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An Exact Solution to a Quaternion Matrix Equation with an Application

Long-Sheng Liu [1], Qing-Wen Wang [1,2,*], Jiang-Feng Chen [1] and Yu-Zhu Xie [1]

[1] Department of Mathematics, Shanghai University, Shanghai 200444, China; liulongsheng@shu.edu.cn (L.-S.L.); c20720012@shu.edu.cn (J.-F.C.); yuzhu_xie@shu.edu.cn (Y.-Z.X.)
[2] Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, China
[*] Correspondence: wqw@t.shu.edu.cn or wqw369@yahoo.com

Abstract: In this paper, we establish the solvability conditions and the formula of the general solution to a Sylvester-like quaternion matrix equation. As an application, we give some necessary and sufficient conditions for a system of quaternion matrix equations to be consistent, and present an expression of the general solution of the system when it is solvable. We present an algorithm and an example to illustrate the main results of this paper. The findings of this paper generalize the known results in the literature.

Keywords: rank; generalized inverse; quaternion; linear matrix equation; solvability; necessary and sufficient conditions; applications

MSC: 15A03; 15A09; 15A24; 15B33

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1. Introduction

In this paper, we mainly investigate the following matrix equation:

$$A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 + A_2X_4B_3 = C_c \tag{1}$$

over the real quaternion algebra, $\mathbb{H}$, where A_1, A_2, B_1, B_2, B_3 and C_c are given matrices, while X_i $(i = \overline{1,4})$ are unknown.

The quaternion algebra, $\mathbb{H}$, is a non-commutative division ring. It has many applications in computer science, orbital mechanics, signal and color image processing, and control theory, and so on (see, e.g., [1–6]).

Linear matrix equation is one of active topics in mathematics. Besides mathematics, they also have important applications in other fields, such as descriptor systems control theorem [7], neural network [8], feedback [9], and graph theory [10]. There have been a large number of papers on this topic (see, e.g., [1–5,11–16]). We know that the following linear matrix equation:

$$A_1X_1B_1 = C \tag{2}$$

is both classical and fundamental, which was studied by many authors. For instance, Ben-Israel and Greville [17] gave a necessary and sufficient condition for the solvability to (2). Peng [18] presented some necessary and sufficient conditions for (2) to have a centrosymmetric solution by using generalized singular value decomposition. Huang [19] investigated the skew-symmetric solution and optimal approximate solution of (2). Recently, Xie and Wang [20] derived a necessary and sufficient condition for (2) to have a reducible solution. Furthermore, Xie and Wang [20] studied the following matrix equation:

$$A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_1 \tag{3}$$

which is the special case of (1). They provided some necessary and sufficient conditions for (3) to be consistent and gave an expression of its general solution when it is solvable. Motivated by the above, in this paper, we aim to establish some necessary and sufficient conditions for (1) to have a solution and derive an expression of its general solution when it is solvable. As an application of (1), we investigate the system of the following matrix equations:

$$\begin{aligned} E_1X_1 = F_1,\ X_1G_1 = H_1, \\ E_1X_2 = F_2,\ X_2G_2 = H_2, \\ E_2X_3 = F_3,\ X_3G_2 = H_3, \\ E_2X_4 = F_4,\ X_4G_3 = H_4, \\ E_{11}X_1F_{11}+E_{11}X_2F_{22} + E_{22}X_3F_{22} + E_{22}X_4F_{33} = T \end{aligned} \tag{4}$$

over $\mathbb{H}$, where $X_i(i = 1, \cdots, 4)$ are unknown quaternion matrices and the others are given.

The rest of this paper is structured as follows. In Section 2, we give preliminaries. In Section 3, we establish some necessary and sufficient conditions for the matrix Equation (1) to have a solution, and derive an expression of the general solution to (1) when it is solvable. as an application of (1), we derive some necessary and sufficient conditions for the system of matrix Equations (4) to have a solution as well as an expression of its general solution. Finally, we give a brief conclusion to close this paper in In Section 4.

2. Preliminaries

Throughout this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all $m \times n$ quaternion matrices by $\mathbb{H}^{m\times n}$, where we obtain the following:

$$\mathbb{H} = \{u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}|\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ u_0, u_1, u_2, u_3 \in \mathbb{R}\}.$$

Denoted by the rank of A by $r(A)$. I and 0 represent an identity matrix and a zero matrix of appropriate sizes, respectively. An inner inverse of A is denoted by A^- which satisfies $AA^-A = A$. L_A and R_A stand for the projectors $L_A = I - A^-A$ and $R_A = I - AA^-$, induced by A, respectively. It is easy to know that $L_A = (L_A)^2$, $R_A = (R_A)^2$.

Lemma 1 ([20])**.** *Let A_1, A_2, B_1, B_2 and C_1 be given matrices over $\mathbb{H}$ with suitable sizes. Put the following:*

$$B_3 = B_1L_{B_2},\ M = R_{A_1}A_2,\ C = C_1L_{B_2} + R_{A_1}C_1,\ N = B_2L_{B_3}.$$

Then, the following statements are equivalent:

(1) *Equation (3) is consistent.*

(2)

$$R_MR_{A_1}C = 0,\ CL_{B_3}L_N = 0,\ R_{A_1}CL_{B_2} = 0,\ R_MCL_{B_3} = 0.$$

(3)

$$r\begin{pmatrix} 2C_1 & A_2 & A_1 \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ r\begin{pmatrix} 2C_1 & A_2 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r(B_2),$$
$$r\begin{pmatrix} 2C_1 & A_1 \\ B_1 & 0 \\ B_2 & 0 \end{pmatrix} = r(A_1) + r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ r\begin{pmatrix} C_1 & A_1 \\ B_2 & 0 \end{pmatrix} = r(A_1) + r(B_2).$$

In this case, the general solution of (3) *can be expressed as follows:*

$$\begin{aligned} X_1 &= A_1^- C B_3^- + L_{A_1} V_1 + V_2 R_{B_3}, \\ X_2 &= A_1^- (C_1 - A_1 X_1 B_1 - A_2 X_3 B_2) B_2^- + T_1 R_{B_2} - L_{A_1} T_2, \\ X_3 &= M^- C B_2^- + L_M U_1 + U_2 R_{B_2}, \end{aligned}$$

where U_1, U_2, V_1, V_2, T_1 *and* T_2 *are arbitrary matrices over* $\mathbb{H}$*, with appropriate sizes.*

The following lemma is due to Marsaglia and Styan [21], which can be generalized to $\mathbb{H}$.

Lemma 2 ([21])**.** *Let* $A \in \mathbb{H}^{m\times n}$*,* $B \in \mathbb{H}^{m\times k}$*,* $C \in \mathbb{H}^{l\times n}$*,* $D \in \mathbb{H}^{j\times k}$ *and* $E \in \mathbb{H}^{l\times i}$ *be given. Then, we have the following rank equality:*

$$r\begin{pmatrix} A & BL_D \\ R_E C & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 3 ([22])**.** *Let* A_{ii}, B_{ii} *and* C_{ii} $(i = \overline{1,2})$ *be given matrices over* $\mathbb{H}$ *with appropriate sizes. Put the following:*

$$\begin{aligned} &A_1 = A_{22} L_{A_{11}}, \quad B_1 = R_{B_{11}} B_{22}, \quad C_1 = C_{22} - A_{22} A_{11}^- C_{11} B_{11}^- B_{22}, \quad D_1 = R_{A_1} A_{22}, \\ &\phi = A_{11}^- C_{11} B_{11}^- + L_{A_{11}} A_1^- C_1 B_{22}^- - L_{A_{11}} A_1^- A_{22} D_1^- R_{A_1} C_1 B_{22}^- + D_1^- R_{A_1} C_1 B_1^- R_{B_{11}}. \end{aligned}$$

Then, the system of matrix equations $A_{ii} X B_{ii} = C_{ii}$ $(i = \overline{1,2})$ *has a solution if—and only if—the following is true:*

$$R_{A_{ii}} C_{ii} = 0, \quad C_{ii} L_{B_{ii}} = 0, \ (i = \overline{1,2}), \quad R_{A_1} C_1 L_{B_1} = 0.$$

In this case, the general solution to the system can be expressed as follows:

$$X = \phi + L_{A_{11}} L_{A_1} U_1 + U_2 R_{B_1} R_{B_{11}} + L_{A_{11}} U_3 R_{B_{22}} + L_{A_{22}} U_4 R_{B_{11}},$$

where U_i $(i = \overline{1,4})$ *are arbitrary matrices over* $\mathbb{H}$*, with appropriate sizes.*

Lemma 4 ([23])**.** *Let* $A_1 \in \mathbb{H}^{m_1\times n_1}, B_1 \in \mathbb{H}^{r_1\times s_1}, C_1 \in \mathbb{H}^{m_1\times r_1}$ *and* $C_2 \in \mathbb{H}^{n_1\times s_1}$ *be given. Then, we obtain the following system:*

$$A_1 X_1 = C_1, \quad X_1 B_1 = C_2 \tag{5}$$

which is consistent if—and only if—the following is true:

$$R_{A_1} C_1 = 0, \quad C_2 L_{B_1} = 0, \quad A_1 C_2 = C_1 B_1.$$

Under these conditions, a general solution to (5) *can be expressed as follows:*

$$X_1 = A_1^- C_1 + L_{A_1} C_2 B_1^- + L_{A_1} U_1 R_{B_1},$$

where U_1 *is an any matrix with conformable dimension.*

Lemma 5 ([24]). *Consider the following matrix equation:*

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{6}$$

over $\mathbb{H}$*, where* $A_1, B_1, C_3, D_3, C_4, D_4$ *and* E_1 *be given matrices of suitable sizes. Put the following:*

$$A = R_{A_1}C_3,\ B = D_3L_{B_1},\ C = R_{A_1}C_4,\ D = D_4L_{B_1},$$
$$E = R_{A_1}E_1L_{B_1},\ M = R_AC,\ N = DL_B,\ S = CL_M.$$

Then, the following statements are equivalent:

(1) *The matrix Equation* (6) *has a solution.*

(2)

$$R_MR_AE = 0,\ EL_BL_N = 0,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

(3)

$$r\begin{pmatrix} E_1 & C_4 & C_3 & A_1 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(C_4,\ C_3,\ A_1),$$
$$r\begin{pmatrix} E_1 & A_1 \\ D_3 & 0 \\ D_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \\ B_1 \end{pmatrix} + r(A_1),$$
$$r\begin{pmatrix} E_1 & C_3 & A_1 \\ D_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(C_3,\ A_1) + r\begin{pmatrix} D_4 \\ B_1 \end{pmatrix},$$
$$r\begin{pmatrix} E_1 & C_4 & A_1 \\ D_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(C_4,\quad A_1) + r\begin{pmatrix} D_3 \\ B_1 \end{pmatrix}.$$

In this case, the general solution to the matrix Equation (6) *can be expressed as follows:*

$$X_1 = A_1^-(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^-T_7B_1 + L_{A_1}T_6,$$
$$X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^- + A_1A_1^-T_7 + T_8R_{B_1},$$
$$X_3 = A^-EB^- - A^-CM^-EB^- - A^-SC^-EN^-DB^- - A^-ST_2R_NDB^- + L_AT_4 + T_5R_B,$$
$$X_4 = M^-ED^- + S^-SC^-EN^- + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

where $T_1, \ldots, T_8$ *are arbitrary matrices of appropriate sizes over* $\mathbb{H}$*.*

Lemma 6 ([22]). *Let* A_1, B_1 *and* C_1 *be given matrices over* $\mathbb{H}$ *with suitable sizes. Then, the matrix equation* $A_1X_1B_1 = C_1$ *is consistent if—and only if—*$R_{A_1}C_1 = 0,\ C_1L_{B_1} = 0$*. In this case, the general solution of the matrix equation can be expressed as follows:*

$$X_1 = A_1^-C_1B_1^- + L_{A_1}V + UR_{B_1},$$

where U *and* V *are any matrices with compatible dimensions over* $\mathbb{H}$*.*

3. The General Solution to the Matrix Equation (1)

For convenience, we define the notation as follows: let A_i, B_j $(i = \overline{1,2},\ j = \overline{1,3})$ be given matrices of suitable sizes over $\mathbb{H}$ and put the following:

$$\begin{aligned}
&B_4 = B_1 L_{B_2},\ B_5 = B_2 L_{B_4},\ A_{33} = R_{A_1} A_2,\ C_{44} = C_c L_{B_2} + R_{A_1} C_c,\ B_{33} = B_3 L_{B_2}, C_{33} = R_{A_1} C_c L_{B_2},\\
&A_{11} = A_2 + A_{33}, B_{11} = B_3 L_{B_4} L_{B_5},\ C_{11} = (I + R_{A_1}) C_c L_{B_4} L_{B_5}, A_{22} = R_{A_{33}} A_2,\ B_{22} = B_3 L_{B_2} L_{B_4},\\
&C_{22} = R_{A_{33}} C_{44} L_{B_4},\ M_1 = A_{22} L_{A_{11}}, N_1 = R_{B_{11}} B_{22},\ C_1 = C_{22} - A_{22} A_{11}^{-} C_{11} B_{11}^{-} B_{22},\ D_1 = R_{A_1} A_{22},\\
&\phi = A_{11}^{-} C_{11} B_{11}^{-} + L_{A_{11}} M_1^{-} C_1 B_{22}^{-} - L_{A_{11}} M_1^{-} A_{22} D_1^{-} R_{M_1} C_1 B_{22}^{-} + D_1^{-} R_{M_1} C_1 N_1^{-} R_{B_{11}},\\
&M_2 = \left[L_{A_{11}} L_{M_1}, L_{A_{33}} \right],\ N_2 = \begin{bmatrix} R_{N_1} R_{B_{11}} \\ R_{B_{33}} \end{bmatrix},\ A = R_{M_2} L_{A_{11}},\ B = R_{B_{22}} L_{N_2},\ C = R_{M_2} L_{A_{22}},\\
&D = R_{B_{11}} L_{N_2},\ E_1 = A_{33}^{-} C_{33} B_{33}^{-} - \phi,\ E = R_{M_2} E_1 L_{N_2},\ M = R_A C, \quad N = D L_B,\ S = C L_M,
\end{aligned} \tag{7}$$

and

$$S_1 = [I_m, 0], \quad S_2 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad S_3 = [0, I_m], \quad S_4 = \begin{bmatrix} 0 \\ I_n \end{bmatrix},$$

where I_m and I_n denote the unit matrices of order n and m, respectively. Then, we obtain the main theorem of this paper.

Theorem 1. *Consider* (1) *with the notation in* (7)*. The following statements are equivalent:*

(1) *The matrix Equation* (1) *has a solution.*

(2)

$$R_{A_{33}} R_{A_1} C_{44} = 0, \quad R_{A_{11}} C_{11} = 0, \quad C_{ii} L_{B_{ii}} = 0,\ (i = \overline{1,3}), \tag{8}$$

$$R_{M_1} C_1 L_{N_1} = 0, \quad R_A E L_D = 0, \quad R_M R_A E = 0, \quad E L_B L_N = 0, \quad R_C E L_B = 0. \tag{9}$$

(3)

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r(B_2), \tag{10}$$

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{11}$$

$$r\begin{pmatrix} 2C_c & A_1 \\ B_3 & 0 \\ B_2 & 0 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r\begin{pmatrix} B_3 \\ B_2 \\ B_1 \end{pmatrix}, \tag{12}$$

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_2 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r\begin{pmatrix} B_3 \\ B_2 \\ B_1 \end{pmatrix}, \tag{13}$$

$$r\begin{pmatrix} C_c & A_1 \\ B_3 & 0 \\ B_2 & 0 \end{pmatrix} = r(A_1) + r\begin{pmatrix} B_3 \\ B_2 \end{pmatrix}, \tag{14}$$

$$r\begin{pmatrix} 2C_c & 0 & 0 & A_2 & A_1 & 0 \\ B_3 & 0 & B_3 & 0 & 0 & 0 \\ 0 & A_2 & -2C_c & 0 & 0 & A_1 \\ B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 \\ B_2 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \end{pmatrix}, \tag{15}$$

$$r\begin{pmatrix} 0 & B_3 & B_3 & 0 & 0 \\ -2A_2 & 2C_c & 0 & A_1 & 0 \\ A_2 & 0 & C_c & 0 & A_1 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 & 0 \\ A_2 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 \\ B_2 & 0 \\ B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{16}$$

$$r\begin{pmatrix} 0 & 0 & -B_3 & B_3 & B_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2C_c & 0 & 0 & A_2 & 0 & A_1 & 0 \\ A_2 & 0 & 0 & 2C_c & 0 & 0 & 0 & 0 & A_1 \\ 0 & A_2 & 0 & 0 & C_c & 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} A_2 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 & B_3 \\ B_2 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix}, \tag{17}$$

$$
\begin{aligned}
&r\begin{pmatrix}
0 & B_3 & B_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -B_3 & B_3 & 0 & 0 & 0 & 0\\
2A_2 & 0 & 2C_c & 0 & 0 & 0 & A_1 & 0\\
0 & 2C_c & 0 & 0 & 0 & A_2 & 0 & A_1\\
A_2 & 0 & 0 & C_c & A_1 & 0 & 0 & 0\\
0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0
\end{pmatrix}\\
&= r\begin{pmatrix}
0 & A_2 & 0 & A_1 & 0\\
2A_2 & 0 & 0 & 0 & A_1\\
A_2 & 0 & A_1 & 0 & 0
\end{pmatrix} + r\begin{pmatrix}
B_3 & B_3 & 0\\
0 & B_3 & B_3\\
B_2 & 0 & 0\\
B_1 & 0 & 0\\
0 & B_2 & 0\\
0 & B_1 & 0\\
0 & 0 & B_2
\end{pmatrix},
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&r\begin{pmatrix}
0 & 0 & B_3 & B_3 & B_3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0\\
0 & 0 & 2C_c & 0 & 0 & 0 & A_1 & A_2 & 0\\
A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_1\\
0 & A_2 & 0 & 0 & 0 & C_c & 0 & 0 & 0\\
0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0
\end{pmatrix}\\
&= r\begin{pmatrix}
0 & 0 & A_2 & A_1 & 0\\
A_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & A_1\\
0 & A_2 & 0 & 0 & 0
\end{pmatrix} + r\begin{pmatrix}
B_3 & B_3 & B_3 & 0\\
0 & 0 & 0 & B_3\\
B_2 & 0 & 0 & 0\\
B_1 & 0 & 0 & 0\\
0 & B_2 & 0 & 0\\
0 & B_1 & 0 & 0\\
0 & 0 & B_1 & 0\\
0 & 0 & B_2 & 0\\
0 & 0 & 0 & B_2
\end{pmatrix}.
\end{aligned} \tag{19}
$$

In this case, the general solution to (1) *can be expressed as follows:*

$$
\begin{aligned}
&X_1 = A_1^{-}\left[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\right]B_4^{-} + L_{A_1}V_1 + V_2R_{B_4},\\
&X_2 = A_1^{-}(C_c - A_1X_1B_1 - A_2X_3B_2 - A_2X_4B_3)B_2^{-} + V_3R_{B_2} - L_{A_1}V_4,\\
&X_3 = M^{-}\left[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\right]B_2^{-} + L_MV_5 + V_6R_{B_2},\\
&X_4 = \phi + L_{A_{11}}L_{M_1}U_1 + U_2R_{N_1}R_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}},\\
&\textit{or}\\
&X_4 = A_{33}^{-}C_{33}B_{33}^{-} + L_{A_{33}}U_5 + U_6R_{B_{33}},
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
&U_1 = S_1\big[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\big],\\
&U_2 = \big[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_2}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\big]S_2,\\
&U_3 = A^-EB^- - A^-CM^-R_AEB^- - A^-SC^-EL_BN^-DB^- - A^-ST_3R_NDB^-\\
&+ L_AT_1 + T_2R_B,\\
&U_4 = M^-R_AED^- + L_MS^-SC^-EL_BN^- + L_ML_ST_4 + L_MT_3R_N + T_5R_D,\\
&U_5 = -S_3\big[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\big],\\
&U_6 = -\big[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\big]S_4,
\end{aligned}
$$

and V_i $(i = \overline{1,6})$, T_i $(i = \overline{1,8})$ are arbitrary matrices over $\mathbb{H}$, with appropriate sizes.

Proof. (1) $\Leftrightarrow$ (2): It is easy to know that Equation (1) can be written as follows:

$$A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_c - A_2X_4B_3. \tag{21}$$

Clearly, Equation (1) is solvable if—and only if—Equation (21) is consistent. By Lemma 1, we obtain that (21) is solvable if—and only if—there exists X_4 in (21) such that we obtain the following:

$$
\begin{aligned}
&R_{A_{33}}R_{A_1}\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big] = 0,\\
&\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]L_{B_3}L_{B_5} = 0,\\
&R_{A_{33}}\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]L_{B_4} = 0,\\
&R_{A_1}\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]L_{B_2} = 0,
\end{aligned}
$$

i.e.,

$$R_{A_{33}}R_{A_1}C_{44} = 0, \tag{22}$$

$$A_{ii}X_4B_{ii} = C_{ii}\ (i = 1,2), \tag{23}$$

and

$$A_{33}X_4B_{33} = C_{33}, \tag{24}$$

respectively. Moreover, when (21) is solvable, we obtain the following:

$$
\begin{aligned}
&X_1 = A_1^-\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]B_4^- + L_{A_1}V_1 + V_2R_{B_4},\\
&X_2 = A_1^-(C_c - A_1X_1B_1 - A_2X_3B_2 - A_2X_4B_3)B_2^- + V_3R_{B_2} - L_{A_1}V_4,\\
&X_3 = M^-\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]B_2^- + L_MV_5 + V_6R_{B_2},
\end{aligned}
$$

where B_4, B_5, A_{33} and M are defined by (7), V_i $(i = \overline{1,6})$, which are arbitrary matrices over $\mathbb{H}$, with appropriate sizes.

Hence, the matrix Equation (21) is solvable if—and only if—(22) holds, and there exists X_4, such that both (23) and (24) are solvable.

Next, we consider the common solution of (23) and (24). On the ond hand, by Lemma 3, the system (23) is solvable if—and only if—the following is true:

$$R_{A_{ii}}C_{ii} = 0, \quad C_{ii}L_{B_{ii}} = 0\ (i = 1,2), \quad R_{M_1}C_1L_{N_1} = 0, \tag{25}$$

in which case, the general solution of (23) can be expressed as follows:

$$X_4 = \phi + L_{A_{11}}L_{M_1}U_1 + U_2R_{N_1}R_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \tag{26}$$

where A_{ii} $(i = 1, 2)$, M_1, N_1 and C_1 are given by (7), and U_i $(i = \overline{1, 4})$ are arbitrary matrices over $\mathbb{H}$, with appropriate sizes. On the other hand, in view of Lemma 6, (24) is solvable if—and only if—the following is true:

$$R_{A_{33}}C_{33} = 0, \ C_{33}L_{B_{33}} = 0, \tag{27}$$

in which case, the general solution of (24) can be expressed as follows:

$$X_4 = A_{33}^{-}C_{33}B_{33}^{-} + L_{A_{33}}U_5 + U_6R_{B_{33}}, \tag{28}$$

where A_{33}, B_{33} and C_{33} are given by (7), and U_5 and U_6 are arbitrary matrices over $\mathbb{H}$, with appropriate sizes.

Clearly, the matrix Equations (23) and (24) have a common solution if—and only if—X_4 of (26) is equal to X_4 of (28). Letting X_4 of (26) be the one of (28), yielding the following:

$$(L_{A_{11}}L_{M_1}, \ L_{A_{33}})\begin{pmatrix} U_1 \\ -U_5 \end{pmatrix} + (U_2, \ -U_6)\begin{pmatrix} R_{N1}R_{B_{11}} \\ R_{B_{33}} \end{pmatrix} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}} = E_1,$$

i.e.,

$$M_2\begin{pmatrix} U_1 \\ -U_5 \end{pmatrix} + (U_2, \ -U_6)N_2 + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}} = E_1. \tag{29}$$

It follows from Lemma 5 that Equation (29) has a solution if—and only if—the following is true:

$$R_AEL_D = 0, \quad R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_CEL_B = 0. \tag{30}$$

In this case, the general solution to (29) can be expressed as follows:

$$\begin{aligned}
U_1 &= S_1\left[M_2^{-}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^{-}T_7N_2 + L_{M_2}T_6\right], \\
U_2 &= \left[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_2}U_4R_{B_{11}})N_2^{-} + M_2M_2^{-}T_7 + T_8R_{N_2}\right]S_2, \\
U_3 &= A^{-}EB^{-} - A^{-}CM^{-}R_AEB^{-} - A^{-}SC^{-}EL_BN^{-}DB^{-} - A^{-}ST_3R_NDB^{-} \\
&+ L_AT_1 + T_2R_B, \\
U_4 &= M^{-}R_AED^{-} + L_MS^{-}SC^{-}EL_BN^{-} + L_ML_ST_4 + L_MT_3R_N + T_5R_D, \\
U_5 &= -S_3\left[M_2^{-}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^{-}T_7N_2 + L_{M_2}T_6\right], \\
U_6 &= -\left[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})N_2^{-} + M_2M_2^{-}T_7 + T_8R_{N_2}\right]S_4,
\end{aligned}$$

where $M_1, M_2, N_1, N_2, A, B, D, E, S, M, N$ are defined as (7), and T_j $(j = \overline{1, 8})$ are arbitrary matrices over $\mathbb{H}$, with appropriate sizes.

So far, we have shown that Equation (21) is solvable if—and only if—(22), (25), (27), and (30) hold. In this case, the general solution of (21) can be expressed as (20).

We now show that $R_{A_{33}}C_{33} = 0 \Leftrightarrow R_{A_{33}}R_{A_1}C_{44} = 0$ and $R_{A_{22}}C_{22} = 0 \Leftrightarrow R_{A_{11}}C_{11} = 0$. In fact, it follows from Lemma 2 that we obtain the following:

$$\begin{aligned}
&R_{A_{33}}C_{33} = 0 \Leftrightarrow r(R_{A_{33}}C_{33}) = 0 \Leftrightarrow r(C_{33}, \ A_{33}) = r(A_{33}) \\
&\Leftrightarrow r(R_{A_1}C_cL_{B_2}, \ R_{A_1}A_2) = r(R_{A_1}A_2) \Leftrightarrow r\begin{pmatrix} C_c & A_2 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2, \ A_1) + r(B_2)
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&R_{A_{33}}R_{A_1}C_{44} = 0 \Leftrightarrow r\big(R_{A_{33}}R_{A_1}(C_cL_{B_2} + R_{A_1}C_c)\big) = 0 \\
&\Leftrightarrow r\big((C_cL_{B_2} + R_{A_1}C_c), \ A_2, \ A_1\big) = r(A_2, \ A_1) \\
&\Leftrightarrow r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2, \ A_1) + r(B_2)
\end{aligned} \tag{32}$$

yielding $R_{A_{33}}C_{33} = 0 \Leftrightarrow R_{A_{33}}R_{A_1}C_{44} = 0$.

On the other hand, we have the following:

$$\begin{aligned}
&R_{A_{11}}C_{11}=0 \Leftrightarrow r(R_{A_{11}}C_{11})=0 \Leftrightarrow r(C_{11},\ A_{11})=r(A_{11})\\
&\Leftrightarrow r[(I+R_{A_1})C_cL_{B_4}L_{B_5},\ (I+R_{A_1})A_2]=r[(I+R_{A_1})A_2]\\
&\Leftrightarrow r\begin{pmatrix} I & 0 & 0\\ 0 & (I+R_{A_1})C_cL_{B_4}L_{B_5} & (I+R_{A_1})A_2\end{pmatrix}=r\begin{pmatrix} I & 0\\ 0 & (I+R_{A_1})A_2\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} I & -C_c & -A_2 & 0\\ I & C_c & A_2 & A_1\\ 0 & B_2 & 0 & 0\\ 0 & B_1 & 0 & 0\end{pmatrix}=r\begin{pmatrix} I & -A_2 & 0\\ I & A_2 & A_1\end{pmatrix}+r\begin{pmatrix} B_2\\ B_1\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} 2C_c & A_2 & A_1\\ B_2 & 0 & 0\\ B_1 & 0 & 0\end{pmatrix}=r(A_2,\ A_1)+r\begin{pmatrix} B_2\\ B_1\end{pmatrix}\Leftrightarrow (11).
\end{aligned} \tag{33}$$

Similarly, we can show that $R_{A_{22}}C_{22}=0 \Leftrightarrow$ (11). Hence, $R_{A_{11}}C_{11}=0 \Leftrightarrow R_{A_{22}}C_{22}=0$. So far, we have proved that (22), (25), (27), and (30) hold if—and only if—(8) and (9) hold. To sum up, Equation (21), and thus Equation (1), are consistent if—and only if—(8) and (9) hold.

(2) $\Leftrightarrow$ (3): We first show that (8) holds if—and only if—(10)–(14) hold. It follows from (32) and (33) that (22) $\Leftrightarrow$ (10) and $R_{A_{11}}C_{11}=0 \Leftrightarrow$ (11). Next, we prove that $C_{11}L_{B_{11}}=0 \Leftrightarrow$ (12). By Lemma 2, as follows:

$$\begin{aligned}
&C_{11}L_{B_{11}}=0 \Leftrightarrow r(C_{11}L_{B_{11}})=0 \Leftrightarrow r\begin{pmatrix} C_{11}\\ B_{11}\end{pmatrix}=r(B_{11})\\
&\Leftrightarrow r\begin{pmatrix} (I+R_{A_1})C_c\\ B_3\\ B_2\\ B_1\end{pmatrix}=r\begin{pmatrix} B_3\\ B_2\\ B_1\end{pmatrix}\Leftrightarrow r\begin{pmatrix} I & 0\\ I & (I+R_{A_1})C_c\\ 0 & B_3\\ 0 & B_2\\ 0 & B_1\end{pmatrix}-r(I)=r\begin{pmatrix} B_3\\ B_2\\ B_1\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} I & -C_c & 0\\ I & C_c & A_1\\ 0 & B_3 & 0\\ 0 & B_2 & 0\\ 0 & B_1 & 0\end{pmatrix}-r(I)=r\begin{pmatrix} B_3\\ B_2\\ B_1\end{pmatrix}+r(A_1)\Leftrightarrow r\begin{pmatrix} 2C_c & A_1\\ B_3 & 0\\ B_2 & 0\\ B_1 & 0\end{pmatrix}=r\begin{pmatrix} B_3\\ B_2\\ B_1\end{pmatrix}+r(A_1)\Leftrightarrow (12).
\end{aligned}$$

Similarly, we can show that $C_{22}L_{B_{22}}=0 \Leftrightarrow$ (13) and $C_{33}L_{B_{33}}=0 \Leftrightarrow$ (14). Therefore, (8) holds if—and only if—(10)–(14) all hold.

We now turn our attention to show that (9) holds if—and only if—(15)–(19) hold. $R_{M_1}C_1L_{N_1}=0 \Leftrightarrow$ (15). In fact, it follows from Lemma 2 that the following is true:

$$
\begin{aligned}
&R_{M_1}C_1L_{N_1}=0 \Leftrightarrow r(R_{M_1}C_1L_{N_1})=0\\
&\Leftrightarrow r\begin{pmatrix} C_1 & M_1\\ N_1 & 0\end{pmatrix}=r(M_1)+r(N_1)\\
&\Leftrightarrow r\begin{pmatrix} C_{22} & A_{22} & 0\\ B_{22} & 0 & B_{11}\\ 0 & A_{11} & -C_{11}\end{pmatrix}=r\begin{pmatrix} A_{22}\\ A_{11}\end{pmatrix}+r(B_{22},\ B_{11})\\
&\Leftrightarrow r\begin{pmatrix} C_cL_{B_2}+R_{A_1}C_c & A_2 & 0 & A_{33}\\ B_3L_{B_2} & 0 & B_3 & 0\\ 0 & (I+R_{A_1})A_2 & -(I+R_{A_1})C_c & 0\\ B_1L_{B_2} & 0 & 0 & 0\\ 0 & 0 & B_4 & 0\\ 0 & 0 & B_2 & 0\end{pmatrix}\\
&=r\begin{pmatrix} B_3L_{B_2} & B_3\\ B_1L_{B_2} & 0\\ 0 & B_4\\ 0 & B_2\end{pmatrix}+r\begin{pmatrix} A_2 & A_{33}\\ (I+R_{A_1})A_2 & 0\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} I & 0 & 0 & 0 & 0\\ I & C_cL_{B_2}+R_{A_1}C_c & A_2 & 0 & A_{33}\\ 0 & B_3L_{B_2} & 0 & B_3 & 0\\ I & 0 & (I+R_{A_1})A_2 & -(I+R_{A_1})C_c & 0\\ 0 & B_1L_{B_2} & 0 & 0 & 0\\ 0 & 0 & 0 & B_4 & 0\\ 0 & 0 & 0 & B_2 & 0\end{pmatrix}\\
&=r\begin{pmatrix} B_3L_{B_2} & B_3\\ B_1L_{B_2} & 0\\ 0 & B_4\\ 0 & B_2\end{pmatrix}+r\begin{pmatrix} I & 0 & 0\\ 0 & A_2 & A_{33}\\ I & (I+R_{A_1})A_2 & 0\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} I & -C_c & -A_2 & C_c & 0 & 0 & 0\\ I & C_c & A_2 & 0 & A_2 & A_1 & 0\\ 0 & B_3 & 0 & B_3 & 0 & 0 & 0\\ I & 0 & A_2 & C_c & 0 & 0 & A_1\\ 0 & B_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & B_4 & 0 & 0 & 0\\ 0 & 0 & 0 & B_2 & 0 & 0 & 0\\ 0 & B_2 & 0 & 0 & 0 & 0 & 0\end{pmatrix}=r\begin{pmatrix} B_3 & B_3\\ B_1 & 0\\ 0 & B_4\\ 0 & B_2\\ B_2 & 0\end{pmatrix}+r\begin{pmatrix} I & -A_2 & 0 & 0 & 0\\ 0 & A_2 & A_2 & A_1 & 0\\ I & A_2 & 0 & 0 & A_1\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix} 2C_c & 0 & 0 & A_2 & A_1 & 0\\ B_3 & 0 & B_3 & 0 & 0 & 0\\ 0 & A_2 & 2C_c & 0 & 0 & A_1\\ B_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & B_1 & 0 & 0 & 0\\ 0 & 0 & B_2 & 0 & 0 & 0\\ B_2 & 0 & 0 & 0 & 0 & 0\end{pmatrix}=r\begin{pmatrix} A_2 & A_1 & 0 & 0\\ 0 & 0 & A_2 & A_1\end{pmatrix}+r\begin{pmatrix} B_3 & B_3\\ B_2 & 0\\ B_1 & 0\\ 0 & B_2\\ 0 & B_1\end{pmatrix}\Leftrightarrow (15).
\end{aligned}
$$

Similarly, we can show that $R_MR_AE=0\Leftrightarrow$ (16), $EL_BL_N=0\Leftrightarrow$ (17), $R_AEL_D=0\Leftrightarrow$ (18) and $R_CEL_B=0\Leftrightarrow$ (19). The proof is completed. Next, we use an Algorithm 1 for calculating Equation (1) to illustrate this theorem. □

3.1. Algorithm with a Numerical Example

In this section, we present an algorithm and an example to illustrate Theorem 1.

Algorithm 1: Algorithm for calculating Equation (1)

(1) Feed the values of A_i, B_j $(i = 1, 2, j = \overline{1,3})$ and C_c with conformable shapes over $\mathbb{H}$.
(2) Compute the symbols in (7).
(3) Check whether (8), (9) or rank equalities in (10)–(19) hold or not. If no, then return "inconsisten".
(4) Otherwise, compute X_i $(i = \overline{1,4})$.

Example 1. *Consider the matrix Equation* (1)*. Put the following:*

$$A_1 = \begin{pmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \\ a_{131} & a_{132} \end{pmatrix}, \; B_1 = \begin{pmatrix} b_{111} & b_{112} & b_{113} \\ b_{121} & b_{122} & b_{123} \end{pmatrix}, \; B_2 = \begin{pmatrix} b_{211} & b_{212} & b_{213} \\ b_{221} & b_{222} & b_{223} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} a_{211} & a_{212} \\ a_{221} & a_{222} \\ a_{231} & a_{232} \end{pmatrix}, \; B_3 = \begin{pmatrix} b_{311} & b_{312} & b_{313} \\ b_{321} & b_{322} & b_{323} \end{pmatrix}, \; C_c = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

where

$$\begin{aligned}
&a_{111} = 0.3181 + 0.5447\mathbf{i} + 0.2187\mathbf{j} + 0.3685\mathbf{k}, \; a_{112} = 0.6456 + 0.7210\mathbf{i} + 0.0636\mathbf{j} + 0.7720\mathbf{k},\\
&a_{121} = 0.1192 + 0.6473\mathbf{i} + 0.1058\mathbf{j} + 0.7635\mathbf{k}, \; a_{122} = 0.4795 + 0.5225\mathbf{i} + 0.4046\mathbf{j} + 0.9329\mathbf{k},\\
&a_{131} = 0.9398 + 0.5439\mathbf{i} + 0.1097\mathbf{j} + 0.6279\mathbf{k}, \; a_{132} = 0.6393 + 0.9937\mathbf{i} + 0.4484\mathbf{j} + 0.9727\mathbf{k},\\
&b_{111} = 0.1920 + 0.8611\mathbf{i} + 0.3477\mathbf{j} + 0.2428\mathbf{k}, \; b_{112} = 0.6963 + 0.3935\mathbf{i} + 0.5861\mathbf{j} + 0.6878\mathbf{k},\\
&b_{113} = 0.5254 + 0.7413\mathbf{i} + 0.0445\mathbf{j} + 0.7363\mathbf{k}, \; b_{121} = 0.1389 + 0.4849\mathbf{i} + 0.1500\mathbf{j} + 0.4424\mathbf{k},\\
&b_{122} = 0.0938 + 0.6714\mathbf{i} + 0.2621\mathbf{j} + 0.3592\mathbf{k}, \; b_{123} = 0.5303 + 0.5201\mathbf{i} + 0.7549\mathbf{j} + 0.3947\mathbf{k},\\
&b_{211} = 0.6834 + 0.2703\mathbf{i} + 0.7691\mathbf{j} + 0.7904\mathbf{k}, \; b_{212} = 0.4423 + 0.8217\mathbf{i} + 0.8085\mathbf{j} + 0.3276\mathbf{k},\\
&b_{213} = 0.3309 + 0.8878\mathbf{i} + 0.3774\mathbf{j} + 0.4386\mathbf{k}, \; b_{221} = 0.7040 + 0.1971\mathbf{i} + 0.3968\mathbf{j} + 0.9493\mathbf{k},\\
&b_{222} = 0.0196 + 0.4299\mathbf{i} + 0.7551\mathbf{j} + 0.6713\mathbf{k}, \; b_{223} = 0.4243 + 0.3912\mathbf{i} + 0.2160\mathbf{j} + 0.8335\mathbf{k},\\
&a_{211} = 0.7689 + 0.5880\mathbf{i} + 0.7900\mathbf{j} + 0.6787\mathbf{k}, \; a_{212} = 0.9899 + 0.4070\mathbf{i} + 0.0900\mathbf{j} + 0.4950\mathbf{k},\\
&a_{221} = 0.1673 + 0.1548\mathbf{i} + 0.3185\mathbf{j} + 0.4952\mathbf{k}, \; a_{222} = 0.5144 + 0.7487\mathbf{i} + 0.1117\mathbf{j} + 0.1476\mathbf{k},\\
&a_{231} = 0.8620 + 0.1999\mathbf{i} + 0.5341\mathbf{j} + 0.1897\mathbf{k}, \; a_{232} = 0.8843 + 0.8256\mathbf{i} + 0.1363\mathbf{j} + 0.0550\mathbf{k},\\
&b_{311} = 0.8507 + 0.8790\mathbf{i} + 0.5277\mathbf{j} + 0.5747\mathbf{k}, \; b_{312} = 0.9296 + 0.0005\mathbf{i} + 0.8013\mathbf{j} + 0.7386\mathbf{k},\\
&b_{313} = 0.5828 + 0.6126\mathbf{i} + 0.4981\mathbf{j} + 0.2467\mathbf{k}, \; b_{321} = 0.5606 + 0.9889\mathbf{i} + 0.4795\mathbf{j} + 0.8452\mathbf{k},\\
&b_{322} = 0.6967 + 0.8654\mathbf{i} + 0.2278\mathbf{j} + 0.5860\mathbf{k}, \; b_{323} = 0.8154 + 0.9900\mathbf{i} + 0.9009\mathbf{j} + 0.6664\mathbf{k},\\
&c_{11} = -38.7863 + 4.0617\mathbf{i} + 0.5536\mathbf{j} - 3.4984\mathbf{k}, \; c_{12} = -35.3609 - 9.0836\mathbf{i} - 1.7527\mathbf{j} - 6.7689\mathbf{k},\\
&c_{13} = -33.503 - 3.7707\mathbf{i} + 5.2872\mathbf{j} - 6.2708\mathbf{k}, \; c_{21} = -24.7749 - 3.6921\mathbf{i} - 0.5143\mathbf{j} - 10.8211\mathbf{k},\\
&c_{22} = -21.0950 - 11.3075\mathbf{i} - 3.9522\mathbf{j} - 10.8211\mathbf{k}, \; c_{23} = -18.9376 - 7.7280\mathbf{i} + 1.7134\mathbf{j} - 11.9519\mathbf{k},\\
&c_{31} = -36.2182 + 5.3877\mathbf{i} - 2.8839\mathbf{j} - 0.8389\mathbf{k}, \; c_{32} = -33.9232 - 7.3248\mathbf{i} - 4.3221\mathbf{j} - 3.9587\mathbf{k},\\
&c_{33} = -31.8026 - 1.6492\mathbf{i} + 1.2342\mathbf{j} - 6.7683\mathbf{k}.
\end{aligned}$$

Computing directly yields the following:

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r(B_2) = 5,$$

$$r\begin{pmatrix} 2C_c & A_1 \\ B_3 & 0 \\ B_2 & 0 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r\begin{pmatrix} B_3 \\ B_2 \\ B_1 \end{pmatrix} = 5,$$

$$r\begin{pmatrix} 2C_c & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_2 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_2,\ A_1) + r\begin{pmatrix} B_3 \\ B_2 \\ B_1 \end{pmatrix} = 6,$$

$$r\begin{pmatrix} C_c & A_1 \\ B_3 & 0 \\ B_2 & 0 \end{pmatrix} = r(A_1) + r\begin{pmatrix} B_3 \\ B_2 \end{pmatrix} = 5,$$

$$r\begin{pmatrix} 2C_c & 0 & 0 & A_2 & A_1 & 0 \\ B_3 & 0 & B_3 & 0 & 0 & 0 \\ 0 & A_2 & -2C_c & 0 & 0 & A_1 \\ B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 \\ B_2 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \end{pmatrix} = 12,$$

$$r\begin{pmatrix} 0 & B_3 & B_3 & 0 & 0 \\ -2A_2 & 2C_c & 0 & A_1 & 0 \\ A_2 & 0 & C_c & 0 & A_1 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 & 0 \\ A_2 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 \\ B_2 & 0 \\ B_1 & 0 \\ 0 & B_2 \end{pmatrix} = 11,$$

$$r\begin{pmatrix} 0 & 0 & -B_3 & B_3 & B_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2C_c & 0 & 0 & A_2 & 0 & A_1 & 0 \\ A_2 & 0 & 0 & 2C_c & 0 & 0 & 0 & 0 & A_1 \\ 0 & A_2 & 0 & 0 & C_c & 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} A_2 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 & B_3 \\ B_2 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix} = 18,$$

$$r\begin{pmatrix} 0 & B_3 & B_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_3 & B_3 & 0 & 0 & 0 & 0 \\ 2A_2 & 0 & 2C_c & 0 & 0 & 0 & A_1 & 0 \\ 0 & 2C_c & 0 & 0 & 0 & A_2 & 0 & A_1 \\ A_2 & 0 & 0 & C_c & A_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} 0 & A_2 & A_1 & 0 & 0 \\ A_2 & 0 & 0 & A_1 & 0 \\ A_2 & 0 & 0 & 0 & A_1 \end{pmatrix} + r\begin{pmatrix} B_3 & 0 & 0 \\ B_3 & B_3 & 0 \\ 0 & B_3 & B_3 \\ B_2 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix} = 17,$$

$$r\begin{pmatrix} 0 & 0 & B_3 & B_3 & B_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 \\ 0 & 0 & 2C_c & 0 & 0 & 0 & A_1 & A_2 & 0 \\ A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & A_2 & 0 & 0 & 0 & C_c & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} 0 & 0 & A_2 & A_1 & 0 \\ A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 \\ 0 & A_2 & 0 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} B_3 & B_3 & B_3 & 0 \\ 0 & 0 & 0 & B_3 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix} = 24.$$

All rank equalities in (10)–(19) hold. Hence, according to Theorem 1, Equation (1) is consistent, and the general solution to the matrix Equation (1) can be expressed as follows:

$$X_1 = \begin{pmatrix} 0.500 + 3.0117\mathbf{i} + 0.0161\mathbf{j} + 1.2813\mathbf{k} & -2.2870 - 5.3594\mathbf{i} + 2.8008\mathbf{j} - 3.4531\mathbf{k} \\ -0.3281 + 0.1484\mathbf{i} + 1.4375\mathbf{j} - 1.1836\mathbf{k} & 2.2500 + 1.9219\mathbf{i} + 0.4844\mathbf{j} + 7.1406\mathbf{k} \end{pmatrix} +$$

$$V_2 \begin{pmatrix} 0.2734 - 0.0010\mathbf{i} - 0.0020\mathbf{j} - 0.0078\mathbf{k} & -0.0117 - 0.1328\mathbf{i} - 0.0625\mathbf{j} - 0.4063\mathbf{k} \\ 0.0508 + 0.1519\mathbf{i} + 0.1387\mathbf{j} + 0.4042\mathbf{k} & -0.1973 + 0.0039\mathbf{i} - 0.2617\mathbf{j} \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 3\mathbf{i} + 2\mathbf{j} & 2\mathbf{k} + \mathbf{i} \\ 2\mathbf{j} + 3\mathbf{i} & 2\mathbf{i} + \mathbf{j} \end{pmatrix}, \; X_3 = \begin{pmatrix} x_{311} & x_{312} \\ x_{321} & x_{322} \end{pmatrix}, \; X_4 = \begin{pmatrix} x_{0411} & x_{0412} \\ x_{0421} & x_{0422} \end{pmatrix},$$

or

$$X_4 = \begin{pmatrix} x_{411} & x_{412} \\ x_{421} & x_{422} \end{pmatrix} + T_1U_5 + U_6T_2, \; U_1 = S_1(U_{11} - U_{12}T_7U_{13} + U_{14}T_6),$$

$$U_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T_7 + T_8 \begin{pmatrix} u_{211} & u_{212} & u_{213} & u_{214} \\ u_{221} & u_{222} & u_{223} & u_{224} \\ u_{231} & u_{232} & u_{233} & u_{234} \\ u_{241} & u_{242} & u_{243} & u_{244} \end{pmatrix} \right\} S_2,$$

$$U_3 = \begin{pmatrix} u_{311} & u_{312} \\ u_{321} & u_{322} \end{pmatrix}, \; U_4 = \begin{pmatrix} u_{411} & u_{412} \\ u_{421} & u_{422} \end{pmatrix}, \; U_5 = -S_3(U_{11} - U_{12}T_7U_{13} + U_{14}T_6),$$

$$U_6 = -\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T_7 + T_8 \begin{pmatrix} u_{211} & u_{212} & u_{213} & u_{214} \\ u_{221} & u_{222} & u_{223} & u_{224} \\ u_{231} & u_{232} & u_{233} & u_{234} \\ u_{241} & u_{242} & u_{243} & u_{244} \end{pmatrix} \right\} S_4, \; U_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 0.2500 + 0.0447\mathbf{i} - 0.0938\mathbf{j} - 0.2090\mathbf{k} & -0.1348 + 0.3086\mathbf{i} - 0.0781\mathbf{j} + 0.2188\mathbf{k} \\ -0.0938 - 0.3564\mathbf{i} + 0.0313\mathbf{j} + 0.3167\mathbf{k} & 0.4102 - 0.0117\mathbf{i} + 0.1094\mathbf{j} - 0.2109\mathbf{k} \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0.0005 + 0.0012\mathbf{i} + 0.0005\mathbf{j} + 0.0005\mathbf{k} & -0.0002 + 0.0005\mathbf{i} + 0.0005\mathbf{j} - 0.0005\mathbf{k} \\ -0.0005 - 0.0004\mathbf{i} - 0.0010\mathbf{j} + 0.0010\mathbf{k} & 0.0012\mathbf{i} + 0.0020\mathbf{j} + 0.0002\mathbf{k}, \end{pmatrix},$$

$$U_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.2157 - 0.5625\mathbf{i} - 0.5079\mathbf{j} + 0.4975\mathbf{k} & 0.0479 + 0.8423\mathbf{i} + 1.4991\mathbf{j} + 0.0312\mathbf{k} \\ 0.4770 + 2.4543\mathbf{i} + 2.5170\mathbf{j} + 2.3890\mathbf{k} & -2.9781 - 2.4611\mathbf{i} - 0.1109\mathbf{j} - 4.8836\mathbf{k} \end{pmatrix},$$

$$U_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2.0941 - 1.8169\mathbf{i} - 19.7767\mathbf{j} + 7.1464\mathbf{k} & 4.0003 - 7.6426\mathbf{i} - 23.6916\mathbf{j} - 12.6004\mathbf{k} \\ 35.0358 + 22.9322\mathbf{i} - 19.0766\mathbf{j} + 62.5683\mathbf{k} & 50.3227 + 10.8273\mathbf{i} - 41.1511\mathbf{j} + 44.4639\mathbf{k} \end{pmatrix},$$

$$U_{13} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.0005 + 0.0012\mathbf{i} + 0.0005\mathbf{k} & -0.0002 + 0.0005\mathbf{i} + 0.0005\mathbf{j} + 0.0005\mathbf{k} \\ -0.0005 - 0.0004\mathbf{i} - 0.0010\mathbf{j} + 0.0010\mathbf{k} & 0.0012\mathbf{i} + 0.0020\mathbf{j} + 0.0002\mathbf{k} \end{pmatrix},$$

where

$$
\begin{aligned}
x_{311} &= 2.4249 \times 10^{15} + 9.158 \times 10^{14}\mathbf{i} + 3.6562 \times 10^{15}\mathbf{j} - 9.802 \times 10^{14}\mathbf{k},\\
x_{312} &= 2.8915 \times 10^{15} - 3.424 \times 10^{15}\mathbf{i} + 3.6682 \times 10^{15}\mathbf{j} + 3.8670 \times 10^{14}\mathbf{k},\\
x_{321} &= -2.8900 \times 10^{15} - 8.349 \times 10^{14}\mathbf{i} + 4.8121 \times 10^{15}\mathbf{j} + 1.3028 \times 10^{15}\mathbf{k},\\
x_{322} &= -2.876 \times 10^{15} + 3.0166 \times 10^{15}\mathbf{i} + 1.3349 \times 10^{15}\mathbf{j} + 5.6492 \times 10^{14}\mathbf{k},\\
x_{0411} &= -1.0460 \times 10^{14} - 5.5967 \times 10^{14}\mathbf{i} - 4.5730 \times 10^{14}\mathbf{j} + 2.5023 \times 10^{14}\mathbf{k},\\
x_{0412} &= 4.0206 \times 10^{14} + 1.7870 \times 10^{14}\mathbf{i} + 3.5717 \times 10^{14}\mathbf{j} + 1.6691 \times 10^{14}\mathbf{k},\\
x_{0421} &= -1.0061 \times 10^{14} + 3.6673 \times 10^{14}\mathbf{i} + 4.9065 \times 10^{14}\mathbf{j} - 7.1850 \times 10^{14}\mathbf{k},\\
x_{0411} &= -1.0460 \times 10^{14} - 5.5967 \times 10^{14}\mathbf{i} - 4.5730 \times 10^{14}\mathbf{j} + 2.5023 \times 10^{14}\mathbf{k},\\
x_{0422} &= -2.0371 \times 10^{14} - 1.9449 \times 10^{14}\mathbf{i} - 6.7917 \times 10^{14}\mathbf{j} + 3.27 \times 10^{11}\mathbf{k},\\
x_{412} &= -2.5081 \times 10^{15} - 2.526 \times 10^{14}\mathbf{i} - 1.8086 \times 10^{15}\mathbf{j} - 1.6121 \times 10^{15}\mathbf{k},\\
x_{421} &= -1.6826 \times 10^{15} + 3.233 \times 10^{14}\mathbf{i} + 1.2608 \times 10^{14}\mathbf{j} - 5.905 \times 10^{14}\mathbf{k},\\
x_{422} &= 3.376 \times 10^{14} - 2.1855 \times 10^{15}\mathbf{i} - 2.5144 \times 10^{15}\mathbf{j} - 9.287 \times 10^{14}\mathbf{k},\\
u_{311} &= -1.093 \times 10^{35} - 9.730 \times 10^{35}\mathbf{i} + 1.8576 \times 10^{36}\mathbf{j} - 1.9289 \times 10^{36}\mathbf{k},\\
u_{312} &= 8.155 \times 10^{35} + 9.72 \times 10^{34}\mathbf{i} - 1.5978 \times 10^{36}\mathbf{j} - 1.5691 \times 10^{36}\mathbf{k},\\
u_{321} &= -8.932 \times 10^{35} - 1.8508 \times 10^{36}\mathbf{i} + 4.0088 \times 10^{36}\mathbf{j} - 4.8557 \times 10^{36}\mathbf{k},\\
u_{322} &= 1.5025 \times 10^{36} - 7.623 \times 10^{35}\mathbf{i} - 4.0296 \times 10^{36}\mathbf{j} - 3.3906 \times 10^{36}\mathbf{k},\\
u_{411} &= 1.4764 \times 10^{35} + 4.669 \times 10^{34}\mathbf{i} - 5.229 \times 10^{34}\mathbf{j} - 2.3157 \times 10^{35}\mathbf{k},\\
u_{412} &= -3.7703 \times 10^{35} + 2.5079 \times 10^{35}\mathbf{i} - 4.3196 \times 10^{35}\mathbf{j} + 5.1752 \times 10^{35}\mathbf{k},\\
u_{421} &= 1.3750 \times 10^{35} - 1.2281 \times 10^{35}\mathbf{i} - 2.4962 \times 10^{35}\mathbf{j} + 1.7278 \times 10^{35}\mathbf{k},\\
u_{422} &= 4.310 \times 10^{34} + 5.1989 \times 10^{35}\mathbf{i} + 8.0931 \times 10^{35}\mathbf{j} + 2.5832 \times 10^{35}\mathbf{k},\\
u_{211} &= 0.9999 + 0.0001\mathbf{j} - 0.0001\mathbf{k},\ u_{212} = -0.0002 + 0.0001\mathbf{i} + 0.0004\mathbf{j},\\
u_{213} &= 0.0025 - 0.0052\mathbf{i} - 0.0085\mathbf{j} - 0.0084\mathbf{k},\ u_{214} = 0.0052 - 0.0006\mathbf{i} - 0.0064\mathbf{j} + 0.0067\mathbf{k},\\
u_{221} &= -0.0002 - 0.0001\mathbf{i} - 0.0001\mathbf{j} - 0.0002\mathbf{k},\ u_{222} = 0.9994 + 0.0001\mathbf{i} - 0.0005\mathbf{k},\\
u_{223} &= 0.0096 - 0.0130\mathbf{i} - 0.0134\mathbf{j} + 0.0039\mathbf{k},\ u_{224} = 0.0098 + 0.0104\mathbf{i} + 0.0081\mathbf{j} + 0.0108\mathbf{k},\\
u_{231} &= 0.0025 + 0.0052\mathbf{i} - 0.0002\mathbf{j} - 0.0013\mathbf{k},\ u_{232} = 0.0096 + 0.0130\mathbf{i} - 0.0014\mathbf{j} - 0.0031\mathbf{k},\\
u_{233} &= 0.0006 + 0.0001\mathbf{i} - 0.0001\mathbf{k},\ u_{234} = -0.0001 + 0.0005\mathbf{i} - 0.0001\mathbf{j},\\
u_{241} &= 0.0052 + 0.0006\mathbf{i} - 0.0040\mathbf{j} - 0.0043\mathbf{k},\ u_{242} = 0.0098 - 0.0104\mathbf{i} + 0.0036\mathbf{j} - 0.0096\mathbf{k},\\
u_{243} &= -0.0001 - 0.0005\mathbf{i} + 0.0004\mathbf{j} - 0.0001\mathbf{k},\ u_{244} = 0.0005 - 0.0002\mathbf{j} - 0.0003\mathbf{k},
\end{aligned}
$$

V_2 is an arbitrary matrix of order 2×2 over quaternion $\mathbb{H}$, T_6 is an arbitrary matrix of order 4×2 over quaternion $\mathbb{H}$, and T_7 and T_8 are arbitrary matrix of order 2×4 over quaternion $\mathbb{H}$, $S_1 = \begin{pmatrix} I_2 & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$, $S_3 = \begin{pmatrix} 0 & I_2 \end{pmatrix}$, $S_4 = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}$.

3.2. The General Solution to the System (4)

Based on Theorem 1, in this section, we consider the system (4) with the known matrices E_1, E_2, E_{11}, E_{22}, F_i, H_i $(i = \overline{1,4})$, G_j, F_{jj} $(j = \overline{1,3})$, and T over $\mathbb{H}$. For convenience, we define the notation as follows:

$$\begin{aligned}
&A_i = E_{ii}L_{E_i},\ B_j = R_{G_j}F_{jj}(i=1,2, j=\overline{1,3}),\ C_c = T - E_{11}(E_1^- F_1 + L_{E_1}H_1G_1^-)F_{11}\\
&- E_{11}(E_1^- F_2 + L_{E_1}H_2G_2^-)F_{22} - E_{22}(E_2^- F_3 + L_{E_2}H_3G_2^-)F_{22} - E_{22}(E_1^- F_4 + L_{E_2}H_4G_3^-)F_{33},\\
&B_4 = B_1L_{B_2},\ B_5 = B_2L_{B_4},\ A_{33} = R_{A_1}A_2,\ B_{33} = B_3L_{B_2},\\
&C_{33} = R_{A_1}C_cL_{B_2},\ A_{11} = A_2 + A_{33},\ C_{44} = C_cL_{B_2} + R_{A_1}C_c,\ B_{11} = B_3L_{B_4}L_{B_5},\\
&C_{11} = (I + R_{A_1})C_cL_{B_4}L_{B_5},\ A_{22} = R_{A_{33}}A_2,\ B_{22} = B_3L_{B_2}L_{B_4},\ C_{22} = R_{A_{33}}C_{44}L_{B_4},\\
&M_1 = A_{22}L_{A_{11}},\ N_1 = R_{B_{11}}B_{22},\ C_1 = C_{22} - A_{22}A_{11}^-C_{11}B_{11}^-B_{22},\ D_1 = R_{A_1}A_{22},\\
&\phi = A_{11}^-C_{11}B_{11}^- + L_{A_{11}}M_1^-C_1B_{22}^- - L_{A_{11}}M_1^-A_{22}D_1^-R_{M_1}C_1B_{22}^- + D_1^-R_{M_1}C_1N_1^-R_{B_{11}},\\
&M_2 = \left[L_{A_{11}}L_{M_1}, L_{A_{33}}\right],\ N_2 = \begin{bmatrix} R_{N_1}R_{B_{11}} \\ R_{B_{33}} \end{bmatrix},\ A = R_{M_2}L_{A_{11}},\ B = R_{B_{22}}L_{N_2},\ C = R_{M_2}L_{A_{22}},\\
&D = R_{B_{11}}L_{N_2},\ E_1 = A_{33}^-C_{33}B_{33}^- - \phi,\ E = R_{M_2}E_1L_{N_2},\ M = R_AC,\ N = DL_B,\ S = CL_M,
\end{aligned} \tag{34}$$

and

$$S_1 = [I_m, 0], \quad S_2 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad S_3 = [0, I_m], \quad S_4 = \begin{bmatrix} 0 \\ I_n \end{bmatrix}.$$

Thus, we obtain the following—one of the main result of this paper.

Theorem 2. *Consider the system of quaternion matrix Equation (4) with the notation given in (34). The following statements are equivalent:*

(1) *The system of matrix Equation (4) is consistent.*

(2)

$$E_1H_1 = F_1G_1,\ E_1H_2 = F_2G_2,\ E_2H_3 = F_3G_2,\ E_2H_4 = F_4G_3 \tag{35}$$

and

$$R_{E_1}F_1 = 0,\ R_{E_2}F_3 = 0,\ H_1L_{G_1} = 0,\ H_2L_{G_2} = 0,\ H_4L_{G_3} = 0, \tag{36}$$

$$\begin{aligned}
&R_{A_{33}}R_{A_1}C_{44} = 0,\ R_{A_{11}}C_{11} = 0,\quad C_{ii}L_{B_{ii}} = 0(i = \overline{1,3}),\ R_{M_1}C_1L_{N_1} = 0,\\
&R_MR_AE = 0,\quad EL_BL_N = 0,\quad R_AEL_D = 0,\quad R_CEL_B = 0.
\end{aligned} \tag{37}$$

(3) (35) *holds and*

$$\begin{aligned}
&r\begin{pmatrix} F_1 & E_1 \end{pmatrix} = r(E_1),\ r\begin{pmatrix} F_3 & E_2 \end{pmatrix} = r(E_2),\\
&r\begin{pmatrix} H_1 \\ G_1 \end{pmatrix} = r(G_1),\ r\begin{pmatrix} H_2 \\ G_2 \end{pmatrix} = r(G_2),\ r\begin{pmatrix} H_4 \\ G_3 \end{pmatrix} = r(G_3),
\end{aligned} \tag{38}$$

$$r\begin{pmatrix} P_i & S_i \\ 0 & Q_i \end{pmatrix} = r(P_i) + r(Q_i), i = \overline{1,10} \tag{39}$$

where

$$P_1 = \begin{pmatrix} E_{22} & E_{11} \\ E_2 & 0 \\ 0 & E_1 \end{pmatrix},\ Q_1 = \begin{pmatrix} F_{22} & G_2 \end{pmatrix}, S_1 = \begin{pmatrix} 2T & 0 \\ 2W_2 & 0 \\ 2W_1 & 0 \end{pmatrix},\ P_2 = \begin{pmatrix} E_{22} & E_{11} \\ E_2 & 0 \\ 0 & E_1 \end{pmatrix}, \tag{40}$$

$$Q_2 = \begin{pmatrix} F_{11} & G_1 & 0 \\ F_{22} & 0 & G_2 \end{pmatrix}, S_2 = \begin{pmatrix} 2T & 0 & 2V_1 \\ 2W_2 & 0 & 0 \\ 2U_1 & 0 & 0 \end{pmatrix},\ Q_3 = \begin{pmatrix} F_{33} & G_3 & 0 & 0 \\ F_{22} & 0 & G_2 & 0 \\ F_{11} & 0 & 0 & G_1 \end{pmatrix}, \tag{41}$$

$$P_3 = \begin{pmatrix} E_{11} \\ E_1 \end{pmatrix},\ S_3 = \begin{pmatrix} 2T & 2V_2 & 2V_3 & 2V_4 \\ 2W_2 & 0 & 0 & \\ 2U_2 & 0 & 0 & 0 \end{pmatrix},\ S_4 = \begin{pmatrix} 2T & 2V_2 & 2V_1 & 0 \\ 2U_3 & 0 & 0 & 0 \\ 2U_1 & 0 & 0 & 0 \end{pmatrix}, \tag{42}$$

$$P_4 = \begin{pmatrix} E_{22} & E_{11} \\ E_2 & 0 \\ 0 & E_1 \end{pmatrix},\ Q_4 = \begin{pmatrix} F_{33} & G_3 & 0 & 0 \\ F_{22} & 0 & G_2 & 0 \\ F_{11} & 0 & 0 & G_1 \end{pmatrix},\ Q_5 = \begin{pmatrix} F_{33} & G_3 & 0 \\ F_{22} & 0 & G_2 \end{pmatrix}, \tag{43}$$

$$P_5 = \begin{pmatrix} E_{11} \\ E_1 \end{pmatrix},\ S_5 = \begin{pmatrix} T & V_2 & V_3 \\ W_1 & 0 & 0 \end{pmatrix},\ Q_6 = \begin{pmatrix} F_{33} & F_{33} & G_3 & 0 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & G_2 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & G_1 & 0 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & G_2 & 0 \\ 0 & F_{11} & 0 & 0 & 0 & 0 & G_1 \end{pmatrix}, \tag{44}$$

$$P_6 = \begin{pmatrix} E_{22} & E_{11} & 0 & 0 \\ 0 & 0 & E_{22} & E_{11} \\ E_2 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix},\ S_6 = \begin{pmatrix} 2T & 0 & 0 & 2V_3 & 2V_4 & 0 & 0 \\ 0 & 2T & 0 & 0 & 0 & 2V_3 & 2V_4 \\ 2U_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_7 = \begin{pmatrix} E_{22} & E_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{22} & E_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{22} & E_{11} \\ E_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1 \end{pmatrix},\ S_7 = \begin{pmatrix} 2T & 0 & 0 & 2V_3 & 2V_4 & 0 & 0 \\ 0 & 2T & 0 & 0 & 0 & 2V_3 & 2V_4 \\ 2U_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{45}$$

$$Q_7 = \begin{pmatrix} F_{33} & F_{33} & F_{33} & G_3 & 0 & 0 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & 0 & G_2 & 0 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & 0 & G_2 & 0 & 0 \\ 0 & F_{11} & 0 & 0 & 0 & 0 & 0 & G_1 & 0 \\ 0 & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & G_2 \end{pmatrix},\ P_8 = \begin{pmatrix} 2E_{22} & 0 & E_{11} & 0 & 0 \\ E_{22} & E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{22} & E_{11} \\ E_2 & 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 \\ 0 & 0 & E_1 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & 0 & E_1 \end{pmatrix}, \tag{46}$$

$$S_8 = \begin{pmatrix} 0 & 2T & 0 & 0 & 0 & 0 & 0 & 2V_3 & 2V_4 & 0 \\ T & 0 & T & 0 & 0 & V_3 & V_4 & 0 & 0 & V_3 \\ 2T & 0 & 0 & 0 & 0 & 2V_3 & 2V_4 & 0 & 0 & 0 \\ 2U_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_4 & 0 & U_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_5 & V_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$Q_8 = \begin{pmatrix} F_{33} & F_{33} & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{33} & F_{33} & 0 & G_3 & 0 & 0 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & 0 & 0 & G_2 & 0 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & G_2 & 0 & 0 \\ 0 & F_{11} & 0 & 0 & 0 & 0 & 0 & 0 & G_1 & 0 \\ 0 & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & G_2 \end{pmatrix}, \tag{47}$$

$$P_{10} = \begin{pmatrix} 0 & 0 & E_{22} & 0 & 0 & E_{11} & 0 & 0 \\ 2E_{22} & 0 & 0 & 0 & 0 & 0 & 0 & E_{11} \\ 0 & 0 & 0 & E_{22} & 0 & 0 & E_{11} & 0 \\ 0 & E_{22} & 0 & 0 & E_{11} & 0 & 0 & 0 \\ E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_1 \end{pmatrix},\ P_9 = \begin{pmatrix} 0 & E_{11} & 0 \\ E_{22} & 0 & E_{11} \\ E_2 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_1 \end{pmatrix}, \tag{48}$$

$$Q_9 = \begin{pmatrix} F_{33} & F_{33} & G_3 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & G_2 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & G_1 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & G_2 \end{pmatrix},\ S_9 = \begin{pmatrix} 2T & 2T & 2V_2 & 2V_3 & 2V_4 & 2V_3 \\ 0 & T & 0 & 0 & 0 & W_4 \\ 0 & U_4 & 0 & 0 & 0 & 0 \\ 2U_5 & 2U_1 & 0 & 0 & 0 & 0 \\ 0 & U_5 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_{10} = \begin{pmatrix} F_{33} & F_{33} & F_{33} & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{33} & F_{33} & F_{33} & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & 0 & 0 & 0 & G_2 & 0 & 0 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & F_{11} & 0 & 0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 \\ 0 & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_2 \end{pmatrix}, \tag{49}$$

$$S_{10} = \begin{pmatrix} 2T & 0 & 0 & 0 & 0 & 0 & 0 & 2V_3 & 2V_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2U_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{50}$$

and

$$\begin{aligned} &W_1 = F_1F_{11} + F_2F_{22},\ W_2 = F_3F_{22} + F_4F_{33}, \\ &W_3 = E_{22}H_2 + E_{22}H_3,\ W_4 = E_{11}H_2 + E_{22}H_3, \end{aligned} \tag{51}$$

$$V_1 = E_{11}H_2,\ V_3 = E_{22}H_3,\ V_2 = E_{22}H_4,\ V_4 = E_{11}H_1, \tag{52}$$

$$U_1 = F_1F_{11},\ U_2 = F_2F_{11},\ U_3 = F_3F_{22},\ U_4 = F_4F_{33},\ U_5 = F_2F_{22}. \tag{53}$$

In this case, the general solution to the system of the matrix Equation (4) can be expressed as follows:

$$\begin{aligned} X_1 &= E_1^- F_1 + L_{E_1} H_1 G_1^- + L_{E_1} U_1 R_{G_1}, \\ X_2 &= E_1^- F_2 + L_{E_1} H_2 G_2^- + L_{E_1} U_2 R_{G_2}, \\ X_3 &= E_2^- F_3 + L_{E_2} H_3 G_2^- + L_{E_2} U_3 R_{G_2}, \\ X_4 &= E_2^- F_4 + L_{E_2} H_4 G_3^- + L_{E_2} U_4 R_{G_3}, \end{aligned}$$

where

$$
\begin{aligned}
&U_1 = A_1^-\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]B_4^- + L_{A_1}V_1 + V_2R_{B_4},\\
&U_2 = A_1^-(C_c - A_1X_1B_1 - A_2X_3B_2 - A_2X_4B_3)B_2^- + V_3R_{B_2} - L_{A_1}V_4,\\
&U_3 = M^-\big[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\big]B_2^- + L_MV_5 + V_6R_{B_2},\\
&U_4 = \phi + L_{A_{11}}L_{M_1}V_7 + V_8R_{N_1}R_{B_{11}} + L_{A_{11}}V_9R_{B_{22}} + L_{A_{22}}V_{10}R_{B_{11}},\\
&\textit{or}\\
&U_4 = A_{33}^-C_{33}B_{33}^- - L_{A_{33}}V_{11} - V_{12}R_{B_{33}},\\
&V_7 = S_1\big[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\big],\\
&V_8 = \big[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_2}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\big]S_2,\\
&V_9 = A^-EB^- - A^-CM^-R_AEB^- - A^-SC^-EL_BN^-DB^- - A^-ST_3R_NDB^-\\
&+ L_AT_1 + T_2R_B,\\
&V_{10} = M^-R_AED^- + L_MS^-SC^-EL_BN^- + L_ML_ST_4 + L_MT_3R_N + T_5R_D,\\
&V_{11} = S_3\big[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\big],\\
&V_{12} = \big[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\big]S_4,
\end{aligned}
$$

V_i $(i = \overline{1,6})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes, and T_i $(i = \overline{1,8})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Proof. (1) $\Leftrightarrow$ (2) Clearly, the system of matrix Equation (4) is solvable if—and only if—both of the following are consistent:

$$
\begin{aligned}
&E_1X_1 = F_1,\ X_1G_1 = H_1,\\
&E_1X_2 = F_2,\ X_2G_2 = H_2,\\
&E_2X_3 = F_3,\ X_3G_2 = H_3,\\
&E_2X_4 = F_4,\ X_4G_3 = H_4
\end{aligned} \tag{54}
$$

and

$$E_{11}X_1F_{11}+E_{11}X_2F_{22} + E_{22}X_3F_{22} + E_{22}X_4F_{33} = T. \tag{55}$$

It follows from Lemma 4 that the system (54) has a solution if—and only if— (35) holds, and

$$
\begin{aligned}
&R_{E_1}F_1 = 0,\ R_{E_1}F_2 = 0,\ R_{E_2}F_3 = 0,\ R_{E_2}F_4 = 0,\\
&H_1L_{G_1} = 0,\ H_2L_{G_2} = 0,\ H_3L_{G_2} = 0,\ H_4L_{G_3} = 0.
\end{aligned} \tag{56}
$$

Under these conditions, the expression of general solution to (54) can be expressed as follows:

$$
\begin{aligned}
&X_1 = E_1^-F_1 + L_{E_1}H_1G_1^- + L_{E_1}U_1R_{G_1},\\
&X_2 = E_1^-F_2 + L_{E_1}H_2G_2^- + L_{E_1}U_2R_{G_2},\\
&X_3 = E_2^-F_3 + L_{E_2}H_3G_2^- + L_{E_2}U_3R_{G_2},\\
&X_4 = E_2^-F_4 + L_{E_2}H_4G_3^- + L_{E_2}U_4R_{G_3},
\end{aligned} \tag{57}
$$

where U_i $(i = \overline{1,4})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Next, substituting (57) into (55) yields the following:

$$A_1U_1B_1 + A_1U_2B_2 + A_2U_3B_2 + A_2U_4B_3 = C_c, \tag{58}$$

where A_i, B_j, $(i = \overline{1,2}, j = \overline{1,3})$ are defined as (34). By Theorem 1, the matrix Equation (58) is solvable if—and only if—(37) holds. In this case, the general solution to matrix Equation (58) can be expressed as follows:

$$
\begin{aligned}
&U_1 = A_1^- \left[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\right]B_4 + L_{A_1}V_1 + V_2R_{B_4},\\
&U_2 = A_1^-(C_c - A_1X_1B_1 - A_2X_3B_2 - A_2X_4B_3)B_2 + V_3R_{B_2} - L_{A_1}V_4,\\
&U_3 = M^- \left[(C_c - A_2X_4B_3)L_{B_2} + R_{A_1}(C_c - A_2X_4B_3)\right]B_2 + L_MV_5 + V_6R_{B_2},\\
&U_4 = \phi + L_{A_{11}}L_{M_1}V_7 + V_8R_{N_1}R_{B_{11}} + L_{A_{11}}V_9R_{B_{22}} + L_{A_{22}}V_{10}R_{B_{11}},\\
&or\ U_4 = A_{33}^-C_{33}B_{33}^- - L_{A_{33}}V_{11} - V_{12}R_{B_{33}},\\
&V_7 = S_1\left[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\right],\\
&V_8 = \left[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_2}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\right]S_2,\\
&V_9 = A^-EB^- - A^-CM^-R_AEB^- - A^-SC^-EL_BN^-DB^- - A^-ST_3R_NDB^-\\
&+ L_AT_1 + T_2R_B,\\
&V_{10} = M^-R_AED^- + L_MS^-SC^-EL_BN^- + L_ML_ST_4 + L_MT_3R_N + T_5R_D,\\
&V_{11} = S_3\left[M_2^-(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - M_2^-T_7N_2 + L_{M_2}T_6\right],\\
&V_{12} = \left[R_{M_2}(E_1 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})N_2^- + M_2M_2^-T_7 + T_8R_{N_2}\right]S_4,
\end{aligned}
$$

where V_i $(i = \overline{1,6})$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes, A_{ii}, B_{ii}, C_{ii} $(i = \overline{1,3}), M_1, M_2, N_1, N_2, A, B, D, E, S, M$ and N are defined as (34), T_i $(i = \overline{1,8})$ are arbitrary matrices over $\mathbb{H}$, with appropriate sizes.

Hence, the system of matrix Equation (54) and the matrix Equation (55) are consistent if—and only if—(35), (56), and (37) hold,

Now, we show that (56) $\Leftrightarrow$ (36). It follows from Lemma 2 that the following is true:

$$
\begin{aligned}
&R_{E_1}F_1 = 0 \Leftrightarrow r(F_1,\ E_1) = r(E_1),\ R_{E_1}F_2 = 0 \Leftrightarrow r(F_2,\ E_1) = r(E_1),\\
&R_{E_2}F_3 = 0 \Leftrightarrow r(F_3,\ E_2) = r(E_2),\ R_{E_2}F_4 = 0 \Leftrightarrow r(F_4,\ E_2) = r(E_2),\\
&H_1L_{G_1} = 0 \Leftrightarrow r\begin{pmatrix} H_1 \\ G_1 \end{pmatrix} = r(G_1),\ H_2L_{G_2} = 0 \Leftrightarrow r\begin{pmatrix} H_2 \\ G_2 \end{pmatrix} = r(G_2),\\
&H_3L_{G_2} = 0 \Leftrightarrow r\begin{pmatrix} H_3 \\ G_2 \end{pmatrix} = r(G_2),\ H_4L_{G_3} = 0 \Leftrightarrow r\begin{pmatrix} H_4 \\ G_3 \end{pmatrix} = r(G_3).
\end{aligned}
\tag{59}
$$

According to (59), we obtain (56) $\Leftrightarrow$ (36). To sum up, the system of matrix Equation (54) and the matrix Equation (55) are consistent if—and only if—(35)–(37) hold.

(2) $\Leftrightarrow$ (3) In view of (59), we obtain (36) $\Leftrightarrow$ (38).

Next, we turn our attention to show that (37) holds if—and only if—(39) holds. According to Theorem 1, we can find that (37) holds if—and only if—(10)–(19) hold. Hence, we need prove $(9+i) \Leftrightarrow$ (39) $(i = \overline{1,10})$ when we show that (37) holds if—and only if—(39) holds. We need to use the following fact to prove $(9+i) \Leftrightarrow$ (39) $(i = \overline{1,10})$:

It is easy to know that there exists a solution, according to $X_1^0, X_2^0, X_3^0, X_4^0$, such that the following is true:

$$
\begin{aligned}
E_1X_1^0 = F_1,&\ X_1^0G_1 = H_1,\\
E_1X_2^0 = F_2,&\ X_2^0G_2 = H_2,\\
E_2X_3^0 = F_3,&\ X_3^0G_2 = H_3,\\
E_2X_4^0 = F_4,&\ X_4^0G_3 = H_4,\\
E_{11}X_1^0F_{11} + E_{11}X_2^0F_{22} + E_{22}X_3^0F_{22} &+ E_{22}X_4^0F_{33} = T,
\end{aligned}
\tag{60}
$$

where

$$X_1^0 = E_1^- F_1 + L_{E_1} H_1 G_1^-, \ X_2^0 = E_1^- F_2 + L_{E_1} H_2 G_2^-,$$
$$X_3^0 = E_2^- F_3 + L_{E_2} H_3 G_2^-, \ X_4^0 = E_2^- F_4 + L_{E_2} H_4 G_3^-.$$

Let $T_0 = T - (E_{11} X_1^0 F_{11} + E_{11} X_2^0 F_{22} + E_{22} X_3^0 F_{22} + E_{22} X_4^0 F_{33})$. We first show the following: $(9+i) \Leftrightarrow$ (39) for $i = 1$, $(9+i) \Leftrightarrow$ (39) for $i = 2$, $(9+i) \Leftrightarrow$ (39) for $i = 3$, $(9+i) \Leftrightarrow$ (39) for $i = 4$, and $(9+i) \Leftrightarrow$ (39) for $i = 5$.

In fact, when $i = 1$, by Lemma 2, (60) and elementary operations, we obtain the following:

$$(9+i) \Leftrightarrow r\begin{pmatrix} 2T_0 & E_{22}L_{E_2} & E_{11}L_{E_1} \\ R_{G_1}F_{11} & 0 & 0 \\ R_{G_2}F_{22} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_{22}L_{E_2}, & E_{11}L_{E_1} \end{pmatrix} + r\begin{pmatrix} R_{G_1}F_{11} \\ R_{G_2}F_{22} \end{pmatrix} \Leftrightarrow$$
$$r\begin{pmatrix} 2T_0 & E_{22} & E_{11} & 0 & 0 \\ F_{11} & 0 & 0 & G_1 & 0 \\ F_{22} & 0 & 0 & 0 & G_2 \\ 0 & E_2 & 0 & 0 & 0 \\ 0 & 0 & E_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} E_{22} & E_{11} \\ E_2 & 0 \\ 0 & E_{11} \end{pmatrix} + r\begin{pmatrix} F_{11} & G_1 & 0 \\ F_{22} & 0 & G_2 \end{pmatrix} \Leftrightarrow (39).$$

Similarly, we can show that $(9+i) \Leftrightarrow$ (39) for $i = 2$, $(9+i) \Leftrightarrow$ (39) for $i = 3$, $(9+i) \Leftrightarrow$ (39) for $i = 4$ and $(9+i) \Leftrightarrow$ (39) for $i = 5$, where P_i, Q_i, S_i and O_i $(i = \overline{1,5})$ in (39) are defined as (40), (41), (42), (43), and (44), respectively, W_i $(i = \overline{1,3})$ are defined as (51), U_j $(j = \overline{1,5})$ are defined as (53), and V_k $(k = \overline{1,4})$ are defined as (52).

Second, we show that $(9+i) \Leftrightarrow$ (39) for $i = 6$, $(9+i) \Leftrightarrow$ (39) for $i = 7$, $(9+i) \Leftrightarrow$ (39) for $i = 8$, $(9+i) \Leftrightarrow$ (39) for $i = 9$ and $(9+i) \Leftrightarrow$ (39) for $i = 10$. In fact, when $i = 6$, it follows from Lemma 2, (60), and elementary operations, that the following is true:

$$(9+i) \Leftrightarrow r\begin{pmatrix} 2T_0 & 0 & 0 & E_{22}L_{E_2} & E_{11}L_{E_1} & 0 \\ R_{G_3}F_{33} & 0 & R_{G_3}F_{33} & 0 & 0 & 0 \\ 0 & E_{22}L_{E_2} & 2T_0 & 0 & 0 & E_{11}L_{E_1} \\ R_{G_1}F_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{G_1}F_{11} & 0 & 0 & 0 \\ 0 & 0 & R_{G_2}F_{22} & 0 & 0 & 0 \\ R_{G_2}F_{22} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= r\begin{pmatrix} E_{22}L_{E_2} & E_{11}L_{E_1} & 0 & 0 \\ 0 & 0 & E_{22}L_{E_2} & E_{11}L_{E_1} \end{pmatrix} + r\begin{pmatrix} R_{G_3}F_{33} & R_{G_3}F_{33} \\ R_{G_2}F_{22} & 0 \\ R_{G_1}F_{11} & 0 \\ 0 & R_{G_2}F_{22} \\ 0 & R_{G_1}F_{11} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} 2T_0 & 0 & 0 & E_{22} & E_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ F_{33} & 0 & F_{33} & 0 & 0 & 0 & G_3 & 0 & 0 & 0 & 0 \\ 0 & E_{22} & 2T_0 & 0 & 0 & E_{11} & 0 & 0 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & 0 & F_{11} & 0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 \\ 0 & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & G_2 & 0 \\ F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_2 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} E_{22} & E_{11} & 0 & 0 \\ 0 & 0 & E_{22} & E_{11} \\ E_2 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix} + r\begin{pmatrix} F_{33} & F_{33} & G_3 & 0 & 0 & 0 & 0 \\ F_{22} & 0 & 0 & G_2 & 0 & 0 & 0 \\ F_{11} & 0 & 0 & 0 & G_1 & 0 & 0 \\ 0 & F_{22} & 0 & 0 & 0 & G_2 & 0 \\ 0 & F_{11} & 0 & 0 & 0 & 0 & G_1 \end{pmatrix} \Leftrightarrow (39).$$

Similarly, we can show that $(9+i) \Leftrightarrow$ (39) for $i = 7$, $(9+i) \Leftrightarrow$ (39) for $i = 8$, $(9+i) \Leftrightarrow$ (39) for $i = 9$ and $(9+i) \Leftrightarrow$ (39) for $i = 10$, where P_i, Q_i, S_i $(i = 6,7,8)$, and in (39), are defined as (45), (46), (47), respectively, P_i, Q_i and $S_i (i = 9,10)$ in (39), are defined as (48), (49), and (50). W_i $(i = \overline{1,3})$ are defined as (51), U_j $(j = \overline{1,5})$ are defined as (53), and V_k $(k = \overline{1,4})$ are defined as (52). The proof is completed. □

4. Conclusions

We have established some necessary and sufficient conditions for the existence of the solution to quaternion matrix Equation (1), and derived a formula of its general solution when it is solvable. As an application of (1), we have investigated some necessary and sufficient conditions for the system of matrix Equation (4) to be consistent, as well as the expression of its general solution, and presented a numerical example to emphasize our main results.

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