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Efficient Splitting Methods for Solving Tensor Absolute Value Equation

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Abstract: The tensor absolute value equation is a class of interesting structured multilinear systems. In this article, from the perspective of pure numerical algebra, we first consider a tensor-type successive over-relaxation method (SOR) (called TSOR) and tensor-type accelerated over-relaxation method (AOR) (called TAOR) for solving tensor absolute value equations. Furthermore, one type of preconditioned tensor splitting method is also applied for solving structured multilinear systems. Numerical experiments adequately demonstrate the efficiency of the presented methods.

Keywords: tensor absolute value equation (TAVE); splitting iteration; successive over-relaxation method (SOR); accelerated over-relaxation method (AOR); preconditioned technique

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1. Introduction

In this paper, we focus on the following tensor absolute value equation (TAVE):

$$\mathcal{A}x^{m-1} - |x|^{[m-1]} = b, \quad (1)$$

where $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is an order m dimension n tensor, vector $b \in \mathbb{R}^n$ and $|x|^{[m-1]}$ is dimension n with the following:

$$|x|^{[m-1]} := (|x_1|^{m-1}, |x_2|^{m-1}, \dots, |x_n|^{m-1})^T \in \mathbb{R}^n, \quad (2)$$

and $x \in \mathbb{R}^n$ is an undetermined vector.

It is obvious that (1) will reduce to the well-known absolute value equation (AVE) as follows:

$$Ax - |x| = b, \quad (3)$$

when $m = 2$, and A is a matrix with appropriate dimensions. Formula (1), in fact, can also be regarded as multilinear systems:

$$\mathcal{F}x^{m-1} = b \quad (4)$$

with special structure $\mathcal{F} = \mathcal{A} - D_x \cdot \mathcal{I}_m$, where the following is the case.

$$D_x = \text{diag}(\text{sgn}(x)) \in \mathbb{R}^{n \times n}, \mathcal{I}_m \in \mathbb{R}^{[m,n]}. \quad (5)$$

Symbol $' \cdot '$ denotes a matrix-tensor product (see details in the next section).

An essential problem in pure and applied mathematics is solving various linear or nonlinear systems. The rapid and efficient techniques of solution-finding for multilinear systems [1–6]) are becoming increasingly significant in the field of science and engineering

due to their extensive applications. In particular, some approaches with respect to higher order tensors [7–15] for data analysis have been widely studied in the era of Big Data.

Song and Qi [16] introduced a class of complementarity problems, named tensor complementarity problems. It was testified that the well-known absolute value equation (AVE) [17] is equivalent to a generalized linear complementarity problem [18] (see [19,20] for more details).

Normally, it is hard to obtain the exact solution by means of direct methods even for smaller-scale general linear systems, which greatly urges substantial developments of presenting various iterative strategies. Many research works have been investigated fast solvers for multilinear systems. Ding and Wei [21,22] proposed some classical iteratives, such as Jacobi, Gauss–Seidel methods and Newton methods by translating (4) into an optimization problem. In general, the computational cost for the Newton method is expensive on account of the inverse of the matrix involved. Then, Han [23] investigated a homotopy method by the Euler–Newton prediction–correction technique to solve multilinear systems with nonsymmetric $\mathcal{M}$ -tensors, which demonstrated a better result than the Newton method in the sense of convergence performances. The tensor splitting method and its convergence results have been studied by Liu and Li et al. [24]. Furthermore, some comparison results for splitting iteration for solving multilinear systems were investigated widely in [25,26]. It is observed that the results of studies about tensor absolute value equation (TAVE) (1) are relatively rare in the existing literature. Du et al. presented the tensor absolute value equation (TAVE) [27] and introduced the Levenberg–Marquardt method for solving this (1) from the point of an optimization theory. Ling et al. [28] provided some conditions for the existence of solutions of TAVE with the help of degree theory. Moreover, by fixed point theory, the system of TAVE has at least one solution under some workable assumptions.

Fixed-point typed methods and splitting tricks can be applicable for the consideration of some problems with low-rank tensor structure, e.g., the dynamical or steady-state multicomponent coagulation equation [29,30]. Recently, a novel approach was proposed to solve TAVE, where the interval tensor is introduced, which is the main tool for proving the unique solvability of TAVE. Then, a sufficient condition for TAVE was presented to obtain a unique solution [31]. TAVE can be related to global optimization via tensor-formats, e.g., tensor train [32]. In this case, the tensor-matrix can be considered as a diagonal, and the global maximum is exactly the largest eigenvalue of such diagonalized tensor. To our knowledge, there have been not many research works for solving TAVE from the splitting iteration perspective. Based on this consideration, some efficient splitting iteration methods will be studied to solve TAVE. Tensor-type successive overrelaxation method (TSOR) and tensor-type accelerated overrelaxation method (TAOR) are intensively surveyed in this paper. Furthermore, a preconditioner is also constructed ultimately for solving TAVE (1).

The remainder of this paper is organized as follows. In Section 2, some basic and useful notations are described simply. In Section 3, we will propose tensor splitting iteration schemes TAOR and TSOR for solving TAVE. Meanwhile, the analyses of convergence conditions are established in detail. Then, some numerical experiments are provided in Section 4 to illustrate the superiority of the presented iteration methods. Finally, a conclusion is provided in Section 5.

2. Preliminaries

Let $A \in \mathbb{R}^{[2,n]}$ and $B \in \mathbb{R}^{[k,n]}$. The matrix-tensor product $\mathcal{C} = A \cdot B \in \mathbb{R}^{[k,n]}$ is defined by the following.

$$c_{ji_2 \cdots i_k} = \sum_{j_2=1}^n a_{jj_2} b_{j_2 i_2 \cdots i_k}. \quad (6)$$

The above formula can be regarded as follows:

$$\mathcal{C}_{(1)} = (\mathcal{A} \cdot \mathcal{B})_{(1)} = \mathcal{A}\mathcal{B}_{(1)}, \quad (7)$$

where $\mathcal{C}_{(1)}$ and $\mathcal{B}_{(1)}$ are the matrices generated from $\mathcal{C}$ and $\mathcal{B}$ flattened along first index. For more details, see [33,34].

Definition 1. (In [35]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. Then, the majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is the $n \times n$ matrix with the entries.

$$M(\mathcal{A})_{ij} = a_{ij \dots j}, \quad i, j = 1, 2, \dots, n. \quad (8)$$

Definition 2. (In [25]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$, we call $M(\mathcal{A})^{-1}$ the order-2 left-inverse of tensor $\mathcal{A}$, and $\mathcal{A}$ is called left-nonsingularity, where $\mathcal{I}_m$ is an identity tensor with all diagonal elements being 1.

Definition 3. (In [25]). Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is named as a splitting of tensor $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular. A regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $M(\mathcal{E})^{-1} \geq 0$ and $\mathcal{F} \geq 0$ (here, $\leq$ or $\geq$ denotes elementwise). A weak regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $M(\mathcal{E})^{-1}\mathcal{F} \geq 0$. A convergence splitting occurs if the spectral radius of $M(\mathcal{E})^{-1}\mathcal{F}$ is less than 1, i.e., $\rho(M(\mathcal{E})^{-1}\mathcal{F}) < 1$.

Definition 4. (In [36]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector of tensor $\mathcal{A}$ if they satisfy the following systems:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \quad (9)$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^T$. (λ, x) is named as an H-eigenpair if both λ and vector x are real.

Definition 5. Let $\rho(\mathcal{A}) = \max \{|\lambda| \mid \lambda \in \sigma(\mathcal{A})\}$ be the spectral radius of $\mathcal{A}$, where $\sigma(\mathcal{A})$ is the set of all eigenvalues of $\mathcal{A}$.

Definition 6. (In [37]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called a Z-tensor if its off-diagonal entries are non-positive. $\mathcal{A}$ is called an M-tensor if there exists a non-negative tensor $\mathcal{B}$ and a positive real number $\eta \geq \rho(\mathcal{B})$ such that

$$\mathcal{A} = \eta\mathcal{I}_m - \mathcal{B}.$$

If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong M-tensor.

3. Reformulation and Tensor Splitting Iteration

We first transform the TAVE (1) into the equivalent form.

$$\begin{cases} \mathcal{A}x^{m-1} - \mathcal{I}_m y^{m-1} = b, \\ -|x|^{[m-1]} + \mathcal{I}_m y^{m-1} = 0, \end{cases} \quad (10)$$

In other words, we have the following:

$$Hz = f, \quad (11)$$

where

$$H = \begin{pmatrix} M(\mathcal{A}) & -M(\mathcal{I}_m) \\ -D_x & M(\mathcal{I}_m) \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \quad z = \begin{pmatrix} \mathcal{I}_m x^{m-1} \\ \mathcal{I}_m y^{m-1} \end{pmatrix} \in \mathbb{R}^{2n}, \quad f = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{2n}, \quad (12)$$

$\mathcal{I}_m$ is an m -order n -dimensional unit tensor for which its entries are 1 if and only if $i_1 = \dots = i_m$ and otherwise zero, $M(\cdot)$ is the majorization matrix in Definition 1 and symbolic matrix D_x is defined by (5).

If we consider the tensor splitting [38,39]:

$$\mathcal{B} = \mathcal{E} - \mathcal{F},$$

then, it is easy to briefly establish an iterative method for solving multi-linear systems

$$\mathcal{B}x^{m-1} = b.$$

Clearly, the above multi-linear systems can be written as follows.

$$\mathcal{E}x^{m-1} = \mathcal{F}x^{m-1} + b, \quad (13)$$

In other words, we have the following.

$$\mathcal{I}_m x^{m-1} = M(\mathcal{E})^{-1} \mathcal{F} x^{m-1} + M(\mathcal{E})^{-1} b, \quad (14)$$

Here, we use the property of two-order left-nonsingularity of tensor $\mathcal{E}$, and $\mathcal{I}_m$ is an identify tensor with appropriate orders.

Moreover, it conceives the following splitting:

$$\mathcal{A} = \mathcal{A}_D - \mathcal{A}_L - \mathcal{A}_U,$$

where $\mathcal{A}_D$ is the diagonal tensor, and $\mathcal{A}_L$ and $\mathcal{A}_U$ are strictly lower and upper triangular tensors generated by tensor $\mathcal{A}$ in (10) [25].

If we consider the TAOR iterative method for (11), it gives rise to the following:

$$(H_D - \sigma H_L)z_{(k+1)} = [(1 - \omega)H_D + (\omega - \sigma)H_L + \omega H_U]z_{(k)} + \omega f, \quad (15)$$

where parameters satisfy $0 < \omega < 2, \sigma > 0$ and the following.

$$H_D = \begin{pmatrix} M(\mathcal{A}_D) & 0 \\ 0 & M(\mathcal{I}_m) \end{pmatrix}, H_L = \begin{pmatrix} M(\mathcal{A}_L) & 0 \\ D_x & 0 \end{pmatrix}, H_U = \begin{pmatrix} M(\mathcal{A}_U) & M(\mathcal{I}_m) \\ 0 & 0 \end{pmatrix}. \quad (16)$$

Thus, we have the following:

$$\begin{aligned} & \begin{pmatrix} M(\mathcal{A}_D) - \sigma M(\mathcal{A}_L) & 0 \\ -\sigma D_x & M(\mathcal{I}_m) \end{pmatrix} \begin{pmatrix} \mathcal{I}_m x_{(k+1)}^{m-1} \\ \mathcal{I}_m y_{(k+1)}^{m-1} \end{pmatrix} \\ &= \begin{pmatrix} (1 - \omega)M(\mathcal{A}_D) + (\omega - \sigma)M(\mathcal{A}_L) + \omega M(\mathcal{A}_U) & \omega M(\mathcal{I}_m) \\ (\omega - \sigma)D_x & (1 - \omega)M(\mathcal{I}_m) \end{pmatrix} \begin{pmatrix} \mathcal{I}_m x_{(k)}^{m-1} \\ \mathcal{I}_m y_{(k)}^{m-1} \end{pmatrix} + \omega f, \end{aligned}$$

which can be written as follows.

$$\begin{cases} \mathcal{I}_m x_{(k+1)}^{m-1} = [M(\mathcal{A}_D - \sigma \mathcal{A}_L)]^{-1} \left\{ [(1 - \omega)\mathcal{A}_D + (\omega - \sigma)\mathcal{A}_L + \omega \mathcal{A}_U] x_{(k)}^{m-1} + \omega(\mathcal{I}_m y_{(k)}^{m-1} + b) \right\}, \\ \mathcal{I}_m y_{(k+1)}^{m-1} = (\omega - \sigma)|x_k|^{[m-1]} + (1 - \omega)\mathcal{I}_m y_{(k)}^{m-1} + \sigma|x_{k+1}|^{[m-1]}. \end{cases} \quad (17)$$

Based on the above analysis, the following Algorithm 1 is proposed formally.

Algorithm 1 Tensor AOR splitting iterative method (TAOR) for TAVE

Step 1 Input vector b , tensor $\mathcal{A}$. Given a precision $\varepsilon > 0$, parameters $0 < \omega < 2, \sigma > 0$, and initial vector x_0, y_0 . Set $k := 0$.

Step 2 If $\|\mathcal{A}x_{(k)}^{m-1} - |x_{(k)}|^{[m-1]} - b\|_2 < \varepsilon$ stop; otherwise, go to Step 3.

Step 3 Compute iterative schemes

$$\begin{cases} x_{(k+1)} = \left\{ [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} \left\{ [(1-\omega)\mathcal{A}_D + (\omega-\sigma)\mathcal{A}_L + \omega\mathcal{A}_U]x_{(k)}^{m-1} + \omega(\mathcal{I}_m y_{(k)}^{m-1} + b) \right\} \right\}^{\left[\frac{1}{m-1}\right]}, \\ y_{(k+1)} = \left\{ (\omega-\sigma)|x_{(k)}|^{[m-1]} + (1-\omega)\mathcal{I}_m y_{(k)}^{m-1} + \sigma|x_{(k+1)}|^{[m-1]} \right\}^{\left[\frac{1}{m-1}\right]}. \end{cases} \quad (18)$$

Step 4 Set $k := k + 1$, return to Step 2.

Remark 1. If we take $\sigma = \omega$ in (17), the TAOR method will be reduced to the TSOR method with the following iterative schemes.

$$\begin{cases} x_{(k+1)} = \left\{ [M(\mathcal{A}_D - \omega\mathcal{A}_L)]^{-1} \left\{ [(1-\omega)\mathcal{A}_D + \omega\mathcal{A}_U]x_{(k)}^{m-1} + \omega(\mathcal{I}_m y_{(k)}^{m-1} + b) \right\} \right\}^{\left[\frac{1}{m-1}\right]}, \\ y_{(k+1)} = \left\{ (1-\omega)\mathcal{I}_m y_{(k)}^{m-1} + \omega|x_{(k+1)}|^{[m-1]} \right\}^{\left[\frac{1}{m-1}\right]}. \end{cases}$$

Remark 2. Furthermore, let us consider a preconditioned TAVE:

$$P \cdot \mathcal{A}x^{m-1} = Pc,$$

where $c = \mathcal{I}_m y_{(k)}^{m-1} + b$. It follows from the following splitting:

$$\begin{aligned} \hat{\mathcal{A}} &= P \cdot \mathcal{A} \\ &= P \cdot \mathcal{A}_D - P \cdot \mathcal{A}_L - P \cdot \mathcal{A}_U \\ &= \hat{\mathcal{A}}_D - \hat{\mathcal{A}}_L - \hat{\mathcal{A}}_U \end{aligned}$$

that the following is the case.

$$\begin{cases} x_{(k+1)} = \left\{ [M(\hat{\mathcal{A}}_D - \omega\hat{\mathcal{A}}_L)]^{-1} \left\{ [(1-\omega)\hat{\mathcal{A}}_D + \omega\hat{\mathcal{A}}_U]x_{(k)}^{m-1} + \omega P(\mathcal{I}_m y_{(k)}^{m-1} + b) \right\} \right\}^{\left[\frac{1}{m-1}\right]}, \\ y_{(k+1)} = \left\{ (1-\omega)\mathcal{I}_m y_{(k)}^{m-1} + \omega|x_{(k+1)}|^{[m-1]} \right\}^{\left[\frac{1}{m-1}\right]}. \end{cases}$$

In the numerical experiment section, a type of preconditioner will be introduced to further accelerate Algorithm 1.

Next, we will provide some critical lemmas [21,27] about the existence of solution for TAVE.

Lemma 1. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ is a strong $\mathcal{M}$ -tensor, then for every positive vector b , the multilinear system of equations $\mathcal{A}x^{m-1} = b$ has a unique positive solution.

Lemma 2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ be a $\mathcal{Z}$ -tensor. Then, it is a strong $\mathcal{M}$ -tensor if and only if the multilinear system of equations $\mathcal{A}x^{m-1} = b$ has a unique positive solution for every positive vector b .

Lemma 3. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ be an $\mathcal{M}$ -tensor and $b \geq 0$. If there exists $v \geq 0$ such that $\mathcal{A}v^{m-1} \geq b$, then the multilinear system of equations $\mathcal{A}x^{m-1} = b$ has a non-negative solution.

Lemma 4. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ can be written as $\mathcal{A} = \eta\mathcal{I} - \mathcal{B}$ with $\mathcal{B} \geq 0$ and $\eta \geq \rho(\mathcal{B}) + 1$, then for every positive vector b , TAVE (10) has a unique positive solution.

Lemma 5. Let the real quadratic equation $x^2 - px + q = 0$, where p and q are real numbers. Both roots of the real quadratic equation are less than one in modulus if and only if $|q| < 1$ and $|p| < 1 + q$.

For the sake of convergence analysis, it is necessary to provide the definition of iterative errors as follows.

$$\begin{aligned} e_{(k)}^x &: = \mathcal{I}_m x_*^{m-1} - \mathcal{I}_m x_{(k)}^{m-1} \\ &= x_*^{[m-1]} - x_{(k)}^{[m-1]}, \end{aligned} \quad (19)$$

$$\begin{aligned} e_{(k)}^y &: = \mathcal{I}_m y_*^{m-1} - \mathcal{I}_m y_{(k)}^{m-1} \\ &= y_*^{[m-1]} - y_{(k)}^{[m-1]}. \end{aligned} \quad (20)$$

Theorem 1. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ can be written as $\mathcal{A} = \eta \mathcal{I} - \mathcal{B}$ with $\mathcal{B} \geq 0$ and $\eta \geq \rho(\mathcal{B}) + 1$, $b > 0$, then the iterative sequence $\{x_{(k)}\}$ ($k = 0, 1, \dots$) generated by the Algorithm 1 converges to a unique positive solution x_* of the TAVE (10) if the following is the case:

$$|\kappa\theta_1 - \theta_2| < 1, \quad |\kappa + \theta_1 + \theta_3| < 1 + \kappa\theta_1 - \theta_2, \quad (21)$$

where

$$\theta_1 = |1 - \omega|, \theta_2 = |\omega - \sigma|, \theta_3 = \omega\sigma\rho \quad (22)$$

and the following is obtained.

$$\kappa = \left\| [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} [(1 - \omega)M(\mathcal{A}_D) + (\omega - \sigma)M(\mathcal{A}_L) + \omega M(\mathcal{A}_U)] \right\|, \quad (23)$$

$$\rho = \left\| [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} \right\|. \quad (24)$$

Proof. Clearly, the existence of solution x_* for TAVE is guaranteed by Lemmas 1–4.

From (18)–(20), it generates the following.

$$\begin{cases} e_{(k+1)}^x = [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} \left\{ [(1 - \omega)\mathcal{A}_D + (\omega - \sigma)\mathcal{A}_L + \omega\mathcal{A}_U] e_{(k)}^x + \omega e_{(k)}^y \right\}, \\ e_{(k+1)}^y = (\omega - \sigma)(|x_*|^{[m-1]} - |x_{(k)}|^{[m-1]}) + (1 - \omega)e_{(k)}^y + \sigma(|x_*|^{[m-1]} - |x_{(k+1)}|^{[m-1]}). \end{cases} \quad (25)$$

By (25), we have the following.

$$\begin{aligned} \|e_{(k+1)}^x\| &= \left\| [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} [(1 - \omega)M(\mathcal{A}_D) + (\omega - \sigma)M(\mathcal{A}_L) \right. \\ &\quad \left. + \omega M(\mathcal{A}_U)] \right\| \|e_{(k)}^x\| + \omega \left\| [M(\mathcal{A}_D - \sigma\mathcal{A}_L)]^{-1} \right\| \|e_{(k)}^y\| \\ &= \kappa \|e_{(k)}^x\| + \omega\rho \|e_{(k)}^y\|, \end{aligned} \quad (26)$$

$$\begin{aligned} \|e_{(k+1)}^y\| &\leq |\omega - \sigma| \left| |x_*|^{[m-1]} - |x_{(k)}|^{[m-1]} \right| + |1 - \omega| \|e_{(k)}^y\| + \sigma \left| |x_*|^{[m-1]} - |x_{(k+1)}|^{[m-1]} \right| \\ &\leq |\omega - \sigma| \|e_{(k)}^x\| + |1 - \omega| \|e_{(k)}^y\| + \sigma \|e_{(k+1)}^x\|. \end{aligned} \quad (27)$$

Based on (26) and (27), it can be reformulated as follows.

$$\|e_{(k+1)}^y\| \leq (|\omega - \sigma| + \sigma\kappa) \|e_{(k)}^x\| + (|1 - \omega| + \sigma\omega\rho) \|e_{(k)}^y\|. \quad (28)$$

It follows from (26) and (28) that the following is the case.

$$\begin{pmatrix} \|e_{(k+1)}^x\| \\ \|e_{(k+1)}^y\| \end{pmatrix} \leq \begin{pmatrix} \kappa & \omega\rho \\ |\omega - \sigma| + \sigma\kappa & |1 - \omega| + \sigma\omega\rho \end{pmatrix} \begin{pmatrix} \|e_{(k)}^x\| \\ \|e_{(k)}^y\| \end{pmatrix} \quad (29)$$

$$\leq \cdots \leq \begin{pmatrix} \kappa & \omega\rho \\ |\omega - \sigma| + \sigma\kappa & |1 - \omega| + \sigma\omega\rho \end{pmatrix}^{k+1} \begin{pmatrix} \|e_{(0)}^x\| \\ \|e_{(0)}^y\| \end{pmatrix}.$$

Let the following be the case.

$$G = \begin{pmatrix} \kappa & \omega\rho \\ |\omega - \sigma| + \sigma\kappa & |1 - \omega| + \sigma\omega\rho \end{pmatrix}. \quad (30)$$

When $\rho(G) < 1$, it must be $\lim_{k \rightarrow \infty} G^{k+1} = 0$. That is, the following is the case:

$$\lim_{k \rightarrow \infty} \|e_{(k)}^x\| = 0, \quad \lim_{k \rightarrow \infty} \|e_{(k)}^y\| = 0, \quad (31)$$

In other words, we have the following.

$$\lim_{k \rightarrow \infty} x_{(k)} = x_*, \quad \lim_{k \rightarrow \infty} y_{(k)} = y_*. \quad (32)$$

Now, we should focus on inequality $\rho(G) < 1$. First let (λ, v) be an eigen-pair of matrix G . The eigenvalue equation is as follows:

$$Gv = \lambda v, \quad (33)$$

and it implies the following:

$$\begin{cases} kv_1 + \omega\rho v_2 = \lambda v_1, \\ (|\omega - \sigma| + \sigma\kappa)v_1 + (|1 - \omega| + \sigma\omega\rho)v_2 = \lambda v_2, \end{cases} \quad (34)$$

where $v = (v_1^T, v_2^T)^T$. By some calculations, it generates the following.

$$\lambda^2 - [\kappa + |1 - \omega| + \omega\sigma\rho]\lambda + \kappa|1 - \omega| - |\omega - \sigma| = 0. \quad (35)$$

According to Lemma 5, all eigenvalues λ are less than one in modulus if and only if the following is the case.

$$|\kappa\theta_1 - \theta_2| < 1, \quad |\kappa + \theta_1 + \theta_3| < 1 + \kappa\theta_1 - \theta_2. \quad (36)$$

This completes the proof. $\square$

4. Numerical Experiments

In this section, some numerical examples are discussed to validate the performance of the effectiveness of the proposed tensor AOR method ('TAOR') (Algorithm 1) and tensor inexact Levenberg–Marquardt-type method (TILM) (see [27]) for solving the tensor absolute value equation. We compare their performances of convergence by the iteration step (denoted as 'IT'), elapsed CPU time in seconds (denoted as 'CPU') and residual error (denoted as 'ERR') defined by the following.

$$ERR := \|Ax_k^{m-1} - |x_k|^{[m-1]} - b\|. \quad (37)$$

All computations are started from $x_{(0)} = 0$ and $y_{(0)} = 0$ and terminated once ERR at the current iterative $x_{(k)} \leq 10^{-10}$ or the maximal number of iterative k_{max} exceeds 1000.

In what follows, we consider the tensor preconditioned splitting of (10):

$$P \cdot \mathcal{A} = \hat{\mathcal{A}} = \hat{\mathcal{A}}_D - \hat{\mathcal{A}}_L - \hat{\mathcal{A}}_U,$$

where $\hat{\mathcal{A}}_D = A_D \mathcal{I}_m$, $\hat{\mathcal{A}}_L = A_L \mathcal{I}_m$, $\hat{\mathcal{A}}_U = A_U \mathcal{I}_m$, and A_D , $-A_L$, $-A_U$ are the diagonal parts, strictly lower and strictly upper triangle part of $M(P \cdot \mathcal{A})$ and $\mathcal{I}_m$ is a unit tensor as previously mentioned.

A type of preconditioner (as a variant form in [40]) $P_\alpha = I + S_\alpha$ is considered, where the following is the case.

$$S_\alpha = \begin{pmatrix} 0 & -\alpha_1 a_{12\dots 2} & 0 & \cdots & 0 \\ -\alpha_1 a_{21\dots 1} & 0 & -\alpha_2 a_{23\dots 3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -\alpha_{n-1} a_{n-1,n\dots n} \\ 0 & 0 & 0 & -\alpha_{n-1} a_{n,n-1\dots n-1} & 0 \end{pmatrix},$$

$\alpha_i = 0.01, i = 1, 2, \dots, n-1$. I is an identity matrix with appropriate dimension.

All numerical experiments have been carried out by MATLAB R2011b 7.1.3 on a PC equipped with an Intel(R) Core(TM) i7-2670QM, CPU running at 2.20GHZ with 8 GB of RAM in Windows 7 operating system.

Example 1. First, consider the tensor absolute Equation (10) with a strong $\mathcal{M}$ -tensor $\mathcal{A}$ in different cases.

Case 1. $\mathcal{A} = \eta \mathcal{I}_3 - \mathcal{B}$, where $\mathcal{B} \in \mathbb{R}^{[3,n]}$ is a non-negative tensor with $b_{i_1, i_2, i_3} = |\tan(i_1 + i_2 + i_3)|$.

Case 2. $\mathcal{A} = n^3 \mathcal{I}_3 - \mathcal{B}$, where $\mathcal{B} \in \mathbb{R}^{[3,n]}$ is a non-negative tensor with $b_{i_1, i_2, i_3} = |\sin(i_1 + i_2 + i_3)|$.

Case 3. $\mathcal{A} = s \mathcal{I}_3 - \mathcal{B}$, where $\mathcal{B} \in \mathbb{R}^{[3,n]}$ is generated randomly by MATLAB, and $s = (1 + \delta) \max_{1,2,\dots,n} (\mathcal{B} \mathbf{e}^2)_i$, $\mathbf{e} = (1, 1, \dots, 1)^T$.

We provide three different cases for different tensors $\mathcal{A}$ and $\mathcal{B}$ with various sizes. Parameters σ , ω , η , δ and sizes n are concretely shown in Tables or Figures.

The numerical results have been shown in Tables 1–3 and Figures 1 and 2. From the numerical results, we can see that TAOR and TILM are all efficient methods. Case 3 seems to be the best condition. From an iterative parameter point of view in experiments, a better choice of σ and ω may be close to 1.2. It is observed that once parameters σ and ω are selected as 1, TAOR will reduce to TSOR, which is displayed in line 4 in Tables 1 and 2. By all performances of convergence, TSOR is also a quite efficient approach. However, TAOR seems to be a more efficient method than TILM and TSOR methods from all aspects due to the flexible selection of parameters. In particular, in the aspects of IT number and elapsed CPU, TAOR is superior to TILM distinctly, despite these ERRs of the latter descending rapidly in the last few steps. Furthermore, from the residual trend chart with the changing numbers of iteration in Figures 1 and 2, we can find the desired performances of the established methods.

Figure 1. The residual ERR for TAOR and TILM methods with different cases in Example 1, $\sigma = 1.1$, $\omega = 1.3$, $n = 5$.

Table 1. Preconditioned numerical results for Example 1, $n = 5$.

Case		1	2	3
TAOR, $\sigma = 1.1$, $\omega = 1.3$	It	18	6	5
	CPU	0.1413	0.1118	0.0743
	RES	3.2455×10^{-11}	2.6210×10^{-12}	6.7591×10^{-12}
TAOR, $\sigma = 0.1$, $\omega = 1.1$	It	15	5	5
	CPU	0.1215	0.1098	0.0713
	RES	2.8875×10^{-11}	9.1778×10^{-13}	3.9930×10^{-13}
TSOR, $\sigma = 1.0$, $\omega = 1.0$	It	20	6	5
	CPU	0.1662	0.1098	0.0914
	RES	4.0899×10^{-11}	1.8440×10^{-12}	6.6225×10^{-12}
TILM	It	34	25	28
	CPU	0.4920	0.3498	0.4861
	RES	1.4108×10^{-11}	5.5511×10^{-12}	1.1809×10^{-12}

Table 2. Preconditioned numerical results for Example 1, $n = 10$.

Case		1	2	3
TAOR, $\sigma = 1.2$, $\omega = 1.2$	It	10	4	6
	CPU	0.1295	0.1223	0.0807
	RES	1.4571×10^{-12}	6.0335×10^{-14}	4.2387×10^{-12}
TAOR, $\sigma = 0.2$, $\omega = 1.3$	It	10	10	9
	CPU	0.1894	0.1466	0.0960
	RES	1.4571×10^{-12}	6.7092×10^{-12}	1.9524×10^{-11}
TSOR, $\sigma = 1.0$, $\omega = 1.0$	It	10	5	5
	CPU	0.1503	0.1098	0.0914
	RES	1.4571×10^{-12}	7.0335×10^{-14}	6.6225×10^{-12}
TILM	It	45	43	35
	CPU	0.6000	0.5883	0.5320
	RES	7.1598×10^{-11}	6.8692×10^{-16}	4.9104×10^{-13}

Table 3. Preconditioned numerical results for Example 1, $n = 50$.

Case		1	2	3
TAOR, $\sigma = 0.8$, $\omega = 1.3$	It	13	6	8
	CPU	0.4183	0.3241	0.2626
	RES	1.6573×10^{-11}	7.0635×10^{-13}	5.2853×10^{-11}
TSOR, $\sigma = 0.8$, $\omega = 0.8$	It	13	7	9
	CPU	0.6902	0.5926	0.4531
	RES	1.5732×10^{-11}	8.4235×10^{-13}	6.8265×10^{-11}
TILM	It	48	46	39
	CPU	1.835	1.3452	1.2921
	RES	8.1845×10^{-10}	7.9655×10^{-13}	4.9104×10^{-11}

Figure 2. The residual ERR for TAOR and TILM methods with difference cases in Example 1, $\sigma = 1.2$, $\omega = 1.2$ and $n = 10$.

Example 2. Consider the following TAVE:

$$\mathcal{A}x^{m-1} - |x|^{[m-1]} = b, \quad (38)$$

where $\mathcal{A} = 2 * \mathcal{I}_m - \vartheta * B$.

$$B(:, :, 1) = \begin{pmatrix} 0.580 & 0.2432 & 0.1429 \\ 0 & 0.4109 & 0.0701 \\ 0.4190 & 0.3459 & 0.7870 \end{pmatrix}, \quad B(:, :, 2) = \begin{pmatrix} 0.4708 & 0.1330 & 0.0327 \\ 0.1341 & 0.5450 & 0.2042 \\ 0.3951 & 0.3220 & 0.7631 \end{pmatrix},$$

$$B(:, :, 3) = \begin{pmatrix} 0.4381 & 0.1003 & 0 \\ 0.0229 & 0.4338 & 0.0930 \\ 0.5390 & 0.4659 & 0.9070 \end{pmatrix},$$

$\mathcal{I}_m$ is an identity tensor of order 3 dimension 3. In this example, we set $x_* = (-1, 1, -1)^T$. Then, it generates the right hand b .

In this example, we discuss the probable optimal parameters of TAOR. In order to guarantee the strong $\mathcal{M}$ -tensor of $\mathcal{A}$, set $\vartheta = 0.1$. All numerical results are depicted in Figures 3–6.

From Figures 3–6, we find that when parameter ω is fixed on interval $[0.8–1.2]$, the other parameter σ shows better performance nearby 1.2 or 1.3. IT and CPU seem to have similar trends in their process of variation. Note that when ω is selected with 1.2, the changes of IT and CPU tend to be stable between 0.4 and 1.6 for parameter σ . These results show a similar trend with Example 1.

Figure 3. IT (a) and CPU (b) versus parameters σ , $\omega = 0.6$ in Example 2.

Figure 4. IT (a) and CPU (b) versus parameters σ , $\omega = 0.8$ in Example 2.

Figure 5. IT (a) and CPU (b) versus parameters σ , $\omega = 1.0$ in Example 2.

Figure 6. IT (a) and CPU (b) versus parameters σ , $\omega = 1.2$ in Example 2.

5. Conclusions

In this paper, tensor splitting iteration schemes TAOR and TSOR are proposed for solving the tensor absolute value equation (TAVE), which can be regarded as generalizations of AOR and SAOR for linear systems. Some efficient preconditioned techniques are provided to improve the efficiency of solution-finding of TAVE. Meanwhile, we attempt to provide probability selections of optimal parameters from the perspective of tentative examples. The proposed approaches have been demonstrated to be superior to the existing method (rare at present) under some conditions, which can be fully validated in our numerical experiments section. Theoretical optimal parameters will be the considerations of our future research work.

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