Article

Super Vertex (Edge)-Connectivity of Varietal Hypercube

Zhecheng Yu ^{1,2}, Liqiong Xu ^{1,2,*} , Shanshan Yin ^{1,2} and Litao Guo ³

¹ School of Science, Jimei University, Xiamen 361021, China; 202011701007@jmu.edu.cn (Z.Y.); 201911701009@jmu.edu.cn (S.Y.)

² Digital Fujian Big Data Modeling and Intelligent Computing Institute, Xiamen 361021, China

³ Department of Mathematics, Xiamen University of Technology, Xiamen 361024, China; guolitao@xmut.edu.cn

* Correspondence: xulqiong@jmu.edu.cn

Abstract: The reliability measure of networks is of significant importance to the design and maintenance of networks. Based on connectivity, many refined quantitative indicators for the reliability of network systems have been introduced. The super vertex edge-connectivity and cyclic edge-connectivity, as important parameters to evaluate the robustness of networks, are explored extensively. As a variant of the hypercube Q_n , the varietal hypercube VQ_n has better properties than Q_n with the same number of edges and vertices. Wang and Xu have proved that VQ_n is super vertex-connected for $n \geq 1$ and is also super edge-connected if $n \neq 2$. In this paper, we use another method to prove these results. Moreover, we also obtain the super restricted connectivity and the cyclic edge-connectivity of the varietal hypercube VQ_n .

Keywords: reliability; super vertex (edge)-connectivity; cyclic edge-connectivity; vertex-transitivity; edge-transitivity; varietal hypercube

Citation: Yu, Z.; Xu, L.; Yin, S.; Guo, L. Super Vertex (Edge)-Connectivity of Varietal Hypercube. *Symmetry* **2022**, *14*, 304. <https://doi.org/10.3390/sym14020304>

Academic Editors: Guifu Su and Junfeng Du

Received: 25 December 2021

Accepted: 28 January 2022

Published: 2 February 2022

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1. Introduction

In the era of internet-of-things (IoT), wireless sensor networks (WSNs) are utilised in many applications, including smart grids (SGs) [1]. When the number of sensors continues to increase, faulty nodes and faulty links are inevitable in wireless sensor networks, which undoubtedly increase the difficulty of network reliability assessment and will adversely affect its fault tolerance and reliability.

Hence, to the design and maintenance purpose of networks, appropriate measures of reliability should be found. Because the topology of network system can usually be represented by a connected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ can be represented as the set of processors and the set of communication links between processors, respectively. Connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ represent the maximum number of faulty vertices and the maximum number of faulty edges that an interconnected network can tolerate in normal communication, respectively. Obviously, the higher the (edge-) connectivity is, the more reliable an interconnection network is [2]. As is known, $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G . Boesch and Tindell [3] introduced the concepts of super connectivity and super edge-connectivity. A connected graph G is super connected (resp., edge-connected, in short super- κ (resp., super- λ),) if any minimum vertex-cut (resp., edge-cut) isolates a vertex of G .

However, the biggest disadvantage of connectivity and edge-connectivity in reliability evaluation and fault-tolerant performance of networks is that they all default to a vertex where all adjacent vertices or adjacent edges fail at the same time, which rarely happens in the real world. To overcome such shortcomings, several new concepts on the (edge-) connectivity of graphs, which are called conditional (edge-) connectivity, were proposed by Harary [4]. Among them are restricted (edge-)connectivity and cyclic edge-connectivity.

Esfahanian and Hakimi proposed the concept of restricted connectivity [5]. For a given non-negative integer h , a vertex-set (resp., edge-set) F is an h -restricted vertex-cut (resp.,

edge-cut) of a connected graph G if $G - F$ is disconnected, and each component of $G - F$ contains at least $h + 1$ vertices. The minimum size of all h -restricted vertex-cuts (resp., edge-cuts), denoted by $\kappa^{(h)}(G)$ (resp., $\lambda^{(h)}(G)$), is called h -restricted connectivity (resp., edge-connectivity) of the graph G . In the exceptional cases, G is called super restricted vertex (edge)-connected and denoted by super- κ' (resp., super- λ'), if every minimum restricted edge cut isolates one component of order 2. The h -restricted (edge)-connectivity and super- κ' (resp., super- λ') graph for some classic interconnection networks have been investigated recently. For instance, Li et al. [3] determine that the h -restricted connectivity of the arrangement graph $A_{n,k}$ is $n(n-1)/2$ for h is odd and $h = n-3$, otherwise the h -restricted connectivity of the arrangement graph $A_{n,k}$ is $(h+2)(n-2) - \lfloor h^2/2 + h \rfloor$ for $k = 2$ and $0 \leq h \leq n-2$. Chen et al. [6] proved that the hypercube, the twisted cube, the cross cube, the Möbius cube, and the locally twisted cube are super connected and super edge-connected. Furthermore, Xu et al. [7] obtained super edge-connectivity, and the restricted (edge-) connectivity of the hypercube, the twisted cube, the cross cube, Möbius cube, and the locally twisted cube is $2n-2$ for $n \geq 3$. Zhou et al. [8] obtained that the balanced hypercube BH_n is super- λ' but not super- κ' for $n \geq 2$.

In 1978, Bollobás indicated all multigraphs without two vertex-disjoint cycles [9]. So, it is necessary to further study the cyclically separable graphs. An edge set S is a cyclic edge-cut if $G - S$ is disconnected and at least two of its components contain cycles. Obviously, a graph G has a cyclic edge-cut if and only if it has two vertex-disjoint cycles and is a cyclically separable graphs. For a cyclically separable graph G , the cyclic edge-connectivity of G , denoted by $\lambda_c(G)$, is defined as the cardinality of the minimum cyclic edge-cut of G [10]. Let $\omega(X)$ be the number of edges with one end in X and another end in $V(G) - X$, and $\xi(G) = \min\{\omega(X) \mid X \text{ induces a shortest cycle in } G\}$. A cyclically separable graph G with $\lambda_c(G) = \xi(G)$ is called cyclically optimal. Moreover, if removing any minimum cyclic edge-cut of the graph G results in the shortest cycle in a component of the graph G , then the graph G is said to be super cyclically edge-connected [11], in short, super- λ_c . There are some results of the super cyclically edge-connectivity for some classes of the graphs. For example, Zhang [12] showed that the cyclic edge-connectivity of a strongly regular connected graph G (not $K_{3,3}$) of degree $k \geq 3$ with girth g is equal to $(k-2)g$, where $g = 3, 4$ or 5 . Wang and Zhang [13] showed that any k -regular vertex-transitive connected graph with $k \geq 4$ and girth of at least 5 is cyclically optimal, and any edge-transitive connected graph with minimum degree of at least 4 and order of at least 6 is cyclically optimal.

Hypercube Q_n , one of the most important and attractive network topologies so far, is widely used due to its excellent characteristics. In 1994, Cheng and Chuang designed a network topology capable of obtaining an interconnection network that is both efficient and cost-effective, this is, the varietal hypercube VQ_n . In recent years, VQ_n has received considerable attention and many of its properties have been studied. For example, Zhou [14] showed that VQ_n is a Cayley graph. Wang et al. [15] also determined the full automorphism group of the varietal hypercube VQ_n . More relevant results on the varietal hypercube VQ_n are shown in [16–20].

Our work in this paper concerns the n -dimensional varietal hypercube VQ_n . We use another method to prove that VQ_n is super vertex-connected for any $n \geq 1$ and super edge-connected for $n \neq 2$. We also obtain that VQ_n is super- λ_c for $n \geq 3$, super- λ' for $n \geq 3$ and $n \neq 4$, and the cyclic edge-connectivity of VQ_n is $4(n-2)$ for $n \geq 3$.

The paper is organized as follows. In Section 2, we give some propositions related to the vertex-transitive and edge-transitive graph. In Section 3, we introduce the necessary definitions and notations. In Section 4, we use the symmetry properties of the n -dimensional varietal hypercube to discuss some reliability measures of the n -dimensional varietal hypercube. In Section 5, we add some concluding remarks.

2. Related Works

We use a number of known results about some propositions of the vertex-transitive and edge-transitive graphs in the following.

Proposition 1 ([21,22]). Let G be a connected graph that is both vertex-transitive and edge-transitive. Then $\kappa(G) = \delta(G)$, and moreover, G is not super- κ if and only if $G \cong C_n \circ mK_1$ ($n \geq 6$) or $L(Q_3) \circ mK_1$, where $L(Q_3)$ is the line graph of the three dimensional hypercube Q_3 .

Proposition 2 ([23–27]). Let G be an edge-transitive k -regular graph where $k \geq 3$. Then

- (1) G is super- λ .
- (2) $\lambda'(G) = 2k - 2$.
- (3) If G is not isomorphic to K_4 , K_5 or $K_{3,3}$, then $\lambda_c(G) = g(k - 2)$, where g is the girth of G .
- (4) G is not super- λ' if and only if G is isomorphic to the three dimensional hypercube Q_3 or to an edge-transitive 4-regular graph of girth 4.

Let $\mathbb{Z}_n$ be denoted the residues modulo n for any positive integer n . We denote by $\mathcal{G}_{16}$ a graph with vertex set $\{a_i, b_i \mid i \in \mathbb{Z}_8\}$, and for each $i \in \mathbb{Z}_8$, if i is odd then $a_i b_{i+j} \in E(\mathcal{G}_{16})$ with $0 \leq j \leq 2$ and $b_i a_{i+j} \in E(\mathcal{G}_{16})$ with $0 \leq j \leq 2$, and if i is even then $a_i b_{i+j} \in E(\mathcal{G}_{16})$ with $0 \leq j \leq 3$ and $b_i a_{i+j} \in E(\mathcal{G}_{16})$ with $0 \leq j \leq 3$. If $n \geq 3$ and $1 \leq t < n/2$, the generalized Petersen graph $P(n, t)$ is a graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{x_i, x_{i+1}\}, \{x_i, y_i\}, \{y_i, y_{i+t}\} \mid i \in \mathbb{Z}_n\}$. For a 4-regular graph H , the subdivided double $\mathcal{D}(H)$ of H is the bipartite graph with the vertex set $(V(H) \times \mathbb{Z}_2) \cup E(H)$ and an edge between vertices $(v, i) \in V(H) \times \mathbb{Z}_2$ and $e \in E(H)$ whenever v is incident with e in H , where $(V(H) \times \mathbb{Z}_2) = \{(u, 0), (u, 1) \mid u \in V(H)\}$. The graph $G - mK_2$ is obtained by removing m vertex disjoint edges from G .

In [28], Zhou and Feng have classified all λ_c -optimal nonsuper- λ_c edge-transitive regular graphs and obtained some properties of λ_c -optimal nonsuper- λ_c edge-transitive regular graphs.

Proposition 3 ([28]). Let G be an edge-transitive k -regular graph of girth g where $k \geq 3$. Suppose that G is not isomorphic to K_4 , K_5 , or $K_{3,3}$. Then, G is not super- λ_c if and only if at least one of the following holds.

- (1) $g = 6$, $k = 3$ and $G \cong P(10, 7)$ or $P(8, 3)$.
- (2) $g = 4$ and $k = 4$, G is isomorphic to Q_4 (the 4-cube), $K_{5,5} - 5K_2$, $C_n \circ 2K_1$ ($n \geq 4$) or $\mathcal{D}(H)$ for some arc-transitive 4-regular graph H .
- (3) $g = 4$, $k = 5$ and $G \cong K_{6,6} - 6K_2$.
- (4) $g = 4$, $k = 6$ and $G \cong \mathcal{G}_{16}$.
- (5) $g = 3$, $k = 6$ and $G \cong K_8 - 4K_2$ or G is isomorphic to the line graph of an arc-transitive 4-regular triangle-free graph.

3. Preliminaries

In this section, we give some definitions and notations. If definitions and notations are not defined here, we follow [29].

Let $\text{Aut}(G)$ be the automorphism group of the graph G . We say that G is vertex-transitive if for any two vertices $u, v \in V(G)$ there exists an automorphism $\pi \in \text{Aut}(G)$ such that $\pi(u) = v$. If for any pair of edges (resp. arcs), there exists an automorphism that transforms one into the other, then G is edge-transitive (resp. arc-transitive) graph. Clearly, if a graph is arc-transitive, then it must be vertex-transitive and edge-transitive. Let u and v be two vertices of G , and the length of a shortest path is called the distance and denoted by $d_G(u, v)$. The maximum distance between any pair of vertices of a connected graph G is called the diameter of the graph G and denoted by $\text{diam}(G)$.

Suppose that G and H are two graphs, the lexicographic product $G \circ H$ is defined as the graph with vertex set $V(G) \times V(H)$, two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 x_2 \in E(G)$, or $x_1 = x_2$ and $y_1 y_2 \in E(H)$. The graph showed in Figure 1 is $C_6 \circ 2K_1$.

Figure 1. The lexicographic product of $C_6 \circ 2K_1$.

Let n be a positive integer. The definitions of the n -dimensional hypercube Q_n and the n -dimensional variational hypercube VQ_n are stated as follows.

Definition 1 ([30]). The n -dimensional hypercube is a connected graph with 2^n vertices and denoted by Q_n . The vertex set $V(Q_n) = \{x_1x_2 \cdots x_n : x_i = 0 \text{ or } 1, 1 \leq i \leq n\}$. Two vertices $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ in Q_n are adjacent if and only if they differ in exact one position.

Definition 2 ([17]). The n -dimensional variational hypercube, denoted by VQ_n , has 2^n vertices, each labeled by an n -bit binary string and $V(VQ_n) = \{x_nx_{n-1} \cdots x_2x_1 : x_i = 0 \text{ or } 1, i = 1, 2, \dots, n\}$. VQ_1 is a complete graph K_2 of two vertices labeled with 0 and 1, respectively. For $n \geq 2$, VQ_n can be recursively constructed from two copies of VQ_{n-1} , denoted by VQ_{n-1}^0 and VQ_{n-1}^1 , and by adding 2^{n-1} edges between VQ_{n-1}^0 and VQ_{n-1}^1 , where $V(VQ_{n-1}^0) = \{0x_{n-1} \cdots x_2x_1 : x_i = 0 \text{ or } 1, i = 1, 2, \dots, n-1\}$, $V(VQ_{n-1}^1) = \{1x_{n-1} \cdots x_2x_1 : x_i = 0 \text{ or } 1, i = 1, 2, \dots, n-1\}$. The vertex $x = 0x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1 \in V(VQ_{n-1}^0)$ is adjacent to the vertex $y = 1y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1 \in V(VQ_{n-1}^1)$ if and only if

$$x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1 \text{ if } 3 \nmid n, \text{ or} \quad (1)$$

$$x_{n-3} \cdots x_2x_1 = y_{n-3} \cdots y_2y_1 \text{ and } (x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(00, 00), (01, 01), (10, 11), (11, 10)\} \text{ if } 3 \mid n. \quad (2)$$

Obviously, VQ_n is an n -regular and its girth is 4. Moreover, it contains circles of length 5 when $n \geq 3$ [18,19]. The variational hypercubes VQ_1 , VQ_2 , and VQ_3 are illustrated in Figure 2.

As a variant of the hypercube, the n -dimensional variational hypercubes VQ_n , which has the same number of vertices and edges as Q_n , not only has the most ideal characteristics of Q_n , including some characteristics such as recursive structure, strong connectivity, and symmetry but also has a smaller diameter than Q_n , and its average distance is smaller than the hypercube [17].

Lemma 1 ([17]). $\text{diam}(VQ_n) = \lceil 2n/3 \rceil$ for $n \geq 3$.

Lemma 2 ([14]). $\kappa^{(2)}(VQ_n) > \kappa^{(1)}(VQ_n)$ for $n \geq 9$.

Lemma 3 ([15,20,31]). VQ_n is vertex-transitive and edge-transitive.

Lemma 4 ([32]). Let G and H be two graphs. If G is non-trivial, non-complete, and connected, then $\kappa(G \circ H) = \kappa(G)|V(H)|$.

Lemma 5 ([33]). Suppose (x_1, y_1) and (x_2, y_2) are two vertices of $G \circ H$. Then, the distance between (x_1, y_1) and (x_2, y_2) is

$$d_{G \circ H}((x_1, y_1), (x_2, y_2)) = \begin{cases} d_G(x_1, x_2) & \text{if } x_1 \neq x_2, \\ d_H(y_1, y_2) & \text{if } x_1 = x_2 \text{ and } d_G(x_1) = 0, \\ \min\{d_H(y_1, y_2), 2\} & \text{if } x_1 = x_2 \text{ and } d_G(x_1) \neq 0. \end{cases}$$

From the above lemma, we can obtain that the diameter of $G \circ H$ is

$$\text{diam}(G \circ H) = \max\{d_{G \circ H}((x_1, y_1), (x_2, y_2)) : (x_1, y_1), (x_2, y_2) \in V(G) \times V(H)\}.$$

Figure 2. The varietal hypercubes VQ_1 , VQ_2 , VQ_3 , and VQ_4 .

4. Reliability Evaluation of Varietal Hypercubes

In this section, we will study some reliability measures in varietal hypercubes, such as restricted (vertex-)connectivity and cyclic connectivity. Although Xu et al. proved that VQ_n is super vertex-connected for any $n \geq 1$ and super edge-connected for $n \neq 2$ in [19], we prove these results and obtain other results such as VQ_n is super- λ_c for $n \geq 3$, super- λ' for $n \geq 3$, and $n \neq 4$ through symmetric properties of VQ_n .

Theorem 1. VQ_n is super- κ for $n \geq 1$.

Proof. If $n = 1$, then VQ_n is a complete graph K_2 . Obviously, it is super- κ .

If $n = 2$, then VQ_2 is cycle C_4 , and clearly it is super- κ .

Assume that $n \geq 3$. Suppose to the contrary that VQ_n is not super- κ . Since VQ_n is both vertex-transitive and edge-transitive, it follows that $VQ_n \cong C_l \circ mK_1$ ($l \geq 6$) or $L(Q_3) \circ mK_1$ from Proposition 1.

First suppose that $VQ_n \cong C_l \circ mK_1$ ($l \geq 6$); we know that $\kappa(VQ_n) = n$ and $\kappa(C_l \circ mK_1) = 2m$ by Lemma 4, $n = 2m$ and $m > 1$. Moreover, since $|V(C_l \circ mK_1)| = |V(VQ_n)|$, it follows that $2^n = ml$. Thus, m and l are even. According to Lemma 1 and Lemma 5, we know that $\text{diam}(VQ_n) = \lceil 2n/3 \rceil$ and $\text{diam}(C_l \circ mK_1) = l/2$, respectively. Now, $l/2 = \lceil 2n/3 \rceil = \lceil 4m/3 \rceil$ and $l = 2\lceil 4m/3 \rceil$. Now, $2^n = ml = 2m\lceil 4m/3 \rceil$, and we have $2^{n-1} = m\lceil 4m/3 \rceil$.

We discuss the value of m , as follows.

Case 1. $m = 3k$ where k is a positive integer.

Then, $2^{n-1} = m\lceil 4m/3 \rceil = 4mk = 12k^2$. Now, we obtain $2^{n-3} = 3k^2$, a contradiction.

Case 2. $m = 3k + 1$ where k is a positive integer.

Then, $2^{n-1} = m\lceil 4m/3 \rceil = (3k + 1)\lceil (12k + 4)/3 \rceil = (3k + 1)(4k + 2)$. So, we have $2^{n-2} = (3k + 1)(2k + 1)$. In this case, $(2k + 1) \mid 2^{n-2}$; hence, we have $k = 0$ and $m = 1$. This contradicts the fact that m is even.

Case 3. $m = 3k + 2$ where k is a positive integer.

Then, $2^{n-1} = m \lceil 4m/3 \rceil = (3k+2) \lceil (12k+8)/3 \rceil = (3k+2)(4k+3)$. So, we have $2^{n-1} = (3k+2)(4k+3)$ and $(4k+3) \mid 2^{n-1}$, a contradiction.

Now, suppose that $VQ_n \cong L(Q_3) \circ mK_1$, then the number of vertices of these graphs is same and we obtain $2^n = 12m$, so we have $2^{n-2} = 3m$ and $3 \mid 2^{n-2}$ where $n \geq 3$, a contradiction. $\square$

Theorem 2. VQ_n is super- λ for $n \neq 2$.

Proof. Since VQ_1 is a complete graph K_2 , it is super- λ . If $n = 2$, VQ_n is a cycle C_4 . Clearly, it is not super- λ . Suppose that VQ_n is not super- λ for $n \geq 3$. VQ_n is edge-transitive k -regular graph where $k \geq 3$, so the theorem follows from Proposition 2(1). $\square$

In [19], Xu et al. proved that 1-restricted edge-connectivity of VQ_n is $2n - 2$. Here, we present another proof of this result.

Theorem 3. $\lambda'(VQ_n) = 2n - 2$ for $n \geq 2$.

Proof. If $n = 2$, then VQ_n is a cycle C_4 . Clearly, $\lambda'(VQ_n) = 2$. Thus, we may suppose that $n \geq 3$. VQ_n is an edge-transitive k -regular graph where $k \geq 3$, so the theorem follows from Proposition 2(2). $\square$

In the following theorem, we show that every minimum edge-cut of VQ_n for $n \geq 3$ and $n \neq 4$ isolates an edge.

Theorem 4. VQ_n is super- λ' for $n \geq 3$ and $n \neq 4$.

Proof. Assume that VQ_n is not super- λ' for $n \geq 3$ and $n \neq 4$. According to VQ_n is edge-transitive and Proposition 2(4), we know that VQ_n is not super- λ' if and only if VQ_n is isomorphic to Q_3 or to an edge-transitive 4-regular graph of girth 4.

If $n = 3$, Q_3 does not have cycle of length 5 but VQ_3 has, then VQ_3 is not isomorphic to Q_3 .

If $n \geq 5$, since VQ_n is edge-transitive k -regular graph where $k \geq 5$, then it implies that VQ_n is super- λ' when $n \geq 5$.

We have accordingly proved the theorem. $\square$

In the following theorem, we prove that if we remove $4(n - 2)$ edges from the VQ_n , then we will obtain a disconnected graph with at least two components containing cycle for $n \geq 3$.

Theorem 5. $\lambda_c(VQ_n) = 4(n - 2)$ for $n \geq 3$.

Proof. Since VQ_n has at least 8 vertices for $n \geq 3$, it is not isomorphic to K_4 , K_5 or $K_{3,3}$. According to the fact that VQ_n has girth 4, it follows that $\lambda_c(VQ_n) = 4(n - 2)$ by Proposition 2(3). $\square$

The following theorem shows that every minimum cyclic edge-cut of VQ_n isolates a shortest cycle for $n \geq 3$.

Theorem 6. VQ_n is super- λ_c for $n \geq 3$.

Proof. Since VQ_n is an edge-transitive n -regular graph of girth $g = 4$ with 2^n vertices. By the definition of VQ_n , we obtain that VQ_n is not isomorphic to K_4 , K_5 or $K_{3,3}$. If $n = 3$, then it is obvious that VQ_n is super- λ_c .

It is assumed that VQ_n is not super- λ_c for $n \geq 4$. So if $n = 4$, then VQ_n is edge-transitive 4-regular graph. According to Proposition 3(3), it follows that Γ is isomorphic

to Q_4 (the 4-cube), $K_{5,5} - 5K_2$, $C_n \circ 2K_1$ ($n \geq 4$), or $\mathcal{D}(H)$, for some arc-transitive 4-regular graph H . Since VQ_4 has odd cycle of length 5 and 16 vertices, it is not difficult to check that VQ_4 is not isomorphic to Q_4 , $K_{5,5} - 5K_2$, or $\mathcal{D}(H)$, for some 4-regular arc-transitive graph H . Furthermore, if $VQ_n \cong C_l \circ 2K_1$ for $n = 4$ and $l \geq 4$, then $|V(C_l \circ 2K_1)| = |V(VQ_n)|$, it follows that $2l = 2^4 = 16$ and $l = 8$. However, in this case, $\text{diam}(VQ_n) = \lceil 2n/3 \rceil = 3$ and $\text{diam}(C_l \circ 2K_1) = \lfloor l/2 \rfloor = 4$ by Lemma 1 and Lemma 5. So, VQ_n is not isomorphic to $C_l \circ 2K_1$, a contradiction. Now, by Proposition 3(3) and (4), VQ_5 or VQ_6 has 12 or 16 vertices, a contradiction.

Thus, we conclude that VQ_n is super- λ_c for $n \geq 3$. $\square$

The next theorem shows that every minimum vertex-cut of VQ_n isolates an edge for $n \geq 9$.

Theorem 7. VQ_n is super- κ' for $n \geq 9$.

Proof. By Lemma 2, we know that $\kappa^{(2)}(VQ_n) > \kappa^{(1)}(VQ_n)$ for $n \geq 9$. It is assumed that VQ_n is not super- κ' for $n \geq 9$. Hence, there is an 1-restricted vertex-cut S and $|S| = \kappa^{(1)}(VQ_n)$, but the cut is not the neighborhood of any edge. That is, S is also a two-restricted vertex-cut and $|S| \geq \kappa^{(2)}(VQ_n)$. So, we have $|S| > \kappa^{(1)}(VQ_n)$, and this leads to a contradiction. $\square$

5. Concluding Remarks

In order to better measure the performance of the network, the concepts of restricted connectivity, restricted edge-connectivity, and cyclic edge-connectivity have been successively proposed. The research in this paper is based on these concepts and uses some properties of vertex-transitive and edge-transitive graph to get the main result that the varietal hypercube is super restricted vertex-connected, super restricted edge-connected, and super cyclically edge-connected. Moreover, the cyclic edge-connectivity of VQ_n is $4(n-2)$ for $n \geq 3$. In the future, we will continue to research the other properties of the varietal hypercube.

Author Contributions: Z.Y.: Conceptualization, Investigation, Validation, Writing—original draft. L.X.: Conceptualization, Methodology, Investigation, Supervision, Validation, Writing—review & editing. S.Y.: Investigation, Validation. L.G.: Investigation, Validation. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by Natural Science Foundation of Fujian Province, China (Nos. 2021J01860, 2019J01857), National Natural Science Foundation of China (Nos. 11301217, 11961051) and Digital Fujian Big Data Modeling and Intelligent Computing Institute Open Fund Project.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitle.

References

1. Elwekeil, M.; Abdalzaher, M.S.; Seddik, K. Prolonging smart grid network lifetime through optimising number of sensor nodes and packet length. *IET Commun.* **2019**, *13*, 2478–2484. [\[CrossRef\]](#)
2. Xu, J.-M. *Combinatorial Theory in Networks*; Academic Press: Beijing, China, 2013; pp. 3–27.
3. Li, X.-J.; Zeng, X.-Q.; Xu, J.-M. Note on reliability evaluation of arrangement graphs. *Appl. Math. Comput.* **2022**, *418*, 126845. [\[CrossRef\]](#)
4. Harary, F. Conditional connectivity. *Networks* **1983**, *13*, 347–357. [\[CrossRef\]](#)

5. Esfahanian, A.H.; Hakimi, S.L. On computing a conditional edge-connectivity of a graph. *Inf. Process. Lett.* **1988**, *27*, 195–199. [[CrossRef](#)]
6. Chen, Y.-C.; Tan, J.-J.; Hsu, L.-H.; Kao, S.-S. Super-connectivity and super-edge-connectivity for some interconnection networks. *Appl. Math. Comput.* **2003**, *140*, 245–254. [[CrossRef](#)]
7. Xu, J.-M.; Wang, J.-W.; Wang, W.-W. On super and restricted connectivity of some interconnection networks. *Ars Combin.* **2010**, *94*, 25–32.
8. Zhou, J.-X.; Wu, Z.-L.; Yang, S.-C.; Yuan, K.-W. Symmetric property and reliability of balanced hypercube. *IEEE Trans. Comput.* **2014**, *64*, 876–881. [[CrossRef](#)]
9. Bollobás, B. *Extremal Graph Theory*; Academic: Cambridge, MA, USA, 1978; pp. 110–119.
10. Plummer, M.D. On the cyclic connectivity of planar graphs, *Graph Theory and Applications*; Springer: Berlin/Heidelberg, Germany, 1972; pp. 235–242.
11. Zhang, Z.; Wang, B. Super cyclically edge connected transitive graphs. *J. Comb. Optim.* **2011**, *22*, 549–562. [[CrossRef](#)]
12. Zhang, W.-Q. The cyclic edge-connectivity of strongly regular graphs. *Graphs Comb.* **2019**, *35*, 779–785. [[CrossRef](#)]
13. Wang, B.; Zhang, Z. On cyclic edge-connectivity of transitive graphs. *Discret. Math.* **2009**, *309*, 4555–4563. [[CrossRef](#)]
14. Zhou, J.-X. On g-extra connectivity of hypercube-like networks. *J. Comput. Syst. Sci.* **2017**, *88*, 208–219. [[CrossRef](#)]
15. Wang, Y.; Feng, Y.-Q.; Zhou, J.-X. Automorphism group of the varietal hypercube graph. *Graphs Comb.* **2017**, *33*, 1131–1137. [[CrossRef](#)]
16. Barpanda, N.K.; Jena, S.; Dash, R.K. Network reliability evaluation of fault tolerant varietal hypercube network. *Int. J. Adv. Res. Comput. Sci.* **2014**, *5*, 132–135.
17. Cheng, S.-Y.; Chuang, J.-H. Varietal hypercube—A new interconnection network topology for large scale multicomputer. In Proceedings of the 1994 International Conference on Parallel and Distributed Systems, Hsinchu, Taiwan, 19–21 December 1994; pp. 703–708.
18. Cao, J.; Li, X.; Xu, J.-M. Cycles and paths embedded in varietal hypercubes. *J. Univ. Sci. Technol. China* **2014**, *44*, 732–737.
19. Wang, J.-W.; Xu, J.-M. Reliability analysis of varietal hypercube networks. *J. Univ. Sci. Technol. China* **2009**, *39*, 1248–1252.
20. Xiao, L.; Cao, J.; Xu, J.-M. Transitivity of varietal hypercube networks. *Front. Math. China* **2014**, *9*, 1401–1410. [[CrossRef](#)]
21. Du, D.-Z.; Hsu, D.F. *Combinatorial Network Theory*; Springer: London, UK, 1996; pp. 41–60.
22. Meng, J. Connectivity of vertex and edge transitive graphs. *Discret. Appl. Math.* **2003**, *127*, 601–613. [[CrossRef](#)]
23. Boesch, F.; Tindell, R. Circulants and their connectivities. *J. Graph Theory* **1984**, *8*, 487–499. [[CrossRef](#)]
24. Li, Q.-L.; Li, Q. Super edge connectivity properties of connected edge symmetric graphs. *Networks* **1999**, *33*, 157–159. [[CrossRef](#)]
25. Tian, Y.; Meng, J. On super restricted edge-connectivity of edge-transitive graphs. *Discret. Math.* **2010**, *310*, 2273–2279. [[CrossRef](#)]
26. Zhou, J.-X. Super restricted edge connectivity of regular edge-transitive graphs. *Discret. Appl. Math.* **2012**, *160*, 1248–1252. [[CrossRef](#)]
27. Zhou, J.-X. Atoms of cyclic edge connectivity in regular graphs. *J. Comb. Optim.* **2016**, *31*, 382–395. [[CrossRef](#)]
28. Zhou, J.-X.; Feng, Y.-Q. Super-cyclically edge-connected regular graphs. *J. Comb. Optim.* **2013**, *26*, 393–411. [[CrossRef](#)]
29. Bondy, J.A.; Murty, U.S.R. *Graph Theory*; Springer: Berlin/Heidelberg, Germany, 2008; p. 327.
30. Li, H.; Yang, W. Bounding the size of the subgraph induced by m vertices and extra edge-connectivity of hypercubes. *Discret. Appl. Math.* **2013**, *161*, 2753–2757. [[CrossRef](#)]
31. Chang, X.; Ma, J.; Yang, D.-W. Symmetric property and reliability of locally twisted cubes. *Discret. Appl. Math.* **2021**, *288*, 257–269. [[CrossRef](#)]
32. Xu, J.-M.; Yang, C. Connectivity of lexicographic product and direct product of graphs. *Ars Combin.* **2013**, *111*, 3–12.
33. Hammack, R.H.; Imrich, W.; Klavžar, S. *Handbook of Product Graphs*; CRC Press: Boca Raton, FL, USA, 2011; pp. 56–59.