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Some Subordination Results for Atangana-Baleanu Fractional Integral Operator Involving Bessel Functions

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Abstract: We propose, in the present paper, to derive some differential subordination results. The work is developed in the case of analytic functions defined on the open unit disc. The results will be formulated by making use of an Atangana–Baleanu fractional integral operator and Bessel functions. For the newly obtained theorems, certain interesting consequences are also considered. Univalent function selections with specific symmetry properties were involved.

Keywords: dominant; fractional integral operator; differential subordination

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1. Introduction and Preliminary Results

Lately, many research studies have paid special attention to the subject of fractional calculus operators. They have investigated different types of solutions for fractional differential equations such as numerical, approximation and analytic solutions (see, e.g., [1–3]). In our research activity, we have to recall here an important paper that deals with a mixed idea of two arrays, Caputo–Fabrizio fractional derivative and continued fractions. Thus, the authors of [4] establish a new operator, namely, the infinite coefficient-symmetric fractional derivative, Caputo–Fabrizio. In a similar manner, the authors of [5] study other interesting fractional derivations.

Fractional calculus is a branch of non-negative integer calculus order $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ that deals with integrals and derivatives of any arbitrary real or complex order.

It is very useful to consider the extension of fractional calculus to the field of complex analysis. The majority of well-known concepts of the fractional νth differintegrals work for both $\nu \in \mathbb{C}$ and $\nu \in \mathbb{R}$. For instance, the most commonly used Riemann–Liouville definition, written by acronym RL, is the following.

$$ {}_{c}^{RL}I_{z}^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_{c}^{z} (z-w)^{\nu-1} f(w)dw, \quad \text{Re}\,(\nu) > 0. \tag{1}$$

We can find (1) developed in [6]. In addition to the Riemann–Liouville fractional calculus operator, many other well-known operators have occurred in theoretical and pure research while others including modified or new operators can be beneficial in a variety of applications.

An important topic in the field of complex fractional calculus deals with both branch points and branch cuts. We mention here that function $(z-w)^{\nu-1}$ is a singular one and appears in (1) to present a branch point at $w = z$. This point is exactly the point at one end of the contour of integration. Consequently, we consider certain branch choices such that they produce reasonable forms for the expression (1). This issue is discussed, for example, in [6], § 22.

A commonly used technique in the study of certain properties of differential subordinations is that of using different types of operators. Recently in [7], interesting differential subordination and superordination results were obtained using integral operators. By means of certain fractional integral operators, in papers [8,9], some inequalities and properties concerning a subclass of analytic functions are derived. Very recently, the paper [10] was published concerning a confluent (or Kummer) hypergeometric function. This study extends a result obtained in [11]. Regarding the univalence property of the Kummer function, we recall here paper [12]. Moreover, we mention papers [13,14] here, where a fractional integral operator on certain hypergeometric functions was applied.

In addition, following the work of [10], we defined a new function that connected two significant operators: Atangana–Baleanu and Riemann–Liouville operators.

The first Atangana–Baleanu concept of fractional operator was defined in [15], and many other further related results were obtained in papers such as [16–18].

Using analytic continuation, the definition of Atangana–Baleanu can readily be extended to complex values with ν differentiation order.

Definition 1 ([19])**.** *Let c be a complex number that is fixed and let f be a complex function that is analytic on an open star-domain D centered at point c. Consider a multiplier function, denoted by $B(\nu)$, which is also analytic. We denote the extended Atangana–Baleanu integral operator by ${}^{AB}_{c}I^{\nu}_{z} f(z)$. It is defined for any complex number $\nu \in \mathbb{C}$ and any $z \in D \backslash \{c\}$ by the following.*

$$ {}^{AB}_{c}I^{\nu}_{z} f(z) = \frac{1-\nu}{B(\nu)} f(z) + \frac{\nu}{B(\nu)} \cdot {}^{RL}_{c}I^{\nu}_{z} f(z). \tag{2} $$

Proposition 1 ([19])**.** *The integral operator, more precisely, the extended Atangana–Baleanu, which is defined in Definition 1, satisfies the following conditions:*

1. *It is a function that is analytic of both $z \in D \backslash \{c\}$ and $\nu \in \mathbb{C}$. The given functions f and B are analytic and the function B is nonzero;*
2. *It is identical to the original formula if $0 < \nu < 1$ and also $c < z$ in $\mathbb{R}$.*
   Thus, it provides the analytic continuation of the Atangana–Baleanu original integral operator to the complex values of z and ν.

Definition 2 ([20,21])**.** *Let $\delta, b, c \in \mathbb{C}$ and the second-order linear homogenous differential equation of the following:*

$$ z^2 w''(z) + bzw'(z) + \left[cz^2 - \delta^2 + (1-b)\delta\right] w(z) = 0 \tag{3} $$

which is a natural extension of Bessel's equation. Solutions of (3) are referred to as the generalized Bessel function of order δ. The differential Equation (3) allows the study of Bessel, modified Bessel and spherical Bessel functions all together. The function solution, denoted by $w(z)$, has the following series representation.

$$ w_{\delta,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma\left(\delta + k + \frac{b+1}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2k+\delta}. \tag{4} $$

We call this function the generalized Bessel function of the first kind of order δ.

The current study contributes to the field of fractional operators by demonstrating certain differential subordination results for a newly proposed analytic function involving the first form of generalized Bessel functions. Certain properties related to this function are to be studied in further works. Knowing the fact that the series defined above is convergent everywhere, we can say that function $w_{\delta,b,c}(z)$ is generally not univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We will further recall the very known symbols and results.

Consider $\mathcal{H}(\mathcal{U})$, the class of all analytic functions defined in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, and consider subclass $\mathcal{H}[a,n]$ of $\mathcal{H}$ containing functions of the following form.

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots.$$

Consider $\mathcal{A}(p,n)$ to be the subclass of all functions f normalized by the following:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p,n \in \mathbb{N} := \{1,2,3,\dots\}) \tag{5}$$

which are analytic in the open unit disc. The well known notations are used.

$$\mathcal{A}(p,1) := \mathcal{A}_p \quad \text{with} \quad \mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1.$$

Consider the following class:

$$\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + \dots\}$$

by $\mathcal{A}_1 := \mathcal{A}$.

The principle of subordination regarding two analytic functions f and g is presented further.

We remind the reader that function f is subordinate to g and is symbolically written as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad (z \in U) \tag{6}$$

if there exists a so-called Schwarz function $w(z)$ that is analytic in U such that $f(z) = g(w(z)), z \in U$.

If we consider the particular case of a g univalent function in U, the differential subordination (6) is equivalently rewritten as follows.

$$f(0) = g(0) \quad \text{with} \quad f(U) \subset g(U).$$

Consider functions $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$.

Let functions p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ be univalent functions. If p satisfies the second order differential superordination:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \tag{7}$$

then function p is a solution of (7). If function g is subordinate to h, then function h is superordinate to g.

For a differential subordination $q \prec p$, with p satisfying (7), we call for the analytic function q to be a subordinant.

If we consider the univalent subordinant $\widetilde{q}$, which satisfies the differential subordination

$$q \prec \widetilde{q}$$

for all q subordinants, of subordination (7), it is called the best subordinant. Miller and Mocanu deduced conditions on the functions h, q and ϕ, which the following inference holds, in their work [22].

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \ \Rightarrow \ q(z) \prec p(z).$$

Certain classes of first order differential subordination and superordination-preserving integral operators [23] were considered in the article [24] by using the results of Miller and Mocanu [22].

Srivastava and Lashin in [25] conducted a study of starlike functions of complex order and also convex functions of complex order using a Briot–Bouquet differential subordination technique.

Definition 3 ([22]). *Consider Q, the class of all functions f that are analytic and injective on set $\overline{U} - E(f)$, with the following:*

$$E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$$

and $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Theorem 1 ([26]). *Consider the functions θ and ϕ that are analytic in a domain D that contains $q(U)$ with $\phi(w) \neq 0$, $w \in q(U)$. Let q be a univalent function, defined in the open unit disc U, with the following.*

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z).$$

Assume the following:

(1) $Q(z)$ is a starlike univalent function defined in Δ;

(2) $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p, an analytic function defined in U, satisfies $p(0) = q(0)$, $p(U) \subset D$ and the following differential subordination:

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then the following is the case:

$$p(z) \prec q(z)$$

and q is the best dominant.

By making use of Definitions 1 and 2, in paper [27], the next definition is introduced.

Definition 4. *Let the numbers $\delta, b, c, \lambda \in \mathbb{C}$ such that $\Re\lambda > 0$ and*

$$^{AB}I_z^{\lambda} w_{\delta,b,c}(z) = \frac{1-\lambda}{B(\lambda)} w_{\delta,b,c}(z) + \frac{\lambda}{B(\lambda)} \cdot {}^{RL}I_z^{\lambda} w_{\delta,b,c}(z) \tag{8}$$

where we denoted the following:

$$^{AB}I_z^{\lambda} w_{\delta,b,c}(z) = {}_0^{AB} I_z^{\lambda} w_{\delta,b,c}(z), \quad {}^{RL}I_z^{\lambda} w_{\delta,b,c}(z) = {}_0^{RL} I_z^{\lambda} w_{\delta,b,c}(z).$$

Function $B(\lambda)$ is a normalization function with $B(0) = B(1) = 1$.

The following representation is obtained:

$$^{AB}I_z^{\lambda} w_{\delta,b,c}(z) = \frac{1}{B(\lambda)2^{\delta}} \sum_{k=0}^{\infty} \alpha(\delta,b,c,k) \left(1 - \lambda + \frac{\lambda\Gamma(2k+\delta+1)z^{\lambda}}{\Gamma(2k+\delta+\lambda+1)}\right) z^{2k+\delta}$$

where

$$\alpha(\delta,b,c,k) = \frac{(-1)^k c^k}{\Gamma(k+1)\Gamma\left(\delta + k + \frac{b+1}{2}\right)4^k}$$

In this manuscript, by using the principle of differential subordination, we investigate some subordination properties of the fractional integral $^{AB}I_z^{\lambda} w_{\delta,b,c}(z)$. Thus, we provide some applications of the Theorem 1 by finding a special form for the function h. Moreover, by using the newly introduced analytic function, for certain values of the parameters, one obtains well-known results. Some interesting further consequences are also presented.

2. Main Results

Theorem 2. *Consider q an univalent function defined in unit disc U, with* $q(z) \neq 0$, $\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \in \mathcal{H}(\mathcal{U})$. *Let the numbers* $\alpha, \beta, \gamma, \xi, \mu \in \mathbb{C}$, $\xi \neq 0$ *such that* $\operatorname{Re} \lambda > 0$.

Suppose the function $\dfrac{zq'(z)}{q(z)}$ *is a starlike univalent function in U. Consider the following:*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}q(z) + \frac{2\gamma}{\xi}(q(z))^2\right\} > 0 \tag{9}$$

and

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z) := (\alpha+\xi) + (\beta-\xi)\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} + \tag{10}$$

$$+\gamma\left[\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)}\right]^2 + \xi\left[\frac{z\cdot\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)''}{\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}\right].$$

If function q satisfies the following subordination criteria:

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z) \prec \alpha + \beta q(z) + \gamma(q(z))^2 + \xi\frac{zq'(z)}{q(z)} \tag{11}$$

then

$$\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \prec q(z), \quad z \in U,\ z \neq 0 \tag{12}$$

and function q is the best dominant.

Proof. Let function $p(z)$ be defined by the following.

$$p(z) := \frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)}, \quad z \in U,\ z \neq 0.$$

One can obtain the following result from a simple computation.

$$\frac{zp'(z)}{p(z)} = 1 - \frac{z({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z))'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} + \frac{z({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z))''}{\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}.$$

By setting the following:

$$\begin{aligned} \theta(w): &= \alpha + \beta w + \gamma w^2 \text{ and} \\ \phi(w): &= \frac{\xi}{w} \end{aligned}$$

it is not difficult to observe that $\theta(w)$ is an analytic function in $\mathbb{C}$, and $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ with $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Moreover, by allowing the following:

$$Q(z) = zq'(z)\phi(q(z)) = \xi\frac{zq'(z)}{q(z)}$$

and the following:

$$h(z) := \theta(q(z)) + Q(z) = \alpha + \beta q(z) + \gamma(q(z))^2 + \xi\frac{zq'(z)}{q(z)}$$

we establish that $Q(z)$ is a starlike univalent function in U with

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\beta}{\xi}q(z) + \frac{2\gamma}{\xi}(q(z))^2\right\} > 0.$$

The subordination from (12) of Theorem 2, follows by an application of Theorem 1. $\square$

By choosing $q(z) = \dfrac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$ with $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\eta}$, $0 < \eta \le 1$ in Theorem 2, we obtain the following two corollaries.

Corollary 1. *Consider complex numbers* $\alpha, \beta, \gamma, \xi \in \mathbb{C}$, $\xi \neq 0$ *with* $\operatorname{Re}\lambda > 0$ *and* $-1 \le B < A \le 1$ *and*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}\frac{1+Az}{1+Bz} + \frac{2\gamma}{\xi}\left(\frac{1+Az}{1+Bz}\right)^2\right\} > 0. \tag{13}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z) \prec \alpha + \beta\frac{1+Az}{1+Bz} + \gamma\left(\frac{1+Az}{1+Bz}\right)^2 + \xi\frac{(A-B)z}{(1+Az)(1+Bz)} \tag{14}$$

where function $\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z)$ *is defined in* (10)*, then we obtain the following:*

$$\frac{z\left({}^{AB}I_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda}w_{\delta,b,c}(z)} \prec \frac{1+Az}{1+Bz} \tag{15}$$

and function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2. *Let the numbers* $\alpha, \beta, \gamma, \xi \in \mathbb{C}$, $\xi \neq 0$ *with* $\operatorname{Re}\lambda > 0$, $0 < \eta \le 1$ *and the following be the case:*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2\gamma}{\xi}\left(\frac{1+z}{1-z}\right)^{2\eta}\right\} > 0. \tag{16}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z) \prec \alpha + \beta\left(\frac{1+z}{1-z}\right)^{\eta} + \gamma\left(\frac{1+z}{1-z}\right)^{2\eta} + \frac{2\xi\eta z}{1-z^2} \tag{17}$$

where function $\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z)$ *is posted in* (10)*, then we obtain the following:*

$$\frac{z\left({}^{AB}I_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda}w_{\delta,b,c}(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \tag{18}$$

and $\left(\dfrac{1+z}{1-z}\right)^{\eta}$ *is the best dominant.*

For a specific situation $q(z) = e^{\eta Az}$, with $|\eta A| < \pi$, we deduce the next corollary.

Corollary 3. *Let numbers* $A, \alpha, \beta, \gamma, \xi \in \mathbb{C}$, $\xi \neq 0$*and* $|\eta A| < \pi$, $\operatorname{Re}\lambda > 0$ *such that*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}e^{\eta Az} + \frac{2\gamma}{\xi}e^{2\eta Az}\right\} > 0. \tag{19}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z) \prec \alpha + \beta e^{\eta Az} + \gamma e^{2\eta Az} + \xi A\eta z \tag{20}$$

where function $\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z)$ *is posted in* (10), *then we obtain the following:*

$$\frac{z\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)} \prec e^{\eta Az} \tag{21}$$

and function $e^{\eta Az}$ *is the best dominant.*

Corollary 4. *Let the numbers* $\alpha,\beta,\gamma,\xi,\eta\in\mathbb{C},\eta\neq 0,\xi\neq 0$ *and* $\operatorname{Re}\lambda>0,-1\leq B<A\leq 1$ *such that*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}(1+Bz)^{\frac{\eta(A-B)}{B}}+\frac{2\gamma}{\xi}(1+Bz)^{\frac{2\eta(A-B)}{B}}\right\}>0.$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z)\prec\alpha+\beta(1+Bz)^{\frac{\eta(A-B)}{B}}+\gamma(1+Bz)^{\frac{2\eta(A-B)}{B}}+\xi\frac{z\eta(A-B)}{1+Bz} \tag{22}$$

where function $\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\gamma,\xi;z)$ *is posted in* (10), *then we obtain the following:*

$$\frac{z\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)} \prec (1+Bz)^{\frac{\eta(A-B)}{B}} \tag{23}$$

and function $(1+Bz)^{\frac{\eta(A-B)}{B}}$ *is the best dominant.*

One observed that function $q(z) = (1+Bz)^{\frac{\eta(A-B)}{B}}$ is univalent if and only if the following is the case.

$$\left|\frac{\eta(A-B)}{B}-1\right|\leq 1 \text{ or } \left|\frac{\eta(A-B)}{B}+1\right|\leq 1.$$

Theorem 3. *Consider* q *as a univalent function in the unit disc* U *such that* $q(z)\neq 0$ *with* $\alpha,\beta,\xi\in\mathbb{C}$, $\xi\neq 0$, $\operatorname{Re}\lambda>0$ *and* $zq'(z)$ *is starlike univalent.*

Assume function $\frac{zq'(z)}{q(z)}$ *as the starlike univalent function in* U. *Let the below inequality holds*

$$\operatorname{Re}\left\{1+\frac{\beta}{\xi}+\frac{zq''(z)}{q'(z)}\right\}>0 \tag{24}$$

and

$$\begin{aligned}\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\xi;z) &:= \alpha+(\beta+\xi)\frac{z\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)}+ \\ &+\xi z^2\left[\frac{\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)''}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)}-\left(\frac{\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)}\right)^2\right].\end{aligned} \tag{25}$$

If the following subordination holds for function q:

$$\psi_{\lambda}^{\delta,b,c}(\alpha,\beta,\xi;z)\prec\alpha+\beta q(z)+\xi zq'(z) \tag{26}$$

then

$$\frac{z\left({}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)\right)'}{{}^{AB}\mathrm{I}_z^{\lambda}w_{\delta,b,c}(z)}\prec q(z),\quad z\in U,\ z\neq 0 \tag{27}$$

and function q *is the best dominant.*

Proof. Consider the analytic function p as follows

$$p(z) := \frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)}, \quad z \in U,\ z \neq 0.$$

Differentiating in the above relation with respect to z, one obtains

$$p'(z) = \frac{\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} - z\left(\frac{\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)}\right)^2 + \frac{z \cdot \left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)''}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)}. \tag{28}$$

By allowing $\theta(w) := \alpha + \beta w$ and $\phi(w) := \xi$, it can be readily observed that $\theta(w)$ is analytic in $\mathbb{C}$ and $\phi(w)$ is the analytic function in $\mathbb{C} \setminus \{0\}$ with $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Considering

$$Q(z) = zq'(z)\phi(q(z)) = \xi zq'(z)$$

and

$$h(z) := \theta(q(z)) + Q(z) = \alpha + \beta q(z) + \xi zq'(z)$$

we deduce that $Q(z)$ is starlike univalent function in U and the following is the case

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\beta}{\xi} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

In conclusion, by making use of Theorem 1, the differential subordination (27) of Theorem 3 holds. □

We deduce the following corollaries by a direct application of Theorem 3.

Corollary 5. *Consider $\alpha, \beta, \xi \in \mathbb{C}$, $\xi \neq 0$, $\operatorname{Re} \lambda > 0$ with $-1 \leq B < A \leq 1$ such that*

$$\operatorname{Re}\left\{1 + \frac{\beta}{\xi} - \frac{2Bz}{1 + Bz}\right\} > 0. \tag{29}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z) \prec \alpha + \beta\frac{1 + Az}{1 + Bz} + \xi\frac{(A - B)z}{(1 + Bz)^2} \tag{30}$$

where function $\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z)$ is posted in (25), then one obtains

$$\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \prec \frac{1 + Az}{1 + Bz} \tag{31}$$

and $\dfrac{1 + Az}{1 + Bz}$ is the best dominant.

Corollary 6. *Let numbers α, β, ξ, $\xi \neq 0$ with $\operatorname{Re} \lambda > 0$ and $0 < \eta \leq 1$ such that*

$$\operatorname{Re}\left\{1 + \frac{\beta}{\xi} + \frac{2z(\eta + z)}{1 - z^2}\right\} > 0. \tag{32}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z) \prec \alpha + \beta\left(\frac{1 + z}{1 - z}\right)^{\eta} + \xi z\frac{(1 + z)^{\eta - 1}}{(1 - z)^{\eta + 1}} \tag{33}$$

where $\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z)$ is posted in (25), then we obtain

$$\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \prec \left(\frac{1 + z}{1 - z}\right)^{\eta} \tag{34}$$

and $\left(\frac{1+z}{1-z}\right)^{\eta}$ *is the best dominant.*

Corollary 7. *Consider the numbers* A, α, β, ξ, $\xi \neq 0$ *with* $|\eta A| < \pi$ *and* $\operatorname{Re}\lambda > 0$ *such that*

$$\operatorname{Re}\left(1 + \frac{\beta}{\xi} + z\eta A\right) > 0. \tag{35}$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z) \prec \alpha + e^{\eta A z}(\beta + \xi A \eta z) \tag{36}$$

where the function $\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z)$ *is posted in* (25), *then we deduce the following:*

$$\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \prec e^{\eta A z} \tag{37}$$

and function $e^{\eta A z}$ *is the best dominant.*

Corollary 8. *Consider the numbers* $\alpha, \beta, \xi, \eta \in \mathbb{C}, \eta \neq 0, \xi \neq 0$ *and* $\operatorname{Re}\lambda > 0, -1 \leq B < A \leq 1$ *such that*

$$\operatorname{Re}\left\{1 + \frac{\beta}{\xi} + \frac{z[\eta(A-B) - B]}{(1+Bz)}\right\} > 0.$$

If

$$\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z) \prec \alpha + \beta(1+Bz)^{\frac{\eta(A-B)}{B}} + \xi z\eta(A-B)(1+Bz)^{\frac{\eta(A-B)-B}{B}} \tag{38}$$

where the function $\psi_{\lambda}^{\delta,b,c}(\alpha, \beta, \xi; z)$ *is posted in* (25), *then we obtain the following:*

$$\frac{z\left({}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)\right)'}{{}^{AB}I_z^{\lambda} w_{\delta,b,c}(z)} \prec (1+Bz)^{\frac{\eta(A-B)}{B}} \tag{39}$$

and $(1+Bz)^{\frac{\eta(A-B)}{B}}$ *is the best dominant.*

3. Discussion

In the present paper, we have investigated the research of a particular integral operator related to the Riemann–Liouville operator and Atangana–Baleanu integral operator. As we know it so far, for the Atangana-Baleanu fractional integral operator; these results are new in the field. The new results were established based on the second author's study, by applying fractional integral operator on hypergeometric functions. We provide new differential subordinations results that are established on a fractional integral operator involvement. Furthermore, specific statements were developed by means of well-known univalent functions. We provide univalent functions selections with specific symmetry properties. The new results produce certain corollaries that yield the best dominants. In future works, according to the new obtained results, it will be convenient to establish new symmetric properties for a certain differential operator and its integral. In conclusion, by combining two different branches (analytical functions with fractional calculus), the present paper provides new and interesting outcomes.

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