqrt{\psi(i)+\psi^*(j)}} \\ &\leqslant C_2^*(r_1,r_2,k,r)\left(\sum_{i=1}^{k}(k-i+1)(\delta_i\theta_i^{r_1-1})^2\right)^{\frac{1}{2}} \\ &\quad\times\left(\sum_{j=1}^{r}(r-j+1)(\xi_j\phi_j^{r_2-1})^2\right)^{\frac{1}{2}}\end{aligned}$$

where $\psi(j)$ and $\psi^*(j)$ are defined as in Definition 10. Unless (δ_i) or (ξ_j) is null, where

$$C_1^*(r_1,r_2) = (r_1r_2)^2 \text{ and } C_2^*(r_1,r_2,k,r) = r_1r_2\sqrt{kr}.$$

Over several decades, Hilbert-type inequalities have attracted many researchers and several refinements, and the previous results have been extended. We refer the reader to the works on classical refinements and extensions of Hilbert-type inequalities [12–22] and time scale versions of Hilbert-type inequalities [23–26].

Lemma 2. *[27] (Hölder's inequalities) Let $\delta, \xi \in \mathbb{T}$ and $\vartheta, \zeta \in C_{ld}([\delta,\xi]_{\mathbb{T}}, [0,\infty))$. If $r_1, r_2 > 1$ with $\frac{1}{r_1}+\frac{1}{r_2} = 1$, then*

$$\int_{\delta}^{\xi} \vartheta(t)\zeta(t)\nabla t \leqslant \left[\int_{\delta}^{\xi}\vartheta^{r_1}(t)\nabla t\right]^{\frac{1}{r_1}}\left[\int_{\delta}^{\xi}\zeta^{r_2}(t)\nabla t\right]^{\frac{1}{r_2}}.$$

Lemma 3. *[14] (Jensen's inequality) Let $\delta, \xi \in \mathbb{T}$, and $c, d \in \mathbb{R}$. Assume that $\zeta \in C_{ld}\left([\delta,\xi]_{\mathbb{T}}, [c,d]\right)$ and $r \in C_{ld}\left([\delta,\xi]_{\mathbb{T}}, \mathbb{R}\right)$ are nonnegative with $\int_{\delta}^{\xi} r(t)\nabla t > 0$. If $\check{\Phi} \in C_{ld}\left((c,d),\mathbb{R}\right)$ is a convex function, then*

$$\check{\Phi}\left(\frac{\int_{\delta}^{\xi} r(t)\zeta(t)\nabla t}{\int_{\delta}^{\xi} r(t)\nabla t}\right) \leqslant \frac{\int_{\delta}^{\xi} r(t)\check{\Phi}(\zeta(t))\nabla t}{\int_{\delta}^{\xi} r(t)\nabla t}.$$

Lemma 4 ([14])**.** *Suppose the time scales $\mathbb{T}$ with $w, s \in \mathbb{T}$: $w \geqslant \delta$. Let $\vartheta : \mathbb{T} \to \mathbb{R}$ be left-dense continuous function with $\vartheta \geqslant 0$ and $\tilde{\alpha} \geqslant 1$, then*

$$\left(\int_{\delta}^{w}\vartheta(\check{\tau})\nabla\check{\tau}\right)^{\tilde{\alpha}} \leqslant \tilde{\alpha}\int_{\delta}^{w}\vartheta(\eta)\left(\int_{\delta}^{\eta}\vartheta(\check{\tau})\nabla\check{\tau}\right)^{\tilde{\alpha}-1}\nabla\eta. \tag{11}$$

Lemma 5 ([14])**.** *Let $\vartheta : \mathbb{T} \to \mathbb{R}$ be a left-dense continuous function. Then the equality that allows interchanging the order of nabla integration given by*

$$\int_{t_0}^{w}\left(\int_{t_0}^{s}\vartheta(\eta)\nabla\eta\right)\nabla s = \int_{t_0}^{w}\left(\int_{\rho(\eta)}^{w}\vartheta(\eta)\nabla s\right)\nabla\eta = \int_{t_0}^{w}[w-\rho(\eta)]\vartheta(\eta)\nabla\eta \tag{12}$$

holds for all s, w, $t_0 \in \mathbb{T}$.

Lemma 6 ([22])**.** *Let* w *and* $z \in \mathbb{R}$ *be such that* $w + z \geqslant 1$ *and* $0 < \gamma$*; then*

$$(w+z)^{\frac{1}{\gamma}} \leqslant \left(|w|^{\frac{1}{2\beta}} + |z|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{\gamma}} \quad \text{for all} \quad \frac{1}{2} \leq \beta \leq \alpha. \tag{13}$$

In this important article, by implying (5), we study some new dynamic inequalities of Hardy–Hilbert-type by using the nabla integral on time scales. We further show some relevant inequalities as special cases: discrete inequalities and integral inequalities. These inequalities may be be used to get more generalized results of several obtained inequalities before by replacing ψ, ψ^* with specific substitution.

Now, we are ready to state and proof our main results.

2. Main Results

In the following, we will let $r_1 > 1$, $r_2 > 1$ and $\frac{1}{r_2} + \frac{1}{r_1} = 1$.

Theorem 14. *Suppose the time scales* $\mathbb{T}$ *with* $\ell, \epsilon \geqslant 1$ *and* $s, t, t_0, w, z \in \mathbb{T}$*. Assume* $\delta(\check{\tau}) \geq 0$ *and* $\xi(\check{\tau}) \geq 0$ *are r-d continuous* $[t_0, w]_{\mathbb{T}}$ *and* $[t_0, z]_{\mathbb{T}}$*, respectively, and define*

$$\theta(s) := \int_{t_0}^{s} \delta(\check{\tau})\nabla\check{\tau}, \ \text{ and } \ \phi(t) := \int_{t_0}^{t} \xi(\check{\tau})\nabla\check{\tau},$$

then for $s \in [t_0, w]_{\mathbb{T}}$ *and* $t \in [t_0, z]_{\mathbb{T}}$*, we have that*

$$\begin{aligned} &\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}}\nabla s\nabla t \\ &\leqslant C_1(\ell,\epsilon,r_2)\left(\int_{t_0}^{w}(w-\rho(s))\big(\delta(s)\theta^{\epsilon-1}(s)\big)^{r_2}\nabla s\right) \\ &\times\left(\int_{t_0}^{z}(z-\rho(t))\big(\xi(t)\phi^{\ell-1}(t)\big)^{r_2}\nabla t\right) \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t \\ &\leqslant C_2(\ell,\epsilon,r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}} \\ &\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}} \end{aligned} \tag{15}$$

where $C_1(\ell,\epsilon,r_2) = (\epsilon\ell)^{r_2}$ *and* $C_2(\ell,\epsilon,r_1) = \epsilon\ell(w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}$.

Proof. By using the inequality (11), we obtain

$$\theta^{\epsilon}(s) \leqslant \epsilon\int_{t_0}^{s}\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\nabla\check{\eta}, \tag{16}$$

$$\phi^{\ell}(t) \leqslant \ell\int_{t_0}^{t}\xi(\check{\eta})\phi^{\ell-1}(\eta))\nabla\check{\eta}. \tag{17}$$

We use Lemma 2. Then, from (16), we get

$$\theta^{\epsilon}(s) \leqslant \epsilon(s-t_0)^{\frac{1}{r_1}} \left(\int_{t_0}^{s} (\delta(\check{\eta})\theta^{\epsilon-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}}. \tag{18}$$

Apply Lemma 2. Thus, from (17), we get

$$\phi^{\ell}(t) \leqslant \ell(t-t_0)^{\frac{1}{r_1}} \left(\int_{t_0}^{t} (\xi(\check{\eta})\phi^{\ell-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}}. \tag{19}$$

From (18) and (19), we get

$$\begin{aligned} \theta^{\epsilon}(s)\phi^{\ell}(t) \leqslant{}& \epsilon\ell(s-t_0)^{\frac{1}{r_1}}(t-t_0)^{\frac{1}{r_1}} \\ &\times\left(\int_{t_0}^{s} (\delta(\check{\eta})\theta^{\epsilon-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}} \\ &\times\left(\int_{t_0}^{t} (\xi(\check{\eta})\phi^{\ell-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}}. \end{aligned} \tag{20}$$

From inequality (20), we have

$$\begin{aligned} \theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t) \leqslant{}& (\epsilon\ell)^{r_2}(s-t_0)^{\frac{r_2}{r_1}}(t-t_0)^{\frac{r_2}{r_1}} \\ &\times\left(\int_{t_0}^{s} (\delta(\check{\eta})\theta^{\epsilon-1}(\eta))^{r_2} \nabla\check{\eta} \right) \\ &\times\left(\int_{t_0}^{t} (\xi(\check{\eta})\phi^{\ell-1}(\eta))^{r_2} \nabla\check{\eta} \right). \end{aligned} \tag{21}$$

Using Lemma 1 in (20) and (21) gives

$$\begin{aligned} \theta^{\epsilon}(s)\phi^{\ell}(t) \leqslant{}& \epsilon\ell\Big(\psi(s-t_0)+\psi^{*}(t-t_0)\Big)^{\frac{1}{r_1}} \\ &\times\left(\int_{t_0}^{s} (\delta(\check{\eta})\theta^{\epsilon-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}} \\ &\times\left(\int_{t_0}^{t} (\xi(\check{\eta})\phi^{\ell-1}(\eta))^{r_2} \nabla\check{\eta} \right)^{\frac{1}{r_2}}, \end{aligned} \tag{22}$$

$$\begin{aligned} \theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t) \leqslant{}& (\epsilon\ell)^{r_2}\Big(\psi(s-t_0)+\psi^{*}(t-t_0)\Big)^{\frac{r_2}{r_1}} \\ &\times\left(\int_{t_0}^{s} (\delta(\check{\eta})\theta^{\epsilon-1}(\eta))^{r_2} \nabla\check{\eta} \right) \\ &\times\left(\int_{t_0}^{t} (\xi(\check{\eta})\phi^{\ell-1}(\eta))^{r_2} \nabla\check{\eta} \right). \end{aligned} \tag{23}$$

Using Lemma 6 in (22) and (23) gives

$$\theta^{\epsilon}(s)\phi^{\ell}(t) \leqslant \epsilon\ell\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}} \times\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}} \times\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}, \tag{24}$$

$$\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t) \leqslant (\epsilon\ell)^{r_2}\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}} \times\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right) \times\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right). \tag{25}$$

By dividing both sides of (24) and (25) by $\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}$ and $\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}$, respectively, we get that

$$\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant \epsilon\ell\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}} \times\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}, \tag{26}$$

$$\frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}} \leqslant (\epsilon\ell)^{r_2}\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right) \times\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right). \tag{27}$$

From (26) by using Lemma 2, we obtain

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t \leqslant \epsilon\ell(w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)\nabla s\right)^{\frac{1}{r_2}} \times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)\nabla t\right)^{\frac{1}{r_2}}. \tag{28}$$

From (27), we get

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant(\epsilon\ell)^{r_2}\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\delta(\check{\eta})\theta^{\epsilon-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)\nabla s$$
$$\times\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\xi(\check{\eta})\phi^{\ell-1}(\eta)\right)^{r_2}\nabla\check{\eta}\right)\nabla t. \tag{29}$$

Applying Lemma 5 on (28) and (29) gives

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\epsilon\ell(w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{w}(w-\rho(s))\left(\delta(s)\theta^{\epsilon-1}(s)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\xi(t)\phi^{\ell-1}(t)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}},$$

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant(\epsilon\ell)^{r_2}\left(\int_{t_0}^{w}(w-\rho(s))\left(\delta(s)\theta^{\epsilon-1}(s)\right)^{r_2}\nabla t\right)$$
$$\times\left(\int_{t_0}^{z}(z-\rho(s))\left(\xi(t)\phi^{\ell-1}(t)\right)^{r_2}\nabla t\right).$$

This completes the proof. □

Remark 1. *In (15), as a special case, if we take* $\psi(w)=\dfrac{w^2}{2}$*, we have* $\psi^*(w)=\dfrac{w^2}{2}$ *see [7], so we get*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$=\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left((s-t_0))^{\frac{1}{\beta}}+(t-t_0)^{\frac{1}{\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{2}\right)^{\frac{2\alpha}{r_1\beta}}C_2(\ell,\epsilon,r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}. \tag{30}$$

Consequently, for $\alpha = \beta = 1$, inequality (57) produces

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(s+t-2t_0\right)^{\frac{2}{r_1}}}\nabla s\nabla t$$
$$\leqslant \left(\frac{1}{2}\right)^{\frac{2}{r_1}}C_2(\ell,\epsilon,r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}.$$

By putting $r_1 = r_2 = 2$, we get

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{t+s-2t_0}\nabla s\nabla t$$
$$\leqslant \frac{1}{2}\epsilon\ell\left((w-t_0)\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{2}\nabla s\right)^{\frac{1}{2}}$$
$$\times\left((z-t_0)\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{2}\nabla t\right)^{\frac{1}{2}}.$$

which is [14] (Theorem 3.3).

Theorem 15. *Suppose $\xi(\check{\eta})$, $\theta(s)$, $\phi(t)$ and $\delta(\check{\tau})$, are defined as in Theorem 14; thus,*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{r_2}(s)\phi^{r_2}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\int_{t_0}^{w}(w-\rho(s))\delta^{r_2}(s)\nabla s\right)\left(\int_{t_0}^{z}(z-\rho(t))\xi^{r_2}(t)\nabla t\right)$$

and

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta(s)\phi(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant (w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{w}(w-\rho(s))\delta^{r_2}(s)\nabla s\right)^{\frac{1}{r_2}}\left(\int_{t_0}^{z}(z-\rho(t))\xi^{r_2}(t)\nabla t\right)^{\frac{1}{r_2}}.$$

Proof. In (14) and (15) taking $\epsilon = \ell = 1$, this grants our claim. □

In Theorem 14, if we chose $\mathbb{T} = \mathbb{R}$, then the following results:

Corollary 2. *If $\delta(s) \geq 0$, $\xi(t) \geq 0$. Define $\theta(s) := \int_0^s \delta(\check{\eta})d\check{\eta}$ and $\phi(t) := \int_0^t \xi(\check{\eta})d\check{\eta}$; then*

$$\int_{0}^{w}\int_{0}^{z}\frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}}dsdt$$
$$\leqslant C_1(\ell,\epsilon,r_2)\left(\int_{0}^{w}(w-s)\left(\delta(s)\theta^{\epsilon-1}(s)\right)^{r_2}ds\right)$$
$$\times\left(\int_{0}^{z}(z-t)\left(\xi(t)\phi^{\ell-1}(t)\right)^{r_2}dt\right).$$

$$\int_0^w \int_0^z \frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}} + |\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} dsdt$$
$$\leqslant C_3(\ell,\epsilon,r_1)\left(\int_0^w (w-s)\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2} ds\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_0^z (z-t)\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2} dt\right)^{\frac{1}{r_2}}$$

where

$$C_3(\ell,\epsilon,r_1) = \epsilon\ell(wz)^{\frac{1}{r_1}}.$$

In Theorem 14, if we chose $\mathbb{T} = \mathbb{Z}$, then we get (2), and the next result:

Corollary 3. *If $\delta(i) \geq 0$ and $\xi(j) \geq 0$. Define*

$$\theta(i) = \sum_{s=0}^{i} \delta(s), \quad \phi(j) = \sum_{a=0}^{j} \xi(a).$$

Then

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\theta^{r_2\ell}(i)\phi^{r_2\epsilon}(j)}{\left(|\psi(i+1)|^{\frac{1}{2\beta}} + |\psi^*(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}} \leqslant C_1(\epsilon,\ell,r_2)\left(\sum_{i=1}^{N}(N-(i+1))(\delta(i)\theta^{\ell-1}(i))^{r_2}\right)$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))(\xi(j)\phi^{\ell-1}(j))^{r_2}\right)$$

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\theta^{\ell}(i)\phi^{\epsilon}(j)}{\left(|\psi(i+1)|^{\frac{1}{2\beta}} + |\psi^*(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant C_4(\epsilon,\ell,r_1)\left(\sum_{i=1}^{N}(N-(i+1))(\delta(i)\theta^{\ell-1}(i))^{r_2}\right)^{\frac{1}{r_2}}$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))(\xi(j)\phi^{\ell-1}(j))^{r_2}\right)^{\frac{1}{r_2}}$$

where

$$C_4(\epsilon,\ell,r_1) = \epsilon\ell(NM)^{\frac{1}{r_1}}.$$

Corollary 4. *With the hypotheses of Theorem 14, we have:*

$$\int_{t_0}^w \int_{t_0}^z \frac{\theta^{r_2\epsilon}(s)\phi^{r_2\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2r_2\alpha}{r_1}}} \nabla s\nabla t$$
$$\leqslant C_1(\ell,\epsilon,r_2)\left\{\psi\left(\int_{t_0}^w (w-\rho(s))(\delta(s)\theta^{\epsilon-1}(s)^{r_2}\nabla s\right)\right.$$
$$\left. +\psi^*\left(\int_{t_0}^z (z-\rho(t))(\xi(t)\phi^{\ell-1}(t)^{r_2}\nabla t\right)\right\}$$

and

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant C_2(\ell,\epsilon,r_1)\left\{\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2}\nabla s\right)\right.$$
$$\left.+\psi^*\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2}\nabla t\right)\right\}^{\frac{1}{r_2}}.$$

Proof. Use the Fenchel–Young inequality (5) in (14) and (15). This proves the claim. □

Theorem 16. *Assume the time scale $\mathbb{T}$ with s, t, t_0, w, $z \in \mathbb{T}$, $\theta(s)$, and $\phi(t)$ defined as in Theorem 14. Suppose $\vartheta(\check{\tau}) \geq 0$ and $\zeta(\check{\eta}) \geq 0$ are right-dense continuous functions on $[t_0, w]_{\mathbb{T}}$ and $[t_0, z]_{\mathbb{T}}$, respectively. Suppose that $\check{\Phi} \geq 0$ and $\check{\Psi} \geq 0$ are convex, and submultiplicative on $[0, \infty)$. Furthermore, assume that*

$$F(s) := \int_{t_0}^{s}\vartheta(\check{\tau})\nabla\check{\tau}, \text{ and } G(t) := \int_{t_0}^{t}\zeta(\check{\eta})\nabla\check{\eta}; \tag{31}$$

then for $s \in [t_0, w]_{\mathbb{T}}$ and $t \in [t_0, z]_{\mathbb{T}}$, we have that

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant M_1(r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}} \tag{32}$$

where

$$M_1(r_1)=\left\{\int_{t_0}^{w}\left[\frac{\check{\Phi}(F(s))}{F(s)}\right]_1^{r}\nabla s\right\}^{\frac{1}{r_1}}\left\{\int_{t_0}^{z}\left[\frac{\check{\Psi}(G(t))}{G(t)}\right]_1^{r}\nabla t\right\}^{\frac{1}{r_1}}.$$

Proof. From the properties of $\check{\Phi}$ and using (3), we obtain

$$\begin{aligned}\check{\Phi}(\theta(s)) &= \check{\Phi}\left(\frac{F(s)\int_{t_0}^{s}\vartheta(\check{\tau})\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\nabla\check{\tau}}{\int_{t_0}^{s}\vartheta(\check{\tau})\nabla\check{\tau}}\right)\\ &\leqslant \check{\Phi}(F(s)\check{\Phi}\left(\frac{\int_{t_0}^{s}\vartheta(\check{\tau})\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\nabla\check{\tau}}{\int_{t_0}^{s}\vartheta(\check{\tau})\nabla\check{\tau}}\right)\\ &\leqslant \frac{\check{\Phi}(F(s)}{F(s)}\int_{t_0}^{s}\vartheta(\check{\tau})\check{\Phi}\left(\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right)\nabla\check{\tau}.\end{aligned} \tag{33}$$

Using (2) in (33), we see that

$$\check{\Phi}(\theta(s)) \leqslant \frac{\check{\Phi}(F(s))}{F(s)}(s-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}. \tag{34}$$

Additionally, from the convexity and submultiplicative property of $\check{\Psi}$, we get by using (2) and (3):

$$\check{\Psi}(\phi(t)) \leqslant \frac{\check{\Psi}(G(t))}{G(t)}(t-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}. \tag{35}$$

From (34) and (35), we have

$$\begin{aligned}\check{\Phi}(\theta(s))\check{\Psi}(\phi(t)) \quad \leqslant \quad & (s-t_0)^{\frac{1}{r_1}}(t-t_0)^{\frac{1}{r_1}}\left(\frac{\check{\Phi}(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\\ & \times\left(\frac{\check{\Psi}(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}\end{aligned} \tag{36}$$

Using (5) on $(s-t_0)^{\frac{1}{r_1}}(t-t_0)^{\frac{1}{r_1}}$ gives:

$$\begin{aligned}\check{\Phi}(\theta(s))\check{\Psi}(\phi(t)) \quad \leqslant \quad & \left(\psi(s-t_0)+\psi^*(t-t_0)\right)^{\frac{1}{r_1}}\left(\frac{\check{\Phi}(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\\ & \times\left(\frac{\check{\Psi}(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}.\end{aligned} \tag{37}$$

Applying Lemma 6 on the right-hand side of (37), we see that

$$\begin{aligned}\check{\Phi}(\theta(s))\check{\Psi}(\phi(t)) \leqslant & \left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}\\ & \times\left(\frac{\check{\Phi}(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\\ & \times\left(\frac{\check{\Psi}(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}.\end{aligned} \tag{38}$$

From (38), we have

$$\begin{aligned}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant & \left(\frac{\check{\Phi}(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\\ \times\left(\frac{\check{\Psi}(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}.\end{aligned} \tag{39}$$

From (39), we obtain

$$\begin{aligned}&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t\\ &\leqslant \int_{t_0}^{w}\frac{\check{\Phi}(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\nabla s\\ &\qquad\times\int_{t_0}^{z}\frac{\check{\Psi}(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}\nabla t.\end{aligned} \tag{40}$$

From (40), by using (2), we have

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))}{F(s)}\right)_1^r\nabla s\right\}^{\frac{1}{r_1}}\left(\int_{t_0}^{w}\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^{r_2}\nabla\check{\tau}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))}{G(t)}\right)_1^r\nabla t\right\}^{\frac{1}{r_1}}\left(\int_{t_0}^{z}\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^{r_2}\nabla\check{\eta}\nabla t\right)^{\frac{1}{r_2}}. \quad (41)$$

From (41), by using Lemma 5, we obtain

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant M_1(r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}.$$

where

$$M_1(r_1)=\left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))}{F(s)}\right)_1^r\nabla s\right\}^{\frac{1}{r_1}}\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))}{G(t)}\right)_1^r\nabla t\right\}^{\frac{1}{r_1}}.$$

This completes the proof. □

Remark 2. *In Theorem 16, as special case, if we take* $\psi(w)=\dfrac{w^2}{2}$, $\psi^*(w)=\dfrac{w^2}{2}$, *and by following the same procedure employed in Remark 1, then we get [14] (Theorem 3.5).*

In Theorem 16, taking $\mathbb{T}=\mathbb{R}$, we have the result:

Corollary 5. *Assume that* $\delta(s)\geq 0$, $\xi(t)\geq 0$, $\vartheta(\check{\tau})\geq 0$, *and* $\zeta(\check{\eta})\geq 0$. *We define*

$$\theta(s):=\int_0^s\delta(\check{\eta})d\check{\eta},\quad \phi(t):=\int_0^t\xi(\check{\eta})d\check{\eta},\quad F(s):=\int_0^s\vartheta(\check{\tau})d\check{\tau},\quad and\quad G(t):=\int_0^t\zeta(\check{\eta})d\check{\eta}.$$

Then

$$\int_0^w\int_0^z\frac{\check{\Phi}(\theta(s)\check{\Psi}(\phi(t))}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}dsdt\leqslant M_2(r_1)\left(\int_0^w(w-s)\left(\vartheta(s)\check{\Phi}\left(\frac{\delta(s)}{\vartheta(s)}\right)\right)^{r_2}ds\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_0^z(z-t)\left(\zeta(t)\check{\Psi}\left(\frac{\xi(t)}{\zeta(t)}\right)\right)^{r_2}dt\right)^{\frac{1}{r_2}}$$

where

$$M_2(r_1)=\left\{\int_0^w\left(\frac{\check{\Phi}(F(s))}{F(s)}\right)_1^r ds\right\}^{\frac{1}{r_1}}\left\{\int_0^z\left(\frac{\check{\Psi}(G(t))}{G(t)}\right)_1^r dt\right\}^{\frac{1}{r_1}}.$$

In Theorem 16, taking $\mathbb{T}=\mathbb{Z}$, gives (2) and the result:

Corollary 6. *Assume that* $\delta(i) \geq 0$, $\xi(j) \geq 0$, $\vartheta(i) \geq 0$, $\zeta(j) \geq 0$ *are sequences of real numbers. Define*

$$\theta(i) = \sum_{s=0}^{i} \delta(s), \;\; \phi(j) = \sum_{a=0}^{j} \xi(a), \;\; F(i) = \sum_{s=0}^{i} \vartheta(s) \;\; and \;\; G(j) = \sum_{a=0}^{j} \zeta(a).$$

Then

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\check{\Phi}(\theta(i))\check{\Psi}(\phi(j))}{\left(|\psi(i+1)|^{\frac{1}{2\beta}} + |\psi^*(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant M_3(r_1)\left\{\sum_{i=1}^{N}(N-(i+1))\left(\vartheta(i)\check{\Phi}\left[\frac{\delta(i)}{\vartheta(i)}\right]\right)^{r_2}\right\}^{\frac{1}{r_2}}$$
$$\times\left\{\sum_{j=1}^{M}(M-(j+1))\left(\zeta(j)\check{\Psi}\left[\frac{\xi(j)}{\zeta(j)}\right]\right)^{r_2}\right\}^{\frac{1}{r_2}}$$

where

$$M_3(r_1) = \left\{\sum_{i=1}^{N}\left(\frac{\check{\Phi}(F(i))}{F(i)}\right)_1^{r}\right\}^{\frac{1}{r_1}}\left\{\sum_{j=1}^{M}\left(\frac{\check{\Psi}(G(j))}{G(j)}\right)_1^{r}\right\}^{\frac{1}{r_1}}$$

Remark 3. *In Corollary 6, if* $r_1 = r_2 = 2$ *we get the result due to Hamiaz and Abuelela [12] (Theorem 5).*

Corollary 7. *Under the hypotheses of Theorem 16 the following inequalities hold:*

$$\int_{t_0}^{w}\int_{t_0}^{z} \frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant M_1(r_1)\left[\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left(\frac{\delta(s)}{\vartheta(s)}\right)\right)^{r_2}\nabla s\right)\right.$$
$$\left.+\psi^*\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left(\frac{\xi(t)}{\zeta(t)}\right)\right)^{r_2}\nabla t\right)\right]^{\frac{1}{r_2}}.$$

Proof. Use (5) in (32). This proves our claim. □

Lemma 7. *With hypotheses of Theorem 16, we get:*

$$\int_{t_0}^{w}\int_{t_0}^{z} \frac{\check{\Phi}(\theta(s))^2\check{\Psi}(\phi(t))^2}{\left(\psi(s-t_0) + \psi^*(t-t_0)\right)}\nabla s\nabla t$$
$$\leqslant M_4\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^4\nabla s\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^4\nabla t\right\}^{\frac{1}{2}} \tag{42}$$

where

$$M_4 = \left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla t\right\}^{\frac{1}{2}}.$$

Proof. From (34) and (35) and by using the Fenchel–Young inequality with $r_1 = r_2 = 2$, we have

$$\begin{aligned}
&\check{\Phi}(\theta(s))^2\check{\Psi}(\phi(t))^2 \\
&\leqslant \left(\psi(s-t_0)+\psi^*(t-t_0)\right)\left(\frac{\check{\Phi}(F(s))^2}{(F(s))^2}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^2\nabla\check{\tau}\right)\right. \\
&\quad\times\left(\frac{\check{\Psi}(G(t))^2}{(G(t))^2}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^2\nabla\check{\eta}\right).
\end{aligned} \tag{43}$$

From (43), by using (2) with $r_1 = r_2 = 2$, we obtain

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))^2\check{\Psi}(\phi(t))^2}{\left(\psi(s-t_0)+\psi^*(t-t_0)\right)}\nabla s\nabla t \\
&\leqslant \int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^2}{(F(s))^2}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^2\nabla\check{\tau}\right)\nabla s \\
&\times\int_{t_0}^{z}\frac{\check{\Psi}(G(t))^2}{(G(t))^2}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^2\nabla\check{\eta}\right)\nabla t \\
&\leqslant \int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^2}{(F(s))^2}\right)(s-t_0)^{\frac{1}{2}}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^4\nabla\check{\tau}\right)^{\frac{1}{2}}\right)\nabla s \\
&\times\int_{t_0}^{z}\frac{\check{\Psi}(G(t))^2}{(G(t))^2}(t-t_0)^{\frac{1}{2}}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^4\nabla\check{\eta}\right)^{\frac{1}{2}}\nabla t \\
&\leqslant\left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}\left[\frac{\delta(\check{\tau})}{\vartheta(\check{\tau})}\right]\right)^4\nabla\check{\tau}\right)\nabla s\right\}^{\frac{1}{2}} \\
&\times\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla t\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}\left[\frac{\xi(\check{\eta})}{\zeta(\check{\eta})}\right]\right)^4\nabla\check{\eta}\right)\nabla t\right\}^{\frac{1}{2}}.
\end{aligned} \tag{44}$$

By applying Lemma 5 on (44), we obtain

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))^2\check{\Psi}(\phi(t))^2}{\left(\psi(s-t_0)+\psi^*(t-t_0)\right)}\nabla s\nabla t \\
&\leqslant\left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^4\nabla s\right\}^{\frac{1}{2}} \\
&\times\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla t\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^4\nabla t\right\}^{\frac{1}{2}} \\
&= M_4\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^4\nabla s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^4\nabla t\right\}^{\frac{1}{2}}.
\end{aligned}$$

where

$$M_4=\left\{\int_{t_0}^{w}\left(\frac{\check{\Phi}(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\frac{\check{\Psi}(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla t\right\}^{\frac{1}{2}}.$$

□

This proves our claim.

Theorem 17. *Let δ, ξ, G, F, ζ, ϑ, $\check{\Psi}$, and $\check{\Phi}$ be as in Theorem 16. Furthermore, assume that for t, s, t_0, w, $z \in \mathbb{T}$*

$$\theta(s) := \frac{1}{F(s)}\int_{t_0}^{s}\delta(\check{\tau})\vartheta(\check{\tau})\nabla\check{\tau}, \ \ \textit{and} \ \ \phi(t) := \frac{1}{G(t)}\int_{t_0}^{t}\xi(\check{\eta})\zeta(\check{\eta})\nabla\check{\eta}; \tag{45}$$

then for $s \in [t_0, w]_{\mathbb{T}}$ and $t \in [t_0, z]_{\mathbb{T}}$, we have that

$$\begin{aligned}\int_{t_0}^{w}\int_{t_0}^{z}&\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))F(s)G(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t \\ &\leqslant M_5(r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left(\delta(s)\right)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}} \\ &\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left(\xi(t)\right)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}\end{aligned} \tag{46}$$

where

$$M_5(r_1) = (w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}.$$

Proof. From (45), we see that

$$\check{\Phi}(\theta(s)) = \check{\Phi}\left(\frac{1}{F(s)}\int_{t_0}^{s}\vartheta(\check{\tau})\delta(\check{\tau})\nabla\check{\tau}\right). \tag{47}$$

By applying (2) to (47), we obtain

$$\check{\Phi}(\theta(s)) \leqslant \frac{(s-t_0)^{\frac{1}{r_1}}}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}. \tag{48}$$

From (48), we get that

$$\check{\Phi}(\theta(s))F(s) \leqslant (s-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}. \tag{49}$$

Similarly, we obtain

$$\check{\Psi}(\phi(t))G(t) \leqslant (t-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}. \tag{50}$$

From (49) and (50), we observe that

$$\begin{aligned}\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s) &\leqslant (s-t_0)^{\frac{1}{r_1}}(t-t_0)^{\frac{1}{r_1}} \\ \times\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}&\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}.\end{aligned} \tag{51}$$

Applying Lemma 1 on the term $(s-t_0)^{\frac{1}{r_1}}(t-t_0)^{\frac{1}{r_1}}$ gives:

$$\begin{aligned}\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s) \quad &\leqslant \left(\psi(s-t_0)+\psi^*(t-t_0)\right)^{\frac{1}{r_1}}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}} \\ &\times\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}.\end{aligned} \tag{52}$$

From 6 and (52), we obtain

$$\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s) \leqslant \left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}$$
$$\times\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}. \tag{53}$$

Through dividing both sides of (53) by $\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}$, we get that

$$\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant \left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}. \tag{54}$$

Taking the double nabla-integral for (54) yields:

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant \left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)^{\frac{1}{r_2}}\nabla s\right)$$
$$\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)^{\frac{1}{r_2}}\nabla t\right). \tag{55}$$

Using (2) in (55), yield:

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant (w-t_0)^{\frac{1}{r_1}}(z-t_0)^{\frac{1}{r_1}}\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)\nabla t\right)^{\frac{1}{r_2}}$$
$$= M_5(r_1)\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\check{\tau})\check{\Phi}[\delta(\check{\tau})]\right)^{r_2}\nabla\check{\tau}\right)\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\check{\eta})\check{\Psi}[\xi(\check{\eta})]\right)^{r_2}\nabla\check{\eta}\right)\nabla t\right)^{\frac{1}{r_2}}. \tag{56}$$

From Lemma 5 and (56), we get:

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\breve{\Phi}(\theta(s))\breve{\Psi}(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$

$$= M_5(r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\breve{\Phi}[\delta(s)]\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$

$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\breve{\Psi}[\xi(t)]\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}.$$

This completes the proof. □

Remark 4. *In Theorem 17, as a special case, if we take $\psi(w)=\frac{w^2}{2}$, $\psi^*(w)=\frac{w^2}{2}$, and by following the same procedure employed in Remark 1, then we get [14] (Theorem 3.7).*

Taking $\mathbb{T}=\mathbb{R}$ in Theorem 17, we have:

Corollary 8. *Assume $\zeta(t)\geq 0$, $\xi(t)\geq 0$, $\vartheta(s)\geq 0$, $\delta(s)\geq 0$. Define*

$$\theta(s):=\frac{1}{F(s)}\int_0^s\vartheta(\check{\tau})\delta(\check{\tau})d\check{\tau}\ \text{ and }\ \phi(t):=\frac{1}{G(t)}\int_0^t\zeta(\check{\tau})\xi(\check{\tau})d\check{\tau},$$

$$F(s):=\int_0^s\vartheta(\check{\tau})d\check{\tau}\ \text{ and }\ G(t):=\int_0^t\zeta(\check{\tau})d\check{\tau}.$$

Then

$$\int_0^w\int_0^z\frac{\breve{\Phi}(\theta(s))\breve{\Psi}(\phi(t))F(s)G(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}dsdt \leqslant M_6(r_1)\left(\int_0^w(w-s)\left(\vartheta(s)\breve{\Phi}\left(\delta(s)\right)\right)^{r_2}ds\right)^{\frac{1}{r_2}}.$$

$$\times\left(\int_0^z(z-t)\left(\zeta(t)\breve{\Psi}\left(\xi(t)\right)\right)^{r_2}dt\right)^{\frac{1}{r_2}}$$

where

$$M_6(r_1)=(w)^{\frac{1}{r_1}}(z)^{\frac{1}{r_1}}$$

Taking $\mathbb{T}=\mathbb{Z}$ in Theorem 17 gives:

Corollary 9. *Assume $\zeta(i)\geq 0$, $\xi(i)\geq 0$, $\vartheta(i)\geq 0$, $\delta(i)\geq 0$. Define*

$$\theta(i):=\frac{1}{F(i)}\sum_{s=0}^{i}\vartheta(s)\delta(s)\ \text{ and }\ \phi(j):=\frac{1}{G(j)}\sum_{a=0}^{j}\zeta(a)\xi(a).$$

$$F(i):=\sum_{s=0}^{i}\vartheta(s)\ \text{ and }\ G(j):=\sum_{a=0}^{j}\zeta(a).$$

Then

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{\check{\Phi}(\theta(i))\check{\Psi}(\phi(j))F(i)G(j)}{\left(|\psi(i+1)|^{\frac{1}{2\beta}}+|\psi^*(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}} \leqslant M_7(r_1)\left(\sum_{i=1}^{N}(N-(i+1))\left(\vartheta(i)\check{\Phi}\left(\delta(i)\right)\right)^{r_2}\right)^{\frac{1}{r_2}}.$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))\left(\zeta(j)\check{\Psi}\left(\xi(j)\right)\right)^{r_2}\right)^{\frac{1}{r_2}}$$

where

$$M_7(r_1)=(NM)^{\frac{1}{r_1}}.$$

Remark 5. *In Corollary 9, if $r_1=r_2=2$, we get the result due to Hamiaz and Abuelela [12] (Theorem 7).*

Corollary 10. *With the hypotheses of Theorem 17, we get:*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\check{\Phi}(\theta(s))\check{\Psi}(\phi(t))F(s)G(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant M_5(r_1)\Bigg\{\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\check{\Phi}\left(\delta(s)\right)\right)^{r_2}\nabla s\right)$$
$$+\psi^*\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\check{\Psi}\left(\xi(t)\right)\right)^{r_2}\nabla t\right)\Bigg\}^{\frac{1}{r_2}}$$

Proof. We apply the Fenchel–Young inequality (5) in (46). This completes the proof. □

3. Some Applications

We can apply our inequalities to obtain different formulas of inequalities by suggesting $\psi^*(z)$ and $\psi(w)$ by some functions:

In (15), as a special case, if we take $\psi(w)=\dfrac{w^2}{2}$, we have $\psi^*(w)=\dfrac{w^2}{2}$ (see [7]), so we get

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$=\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\left((s-t_0))^{\frac{1}{\beta}}+(t-t_0)^{\frac{1}{\beta}}\right)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{2}\right)^{\frac{\alpha}{r_1\beta}}C_2(\ell,\epsilon,r_1)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-1}(s)\delta(s)\right)^{r_2}\nabla s\right)^{\frac{1}{r_2}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-1}(t)\xi(t)\right)^{r_2}\nabla t\right)^{\frac{1}{r_2}}. \tag{57}$$

Consequently, for $\alpha = \beta = 1$, inequality (57) produces

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\Big((s-t_0))+(t-t_0)\Big)^{\frac{2}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{2}\right)^{\frac{1}{r_1}}C_2(\ell,\epsilon,r_1)\bigg(\int_{t_0}^{w}(w-\rho(s))\Big(\theta^{\epsilon-1}(s)\delta(s)\Big)^{r_2}\nabla s\bigg)^{\frac{1}{r_2}}$$
$$\times\bigg(\int_{t_0}^{z}(z-\rho(t))\Big(\phi^{\ell-1}(t)\xi(t)\Big)^{r_2}\nabla t\bigg)^{\frac{1}{r_2}}. \tag{58}$$

On the other hand, if we take $\psi(i) = \frac{i^r}{r}, r > 1$, then $\psi^*(j) = \frac{j^a}{a}$, where $\frac{1}{r} + \frac{1}{a} = 1$ and $i, j\ \mathbb{R}_+$, then (15) gives

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\Big(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\Big)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$=\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\Big((a(s-t_0)^r)^{\frac{1}{2\beta}}+(r(t-t_0)^a)^{\frac{1}{2\beta}}\Big)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{rk}\right)^{\frac{\alpha}{r_1\beta}}C_2(\ell,\epsilon,r_1)\bigg(\int_{t_0}^{w}(w-\rho(s))\Big(\theta^{\epsilon-1}(s)\delta(s)\Big)^{r_2}\nabla s\bigg)^{\frac{1}{r_2}}$$
$$\times\bigg(\int_{t_0}^{z}(z-\rho(t))\Big(\phi^{\ell-1}(t)\xi(t)\Big)^{r_2}\nabla t\bigg)^{\frac{1}{r_2}}. \tag{59}$$

Clearly, when $\beta = \frac{1}{2\alpha}$, the inequality (59) becomes

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\Big((a(s-t_0)^r)^{\alpha}+(r(t-t_0)^a)^{\alpha}\Big)^{\frac{2\alpha}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{rk}\right)^{\frac{2\alpha^2}{r_1}}C_2(\ell,\epsilon,r_1)\bigg(\int_{t_0}^{w}(w-\rho(s))\Big(\theta^{\epsilon-1}(s)\delta(s)\Big)^{r_2}\nabla s\bigg)^{\frac{1}{r_2}}$$
$$\times\bigg(\int_{t_0}^{z}(z-\rho(t))\Big(\phi^{\ell-1}(t)\xi(t)\Big)^{r_2}\nabla t\bigg)^{\frac{1}{r_2}}. \tag{60}$$

If $\beta = \alpha = 1$. From (59), we get

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon}(s)\phi^{\ell}(t)}{\Big((a(s-t_0)^r)^{\frac{1}{2}}+(r(t-t_0)^a)^{\frac{1}{2}}\Big)^{\frac{2}{r_1}}}\nabla s\nabla t$$
$$\leqslant\left(\frac{1}{rk}\right)^{\frac{1}{r_1}}C_2(\ell,\epsilon,r_1)\bigg(\int_{t_0}^{w}(w-\rho(s))\Big(\theta^{\epsilon-1}(s)\delta(s)\Big)^{r_2}\nabla s\bigg)^{\frac{1}{r_2}}$$
$$\times\bigg(\int_{t_0}^{z}(z-\rho(t))\Big(\phi^{\ell-1}(t)\xi(t)\Big)^{r_2}\nabla t\bigg)^{\frac{1}{r_2}}.$$

4. Conclusions and Discussion

In this important work, we discussed some new dynamic inequalities of Hardy–Hilbert-type by using the nabla integral on time scales. We further presented some relevant inequalities as special cases: discrete inequalities and integral inequalities. These results may be used to get more generalized results of several obtained inequalities by replacing the Fenchel–Legendre transform with specific substitution. Furthermore, all results obtained in this manuscript may be generalized by using fractional conformable derivative calculus. Symmetry plays an essential role in determining the correct methods for solutions to dynamic inequalities.

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References

1. Hilger, S. Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg, Würzburg, Germany, 1988.
2. Hilger, S. Analysis on measure chains—A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [CrossRef]
3. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhauser Boston, Inc.: Boston, MA, USA, 2001.
4. Bohner, M.; Bohner, L.; Peterson, A. Oscillation for nonlinear second order dynamic equations on a time scale. *J. Math. Anal. Appl.* **2005**, *301*, 491–507. [CrossRef]
5. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhauser: Boston, MA, USA, 2003.
6. Arnold, V.I. *Lectures on Partial Differential Equations*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2013.
7. Borwein, J.; Lewis, A.S. *Convex Analysis and Nonlinear Optimization: Theory and Examples*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2010.
8. Dong, Q.; Lu, Y.; Yang, J.; He, S. Approximately solving multivalued variational inequalities by using a projection and contraction algorithm. *Numer. Algorithms* **2017**, *76*, 799–812. [CrossRef]
9. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1952.
10. Pachpatte, B.G. A note on Hilbert type inequality. *Tamkang J. Math.* **1998**, *29*, 293–298. [CrossRef]
11. Handley, G.D.; Koliha, J.J.; Pecaric, J. A Hilbert type inequality. *Tamkang J. Math.* **2000**, *31*, 311–316. [CrossRef]
12. Hamiaz, A.; Abuelela, W. Some new discrete Hilbert's inequalities involving Fenchel-Legendre transform. *J. Inequal. Appl.* **2020**, *2020*, 39. [CrossRef]
13. Abuelela, W. On an inequality related to Hilbert's with Laplace transform. *Int. J. Pure Appl. Math.* **2015**, *101*, 87–94. [CrossRef]
14. Anderson, D.R. Dynamic double integral inequalities in two independent variables on time scales. *J. Math. Inequal.* **2008**, *2*, 163–184. [CrossRef]
15. Fatma, M.K.H.; El-Deeb, A.A.; Abdeldaim, A.; Khan, Z.A. On some generalizations of dynamic Opial-type inequalities on time scales. *Adv. Differ. Equ.* **2019**, *2019*, 323.
16. Saker, S.H.; El-Deeb, A.A.; Rezk, H.M.; Agarwal, R.P. On Hilbert's inequality on time scales. *Appl. Anal. Discret. Math.* **2017**, *11*, 399–423. [CrossRef]
17. Tian, Y.; El-Deeb, A.A.; Meng, F. Some nonlinear delay Volterra-Fredholm type dynamic integral inequalities on time scales. *Discret. Dyn. Nat. Soc.* **2018**, *8*, 5841985. [CrossRef]
18. El-Deeb, A.A.-M.; Bazighifan, O.; Awrejcewicz, J. A Variety of Dynamic Steffensen-Type Inequalities on a General Time Scale. *Symmetry* **2021**, *13*, 1738. [CrossRef]
19. El-Deeb, A.A.; Makharesh, S.D.; Baleanu, D. Dynamic Hilbert-Type Inequalities with Fenchel-Legendre Transform. *Symmetry* **2020**, *12*, 582. [CrossRef]
20. El-Deeb, A.A.; Bazighifan, O.; Awrejcewicz, J. On Some New Weighted Steffensen-Type Inequalities on Time Scales. *Mathematics* **2021**, *9*, 2670. [CrossRef]
21. El-Deeb, A.A.; Baleanu, D. New Weighted Opial-Type Inequalities on Time Scales for Convex Functions. *Symmetry* **2020**, *12*, 842. [CrossRef]
22. El-Deeb, A.A.; Rashid, S.; Khan, Z.A.; Makharesh, S.D. New dynamic Hilbert-type inequalities in two independent variables involving Fenchel-Legendre transform. *Adv. Differ. Equ.* **2021**, *2021*, 239. [CrossRef]

23. Chen, Q.; Yang, B. A survey on the study of Hilbert-type inequalities. *J. Inequal. Appl.* **2015**, *2015*, 302. [CrossRef]
24. Huang, Q.; Yang, B. A multiple Hilbert-type integral inequality with a nonhomogeneous kernel. *J. Inequal. Appl.* **2013**, *2013*, 73. [CrossRef]
25. Kim, Y.H. An improvement of some inequalities similar to Hilbert's inequality. *Int. J. Math. Math. Sci.* **2001**, *28*, 211–221. [CrossRef]
26. Pachpatte, B.G. On some new inequalities similar to Hilbert's inequality. *J. Math. Appl.* **1998**, *226*, 166–179. [CrossRef]
27. Agarwal, R.; O'Regan, D.; Saker, S. *Dynamic Inequalities on Time Scales*; Springer: Cham, Switzerland, 2014.