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On Generalization of Different Integral Inequalities for Harmonically Convex Functions

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Abstract: In this study, we first prove a parameterized integral identity involving differentiable functions. Then, for differentiable harmonically convex functions, we use this result to establish some new inequalities of a midpoint type, trapezoidal type, and Simpson type. Analytic inequalities of this type, as well as the approaches for solving them, have applications in a variety of domains where symmetry is important. Finally, several particular cases of recently discovered results are discussed, as well as applications to the special means of real numbers.

Keywords: midpoint and trapezoidal inequality; Simpson's inequality; harmonically convex functions

MSC: 26D10; 26D15; 26A51

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1. Introduction

The Hermite–Hadamard inequality, which was independently found by C. Hermite and J. Hadamard (see, also [1], and [2] (p. 137)), is particularly important in the convex functions theory:

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} \tag{1}$$

where $\mathcal{F} : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function over I, and $\kappa_1, \kappa_2 \in I$, with $\kappa_1 < \kappa_2$. In the case of concave mappings, the above inequality is satisfied in reverse order.

Several researches have concentrated on obtaining trapezoid and midpoint-type inequalities that offer bounds for the right-hand side and left-hand side of the inequality (1), respectively, throughout the previous two decades. In [3,4], for example, authors first obtained trapezoid and midpoint-type inequalities for convex functions. In [5], Sarikaya et al. obtained the inequalities (1) for the Riemann–Liouville fractional integrals and the authors also proved some corresponding trapezoid-type inequalities for fractional integrals. Iqbal et al. presented some fractional midpoint-type inequalities for convex functions in [6]. On the other hand, İşcan defined the harmonically convex functions and obtained Hermite–Hadamard-type inequalities for these kinds of functions in [7]. The author also established some trapezoid-type inequalities for harmonically convex functions in [7]. Furthermore, using the Riemann–Liouville fractional integrals, the authors proved Hermite–Hadamard-type inequalities for harmonically convex functions in [8]. They also proved some fractional trapezoid-type inequalities for mapping whose derivatives in absolute value are harmonically convex. In [9], Şanlı proved several fractional midpoint-type inequalities utilizing differentiable convex functions. In [10], Butt et al. presented a new generalization of Hermite–Hadamard inequalities for harmonically convex functions using the notions of the Jensen–Mercer inequality and, in [11], Butt et al. gave a new definition of general harmonically convex functions and proved Hermite–Hadamard-type inequalities. In [12], the authors used fractional operators and proved some new inequalities for general harmonic convex functions. In [13], the authors established Hermite–Hadamard-type inequalities for harmonically convex functions on n-co-ordinates. Some generalizations of Hermite–Hadamard-type inequalities for harmonically convex functions on fractal sets are also given in [14]. Moreover, Liu and Xu extended this class of functions and defined a general harmonic convexity for interval-valued functions in [15]. In the literature, there are several papers on inequalities for harmonically convex functions. For some recent developments in integral inequalities and harmonical convexity, one can consult [16–26].

Inspired by these ongoing studies, we prove several Simpson's type generalized integral inequalities for differentiable convex functions. The key benefit of these inequalities is that they can be turned into midpoint and trapezoidal-type inequalities for differentiable convex functions without having to prove each one independently. These newly established inequalities are the generalizations of inequalities proved in [7,9].

The following is the structure of this paper: In Section 2, we present the definition of the harmonically convex functions and some related results. In Section 3, we prove several new results for harmonically convex functions depending on parameters. We also prove some new integral inequalities to highlight the relationship between the results reported here and related results in the literature. By specially choosing one of the parameters, we give some new results in Section 4. Some applications of newly established inequalities to special means of real numbers are given in Section 5. In Section 6, we give some recommendations for future studies.

2. Preliminaries

In [7], İşcan gave the concept of harmonically convex functions and proved associated Hermite–Hadamard inequalities as follows:

Definition 1 ([7])**.** *If the mapping $\mathcal{F} : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfies the inequality*

$$\mathcal{F}\left(\frac{1}{\frac{\tau}{\kappa_2} + \frac{1-\tau}{\kappa_1}}\right) \leq \tau\mathcal{F}(\kappa_2) + (1-\tau)\mathcal{F}(\kappa_1), \tag{2}$$

for all $x, y \in I$ and $\tau \in [0,1]$; then, $\mathcal{F}$ is called the harmonically convex function. In the case of harmonically concave mappings, the above inequality is satisfied in reverse order.

Theorem 1 ([7])**.** *For any harmonically convex mapping $\mathcal{F} : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ and $\kappa_1, \kappa_2 \in I$, with $\kappa_1 < \kappa_2$, the following inequality holds:*

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \leq \frac{\kappa_1\kappa_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \frac{\mathcal{F}(x)}{x^2}dx \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \tag{3}$$

In [7], İşcan established the following Lemma to prove trapezoidal-type inequalities for harmonically convex functions.

Lemma 1. *Consider a mapping $\mathcal{F} : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$, which is differentiable mapping on I° (interior of I) and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$. If $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$, then we would have the following equality:*

$$\begin{aligned}
&\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx \\
= \; &\frac{\kappa_1\kappa_2(\kappa_2-\kappa_1)}{2}\int_0^1 \frac{1-2\tau}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)d\tau.
\end{aligned} \tag{4}$$

Theorem 2. *Consider a mapping $\mathcal{F} : I \subset (0,\infty) \to \mathbb{R}$, which is differentiable mapping on I° and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, and $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$. If $|\mathcal{F}'|$ is harmonically convex on $[\kappa_1, \kappa_2]$, then the following inequality satisfies:*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\le \; &\frac{\kappa_1\kappa_2(\kappa_2-\kappa_1)}{2}\theta_1^{1-\frac{1}{q}}\left(\theta_2|\mathcal{F}'(\kappa_1)|^q+\theta_3|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\theta_1 &= \frac{1}{\kappa_1\kappa_2} - \frac{2}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{(\kappa_1+\kappa_2)^2}{4\kappa_1\kappa_2}\right), \\
\theta_2 &= \frac{1}{\kappa_2(\kappa_1-\kappa_2)} + \frac{3\kappa_1+\kappa_2}{(\kappa_2-\kappa_1)^3}\ln\left(\frac{(\kappa_1+\kappa_2)^2}{4\kappa_1\kappa_2}\right), \\
\theta_3 &= \theta_1-\theta_2.
\end{aligned}$$

Recently, Şanli [9] proved the following Lemma to find the left estimates of the inequality (3).

Lemma 2. *Consider a mapping $\mathcal{F} : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$, which is differentiable mapping on I° and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$. If $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$, then we would have the following equality:*

$$\begin{aligned}
\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) - &\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx \\
= \; &\kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}\frac{\tau}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)d\tau\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{\tau-1}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)d\tau\right].
\end{aligned} \tag{6}$$

3. New Parameterized Inequalities for Harmonically Convex Function

Lemma 3. *Consider a mapping $\mathcal{F} : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$, which is differentiable mapping on I° and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$. If $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$, then for $\lambda \in \mathbb{R}$ we would have the following equality:*

$$\begin{aligned}
&\lambda[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)] - (2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx \\
= \; &\kappa_1\kappa_2(\kappa_2-\kappa_1)\int_0^1\frac{m(\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)d\tau,
\end{aligned} \tag{7}$$

where

$$m(\tau) = \begin{cases} \lambda - \tau, & \textit{if } \tau \in \left[0, \frac{1}{2}\right) \\ 1 - \lambda - \tau, & \textit{if } \tau \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. From the fundamental concepts of integration, we had:

$$\begin{aligned}
& \kappa_1\kappa_2(\kappa_2 - \kappa_1)\int_0^1 \frac{m(\tau)}{(\tau\kappa_2 + (1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)d\tau \\
= & \kappa_1\kappa_2(\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}} \frac{\lambda - \tau}{(\tau\kappa_2 + (1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)d\tau\right. \\
& \left. + \int_{\frac{1}{2}}^1 \frac{1 - \lambda - \tau}{(\tau\kappa_2 + (1-\tau)\kappa_1)^2}\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)d\tau\right] \\
= & \left[(\tau - \lambda)\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \mathcal{F}\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)d\tau\right] \\
& + \left[(\tau + \lambda - 1)\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)\Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \mathcal{F}\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2 + (1-\tau)\kappa_1}\right)d\tau\right] \\
= & \lambda[\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)] - (2\lambda - 1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) - \frac{\kappa_1\kappa_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \frac{\mathcal{F}(x)}{x^2}dx
\end{aligned}$$

and the proof was completed. □

Remark 1. *In Lemma 3, if we set $\lambda = \frac{1}{2}$, then we recaptured the identity (4).*

Remark 2. *In Lemma 3, if we assume $\lambda = 0$, then we recaptured the identity (6).*

Theorem 3. *Consider a mapping $\mathcal{F} : I \subset (0, \infty) \to \mathbb{R}$, which is differentiable mapping on I° and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, and $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$. If $|\mathcal{F}'|$ is harmonically convex on $[\kappa_1, \kappa_2]$, then the following inequality satisfies:*

$$\begin{aligned}
& \left|\lambda[\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)] - (2\lambda - 1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) - \frac{\kappa_1\kappa_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq & \kappa_1\kappa_2(\kappa_2 - \kappa_1) \\
& \times \begin{cases} \begin{array}{l} \{[\Delta_1(\kappa_1, \kappa_2; \lambda) + \Delta_3(\kappa_1, \kappa_2; \lambda) + \Delta_5(\kappa_1, \kappa_2; \lambda) + \Delta_7(\kappa_1, \kappa_2; \lambda)]|\mathcal{F}'(\kappa_1)| \\ +[\Delta_2(\kappa_1, \kappa_2; \lambda) + \Delta_4(\kappa_1, \kappa_2; \lambda) + \Delta_6(\kappa_1, \kappa_2; \lambda) + \Delta_8(\kappa_1, \kappa_2; \lambda)]|\mathcal{F}'(\kappa_2)|\} \end{array}, & \textit{if } 0 \leq \lambda < \frac{1}{2} \\ \begin{array}{l} \{[\Delta_9(\kappa_1, \kappa_2; \lambda) + \Delta_{11}(\kappa_1, \kappa_2; \lambda)]|\mathcal{F}'(\kappa_1)| \\ +[\Delta_{10}(\kappa_1, \kappa_2; \lambda) + \Delta_{12}(\kappa_1, \kappa_2; \lambda)]|\mathcal{F}'(\kappa_2)|\} \end{array}, & \textit{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1(\kappa_1,\kappa_2;\lambda) &= \frac{\lambda}{(\kappa_2-\kappa_1)^2}\ln(\kappa_1.(\lambda\kappa_2+(1-\lambda)\kappa_1)) \\
&\quad -\frac{2\lambda}{(\kappa_2-\kappa_1)^2}+\frac{2\kappa_1}{(\kappa_2-\kappa_1)^3}\ln\left(\frac{\lambda\kappa_2+(1-\lambda)\kappa_1}{\kappa_1}\right), \\
\Delta_2(\kappa_1,\kappa_2;\lambda) &= \frac{\lambda}{\kappa_1(\kappa_2-\kappa_1)}+\frac{1}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{\kappa_1}{\lambda\kappa_2+(1-\lambda)\kappa_1}\right)-\Delta_1(\kappa_1,\kappa_2;\lambda), \\
\Delta_3(\kappa_1,\kappa_2;\lambda) &= \frac{2\lambda-1}{2(\kappa_2^2-\kappa_1^2)}+\frac{\lambda}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{(\kappa_1+\kappa_2)(\lambda\kappa_2+(1-\lambda)\kappa_1)}{2}\right) \\
&\quad +\frac{1-2\lambda}{(\kappa_2-\kappa_1)^2}+\frac{2\kappa_1}{(\kappa_2-\kappa_1)^3}\ln\left(\frac{2(\lambda\kappa_2+(1-\lambda\kappa_1))}{\kappa_1+\kappa_2}\right), \\
\Delta_4(\kappa_1,\kappa_2;\lambda) &= \frac{2\lambda-1}{\kappa_2^2-\kappa_1^2}+\frac{1}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{\kappa_1+\kappa_2}{2(\lambda\kappa_2+(1-\lambda\kappa_1))}\right)-\Delta_3(\kappa_1,\kappa_2;\lambda), \\
\Delta_5(\kappa_1,\kappa_2;\lambda) &= \frac{2\lambda-1}{2(\kappa_2^2-\kappa_1^2)}+\frac{\kappa_1+\kappa_2}{(\kappa_2-\kappa_1)^3}\ln\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&\quad +\frac{1-2\lambda}{(\kappa_2-\kappa_1)^2}(1+\ln(\lambda\kappa_1+(1-\lambda)\kappa_2)) \\
&\quad -\frac{2}{(\kappa_2-\kappa_1)^3}(\lambda\kappa_1+(1-\lambda)\kappa_2)\ln(\lambda\kappa_1+(1-\lambda)\kappa_2), \\
\Delta_6(\kappa_1,\kappa_2;\lambda) &= \frac{2\lambda-1}{\kappa_2^2-\kappa_1^2}+\frac{1}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{2(\lambda\kappa_1+(1-\lambda)\kappa_2)}{\kappa_1+\kappa_2}\right)-\Delta_5(\kappa_1,\kappa_2;\lambda), \\
\Delta_7(\kappa_1,\kappa_2;\lambda) &= \frac{1-2\lambda}{(\kappa_2-\kappa_1)^2}\ln(\lambda\kappa_1+(1-\lambda)\kappa_2) \\
&\quad -\frac{2}{(\kappa_2-\kappa_1)^3}(\lambda\kappa_1+(1-\lambda)\kappa_2)\ln(\lambda\kappa_1+(1-\lambda)\kappa_2) \\
&\quad +\frac{\lambda}{\kappa_2(\kappa_2-\kappa_1)}-\frac{2\lambda}{(\kappa_2-\kappa_1)^2}+\frac{\kappa_1+\kappa_2}{\kappa_2-\kappa_1}\ln(\kappa_2), \\
\Delta_8(\kappa_1,\kappa_2;\lambda) &= \frac{1}{(\kappa_2-\kappa_1)^2}\ln\left(\frac{\lambda\kappa_1+(1-\lambda)\kappa_2}{\kappa_2}\right)+\frac{\lambda}{\kappa_2(\kappa_2-\kappa_1)}-\Delta_7(\kappa_1,\kappa_2;\lambda), \\
\Delta_9(\kappa_1,\kappa_2;\lambda) &= \frac{1-2\lambda}{2(\kappa_2^2-\kappa_1^2)}+\frac{\kappa_1+\kappa_2}{(\kappa_2-\kappa_1)^3}\ln\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&\quad -\frac{1}{(\kappa_2-\kappa_1)^2}-\frac{\lambda}{(\kappa_2-\kappa_1)^2}\ln(\kappa_1)-\frac{2}{(\kappa_2-\kappa_1)^3}\kappa_1\ln(\kappa_1), \\
\Delta_{10}(\kappa_1,\kappa_2;\lambda) &= \frac{1-2\lambda}{\kappa_2^2-\kappa_1^2}+\frac{\kappa_1+\kappa_2}{2(\kappa_2-\kappa_1)^2}\ln\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&\quad -\frac{\kappa_2-\kappa_1}{2}+\frac{\lambda}{\kappa_1(\kappa_2-\kappa_1)}-\frac{1}{(\kappa_2-\kappa_1)^2}\kappa_1\ln(\kappa_1)-\Delta_9(\kappa_1,\kappa_2;\lambda), \\
\Delta_{11}(\kappa_1,\kappa_2;\lambda) &= \frac{\lambda}{\kappa_2(\kappa_1-\kappa_2)}+\ln\left(\frac{(1+\lambda)(\kappa_2-\kappa_1)+2\kappa_2}{(\kappa_2-\kappa_1)^3}\right) \\
&\quad -\frac{1}{(\kappa_2-\kappa_1)^2}+\frac{2\lambda-1}{2(\kappa_2^2-\kappa_1^2)}+\ln\left(\frac{\lambda(\kappa_2-\kappa_1)+\kappa_1+\kappa_2}{(\kappa_2-\kappa_1)^3}\right), \\
\Delta_{12}(\kappa_1,\kappa_2;\lambda) &= \frac{\lambda}{\kappa_2(\kappa_1-\kappa_2)}+\frac{1}{(\kappa_2-\kappa_1)^2}\kappa_2\ln(\kappa_2) \\
&\quad -\frac{1}{2(\kappa_2-\kappa_1)}+\frac{2\lambda-1}{\kappa_2^2-\kappa_1^2}-\frac{\kappa_1+\kappa_2}{2(\kappa_2-\kappa_1)^2}\ln\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned}$$

Proof. Taking modulus in (7), we had:

$$\begin{aligned}
& \left|\lambda[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)]-(2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|d\tau\right. \\
& \left.+\int_{\frac{1}{2}}^1\frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|d\tau\right] \\
=\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)[S_1+S_2].
\end{aligned}$$

From the convexity of $|\mathcal{F}'|$ and for $0\leq\lambda\leq\frac{1}{2}$, we have:

$$\begin{aligned}
S_1 &\leq \int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \int_0^{\lambda}\frac{(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&\quad+\int_{\lambda}^{\frac{1}{2}}\frac{(\tau-\lambda)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \left[\int_0^{\lambda}\frac{\tau(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\int_{\lambda}^{\frac{1}{2}}\frac{\tau(\tau-\lambda)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right]\left|\mathcal{F}'(\kappa_1)\right| \\
&\quad+\left[\int_0^{\lambda}\frac{(1-\tau)(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\int_{\lambda}^{\frac{1}{2}}\frac{(1-\tau)(\tau-\lambda)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right]\left|\mathcal{F}'(\kappa_2)\right| \\
&= [\Delta_1(\kappa_1,\kappa_2;\lambda)+\Delta_3(\kappa_1,\kappa_2;\lambda)]\left|\mathcal{F}'(\kappa_1)\right|+[\Delta_2(\kappa_1,\kappa_2;\lambda)+\Delta_4(\kappa_1,\kappa_2;\lambda)]\left|\mathcal{F}'(\kappa_2)\right|.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
S_2 &\leq \int_{\frac{1}{2}}^1\frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \int_{\frac{1}{2}}^{1-\lambda}\frac{(1-\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&\quad+\int_{1-\lambda}^{1}\frac{(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \left[\int_{\frac{1}{2}}^{1-\lambda}\frac{\tau(1-\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\int_{1-\lambda}^{1}\frac{\tau(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right]\left|\mathcal{F}'(\kappa_1)\right| \\
&\quad+\left[\int_{\frac{1}{2}}^{1-\lambda}\frac{(1-\tau)(1-\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\int_{1-\lambda}^{1}\frac{(1-\tau)(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right]\left|\mathcal{F}'(\kappa_2)\right| \\
&= [\Delta_5(\kappa_1,\kappa_2;\lambda)+\Delta_7(\kappa_1,\kappa_2;\lambda)]\left|\mathcal{F}'(\kappa_1)\right|+[\Delta_6(\kappa_1,\kappa_2;\lambda)+\Delta_8(\kappa_1,\kappa_2;\lambda)]\left|\mathcal{F}'(\kappa_2)\right|.
\end{aligned}$$

Now, from the convexity of $|\mathcal{F}'|$ and for $\frac{1}{2}\leq\lambda\leq 1$, we have:

$$\begin{aligned}
S_1 &\leq \int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \int_0^{\frac{1}{2}}\frac{(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \left|\mathcal{F}'(\kappa_1)\right|\int_0^{\frac{1}{2}}\frac{\tau(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\left|\mathcal{F}'(\kappa_2)\right|\int_0^{\frac{1}{2}}\frac{(1-\tau)(\lambda-\tau)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau \\
&= \Delta_9(\kappa_1,\kappa_2;\lambda)\left|\mathcal{F}'(\kappa_1)\right|+\Delta_{10}(\kappa_1,\kappa_2;\lambda)\left|\mathcal{F}'(\kappa_2)\right|
\end{aligned}$$

and

$$\begin{aligned}
S_2 &\leq \int_{\frac{1}{2}}^{1} \frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \int_{\frac{1}{2}}^{1} \frac{(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left[\tau\left|\mathcal{F}'(\kappa_1)\right|+(1-\tau)\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau \\
&= \left|\mathcal{F}'(\kappa_1)\right|\int_{\frac{1}{2}}^{1} \frac{\tau(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau+\left|\mathcal{F}'(\kappa_2)\right|\int_{\frac{1}{2}}^{1} \frac{(1-\tau)(\tau+\lambda-1)}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau \\
&= \Delta_{11}(\kappa_1,\kappa_2;\lambda)\left|\mathcal{F}'(\kappa_1)\right|+\Delta_{12}(\kappa_1,\kappa_2;\lambda)\left|\mathcal{F}'(\kappa_2)\right|.
\end{aligned}$$

Thus, the proof is completed. □

Remark 3. *In Theorem 3, if we set $\lambda = \frac{1}{2}$, then we recaptured the inequality (5) for $q = 1$.*

Remark 4. *In Theorem 3, if we set $\lambda = 0$, then Theorem 3 reduced to [9] (Theorem 3.1 for $q = 1$).*

Corollary 1. *In Theorem 3, if we set $\lambda = \frac{1}{6}$, then we had the following inequality of Simpson's type:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)+\mathcal{F}(\kappa_2)\right]-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\left\{\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_3\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_7\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]\left|\mathcal{F}'(\kappa_1)\right|\right. \\
&\left.+\left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_4\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{6}\right)+\Delta_8\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]\left|\mathcal{F}'(\kappa_2)\right|\right\}.
\end{aligned}$$

Theorem 4. *Consider a mapping $\mathcal{F} : I \subset (0,\infty) \to \mathbb{R}$, which is differentiable mapping on I° and $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, and $\mathcal{F}'$ is integrable over $[\kappa_1, \kappa_2]$. If $|\mathcal{F}'|^q$, $q \geq 1$ is harmonically convex on $[\kappa_1, \kappa_2]$, then the following inequality satisfies:*

$$\begin{aligned}
&\left|\lambda(\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2))-(2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
=\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\begin{cases}
\begin{array}{l}
\Big[(\Delta_1(\kappa_1,\kappa_2;\lambda)+\Delta_2(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
\times\left([\Delta_1(\kappa_1,\kappa_2;\lambda)+\Delta_3(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_1)|^q\right. \\
\left.+[\Delta_2(\kappa_1,\kappa_2;\lambda)+\Delta_4(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
+(\Delta_5(\kappa_1,\kappa_2;\lambda)+\Delta_6(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
\left([\Delta_5(\kappa_1,\kappa_2;\lambda)+\Delta_7(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_1)|^q\right. \\
\left.+[\Delta_6(\kappa_1,\kappa_2;\lambda)+\Delta_8(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\Big]
\end{array}, & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\
\\
\begin{array}{l}
\Big[(\Delta_9(\kappa_1,\kappa_2;\lambda)+\Delta_{10}(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
\times\left(\Delta_9(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_1)|^q+\Delta_{10}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
+(\Delta_{11}(\kappa_1,\kappa_2;\lambda)+\Delta_{12}(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
\left(\Delta_{11}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_1)|^q+\Delta_{12}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\Big]
\end{array}, & \text{if } \frac{1}{2} \leq \lambda \leq 1
\end{cases}
\end{aligned}$$

where $\Delta_1(\kappa_1, \kappa_2; \lambda) - \Delta_{12}(\kappa_1, \kappa_2; \lambda)$ are defined as in Theorem 3.

Proof. Taking modulus in (7) and applying the power mean inequality, we had:

$$\begin{aligned}
&\left|\lambda[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)]-(2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|d\tau\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|d\tau\right]. \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right)^{1-\frac{1}{q}}\right. \\
&\left(\int_0^{\frac{1}{2}}\frac{|\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}d\tau\right)^{1-\frac{1}{q}} \\
&\left.\left(\int_{\frac{1}{2}}^1\frac{|1-\lambda-\tau|}{(\tau\kappa_2+(1-\tau)\kappa_1)^2}\left|\mathcal{F}'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)\right|^q d\tau\right)^{\frac{1}{q}}\right].
\end{aligned}$$

From the convexity of $|\mathcal{F}'|^q$ and $0\leq\lambda<\frac{1}{2}$, we have:

$$\begin{aligned}
&\left|\lambda[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)]-(2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[(\Delta_1(\kappa_1,\kappa_2;\lambda)+\Delta_2(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}}\right. \\
&\times\left([\Delta_1(\kappa_1,\kappa_2;\lambda)+\Delta_3(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_1)|^q+[\Delta_2(\kappa_1,\kappa_2;\lambda)+\Delta_4(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
&+(\Delta_5(\kappa_1,\kappa_2;\lambda)+\Delta_6(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
&\left.\times\left([\Delta_5(\kappa_1,\kappa_2;\lambda)+\Delta_7(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_1)|^q+[\Delta_6(\kappa_1,\kappa_2;\lambda)+\Delta_8(\kappa_1,\kappa_2;\lambda)]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Now, from the convexity of $|\mathcal{F}'|^q$ and $\frac{1}{2}<\lambda\leq 1$, we have:

$$\begin{aligned}
&\left|\lambda[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)]-(2\lambda-1)\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[(\Delta_9(\kappa_1,\kappa_2;\lambda)+\Delta_{10}(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}}\right. \\
&\times\left(\Delta_9(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_1)|^q+\Delta_{10}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
&+(\Delta_{11}(\kappa_1,\kappa_2;\lambda)+\Delta_{12}(\kappa_1,\kappa_2;\lambda))^{1-\frac{1}{q}} \\
&\left.\times\left(\Delta_{11}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_1)|^q+\Delta_{12}(\kappa_1,\kappa_2;\lambda)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and the proof was completed. □

Remark 5. *In Theorem 4, if we set* $\lambda=\frac{1}{2}$*, then we recaptured the inequality (5).*

Remark 6. *In Theorem 4, if we set* $\lambda=0$*, then Theorem 4 reduced to [9] (Theorem 3.1).*

Corollary 2. *In Theorem 4, if we set $\lambda = \frac{1}{6}$, then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) + \mathcal{F}(\kappa_2)\right] - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\left(\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_2\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_3\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]|\mathcal{F}'(\kappa_1)|^q\right. \\
&\left.+\left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_4\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
&+\left(\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_6\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right)^{1-\frac{1}{q}} \\
&\times\left(\left[\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_7\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]|\mathcal{F}'(\kappa_1)|^q\right. \\
&\left.\left.\times+\left[\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{6}\right) + \Delta_8\left(\kappa_1,\kappa_2;\frac{1}{6}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

4. Some Special Cases of Main Results

In this section, we gave some new inequalities as special cases of the newly established results.

Corollary 3. *Under the assumptions of Theorem 3 with $\lambda = 1$, the following inequality held:*

$$\begin{aligned}
&\left|\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\{[\Delta_9(\kappa_1,\kappa_2;1) + \Delta_{11}(\kappa_1,\kappa_2;1)]|\mathcal{F}'(\kappa_1)| + [\Delta_{10}(\kappa_1,\kappa_2;1) + \Delta_{12}(\kappa_1,\kappa_2;1)]|\mathcal{F}'(\kappa_2)|\}.
\end{aligned}$$

Corollary 4. *Under the assumptions of Theorem 3 with $\lambda = \frac{1}{3}$, the following inequality held:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[\mathcal{F}(\kappa_1) + \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) + \mathcal{F}(\kappa_2)\right] - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\left\{\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_3\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_5\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_7\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_1)|\right. \\
&\left.+\left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_4\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_6\left(\kappa_1,\kappa_2;\frac{1}{3}\right) + \Delta_8\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_2)|\right\}.
\end{aligned}$$

Corollary 5. *Under the assumptions of Theorem 3 with $\lambda = \frac{1}{4}$, the following inequality held:*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_1)}{2} + \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right] - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\left\{\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_3\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_5\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_7\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_1)|\right. \\
&\left.+\left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_4\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_6\left(\kappa_1,\kappa_2;\frac{1}{4}\right) + \Delta_8\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_2)|\right\}.
\end{aligned}$$

Corollary 6. *Under the assumptions of Theorem 4 with $\lambda = 1$, the following inequality held:*

$$\begin{aligned}
&\left|2\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1) \\
&\times\left\{(\Delta_9(\kappa_1,\kappa_2;1)+\Delta_{10}(\kappa_1,\kappa_2;1))^{1-\frac{1}{q}}\left(\Delta_9(\kappa_1,\kappa_2;1)|\mathcal{F}'(\kappa_1)|^q + \Delta_{10}(\kappa_1,\kappa_2;1)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+(\Delta_{11}(\kappa_1,\kappa_2;1)+\Delta_{12}(\kappa_1,\kappa_2;1))^{1-\frac{1}{q}}\left(\Delta_{11}(\kappa_1,\kappa_2;1)|\mathcal{F}'(\kappa_1)|^q + \Delta_{12}(\kappa_1,\kappa_2;1)|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 7. *Under the assumptions of Theorem 4 with $\lambda = \frac{1}{3}$, the following inequality held:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[\mathcal{F}(\kappa_1)+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)+\mathcal{F}(\kappa_2)\right] - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\left(\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_3\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_1)|^q + \left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_4\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
&+\left(\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\left[\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_7\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_1)|^q + \left[\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{3}\right)+\Delta_8\left(\kappa_1,\kappa_2;\frac{1}{3}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 8. *Under the assumptions of Theorem 4 with $\lambda = \frac{1}{4}$, the following inequality held:*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_1)}{2}+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right] - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}dx\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2-\kappa_1)\left[\left(\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\left[\Delta_1\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_3\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_1)|^q + \left[\Delta_2\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_4\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}} \\
&+\left(\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\left[\Delta_5\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_7\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_1)|^q + \left[\Delta_6\left(\kappa_1,\kappa_2;\frac{1}{4}\right)+\Delta_8\left(\kappa_1,\kappa_2;\frac{1}{4}\right)\right]|\mathcal{F}'(\kappa_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

5. Application to Special Means

For arbitrary positive numbers κ_1, κ_2 $(\kappa_1 \neq \kappa_2)$, we considered the means as follows:

1. The arithmetic mean;

$$\mathcal{A} = \mathcal{A}(\kappa_1,\kappa_2) = \frac{\kappa_1+\kappa_2}{2}.$$

2. The geometric mean;

$$\mathcal{G} = \mathcal{G}(\kappa_1,\kappa_2) = \sqrt{\kappa_1\kappa_2}.$$

3. The harmonic mean;
$$\mathcal{H} = \mathcal{H}(\kappa_1, \kappa_2) = \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}.$$
4. The logarithmic mean;
$$\mathcal{L} = \mathcal{L}(\kappa_1, \kappa_2) = \frac{\kappa_2 - \kappa_1}{\ln \kappa_2 - \ln \kappa_1}.$$
5. The generalize logarithmic mean;
$$\mathcal{L}_p = \mathcal{L}_p(\kappa_1, \kappa_2) = \left[\frac{\kappa_2^{p+1} - \kappa_1^{p+1}}{(\kappa_2 - \kappa_1)(p+1)}\right]^{\frac{1}{p}}, \; p \in \mathbb{R}\backslash\{-1, 0\}.$$
6. The identric mean.
$$\mathcal{I} = \mathcal{I}(\kappa_1, \kappa_2) = \begin{cases} \frac{1}{e}\left(\frac{\kappa_2^{\kappa_2}}{\kappa_1^{\kappa_1}}\right)^{\frac{1}{\kappa_2 - \kappa_1}}, & \text{if } \kappa_1 \neq \kappa_2, \\ \kappa_1, & \text{if } \kappa_1 = \kappa_2, \end{cases} \quad \kappa_1, \kappa_2 > 0.$$

These means are often employed in numerical approximations and other fields. However, the following straightforward relationship has been stated in the literature.
$$\mathcal{H} \leq \mathcal{G} \leq \mathcal{L} \leq \mathcal{I} \leq \mathcal{A}.$$

Proposition 1. *For $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, the following inequality was true:*
$$\begin{aligned} &\left|2\lambda\mathcal{A}(\kappa_1, \kappa_2) - (2\lambda - 1)\mathcal{H}(\kappa_1, \kappa_2) - \frac{\mathcal{G}^2(\kappa_1, \kappa_2)}{\mathcal{L}(\kappa_1, \kappa_2)}\right| \\ \leq\;& \kappa_1\kappa_2(\kappa_2 - \kappa_1) \\ &\times \begin{cases} \begin{array}{l} \{[\Delta_1(\kappa_1, \kappa_2; \lambda) + \Delta_3(\kappa_1, \kappa_2; \lambda) + \Delta_5(\kappa_1, \kappa_2; \lambda) + \Delta_7(\kappa_1, \kappa_2; \lambda)] \\ +[\Delta_2(\kappa_1, \kappa_2; \lambda) + \Delta_4(\kappa_1, \kappa_2; \lambda) + \Delta_6(\kappa_1, \kappa_2; \lambda) + \Delta_8(\kappa_1, \kappa_2; \lambda)]\kappa_2^{p+1}\} \end{array}, & \text{if } 0 \leq \lambda < \frac{1}{2} \\ \{[\Delta_9(\kappa_1, \kappa_2; \lambda) + \Delta_{11}(\kappa_1, \kappa_2; \lambda)] + [\Delta_{10}(\kappa_1, \kappa_2; \lambda) + \Delta_{12}(\kappa_1, \kappa_2; \lambda)]\}, & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases} \end{aligned}$$

Proof. The inequality in Theorem 3 for mapping $\mathcal{F} : (0, \infty) \to \mathbb{R}$, $\mathcal{F}(x) = x$ leads to this conclusion. □

Proposition 2. *For $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, the following inequality was true:*
$$\begin{aligned} &\left|2\lambda\mathcal{A}\left(\kappa_1^{p+2}, \kappa_2^{p+2}\right) - (2\lambda - 1)\mathcal{H}^{p+2}(\kappa_1, \kappa_2) - \mathcal{G}^2(\kappa_1, \kappa_2)\mathcal{L}_p^p(\kappa_1, \kappa_2)\right| \\ \leq\;& \kappa_1\kappa_2(\kappa_2 - \kappa_1)(p+2) \\ &\times \begin{cases} \begin{array}{l} \Big\{[\Delta_1(\kappa_1, \kappa_2; \lambda) + \Delta_3(\kappa_1, \kappa_2; \lambda) + \Delta_5(\kappa_1, \kappa_2; \lambda) + \Delta_7(\kappa_1, \kappa_2; \lambda)]\kappa_1^{p+1} \\ +[\Delta_2(\kappa_1, \kappa_2; \lambda) + \Delta_4(\kappa_1, \kappa_2; \lambda) + \Delta_6(\kappa_1, \kappa_2; \lambda) + \Delta_8(\kappa_1, \kappa_2; \lambda)]\kappa_2^{p+1}\Big\} \end{array}, & \text{if } 0 \leq \lambda < \frac{1}{2} \\ \begin{array}{l} \Big\{[\Delta_9(\kappa_1, \kappa_2; \lambda) + \Delta_{11}(\kappa_1, \kappa_2; \lambda)]\kappa_1^{p+1} \\ +[\Delta_{10}(\kappa_1, \kappa_2; \lambda) + \Delta_{12}(\kappa_1, \kappa_2; \lambda)]\kappa_2^{p+1}\Big\} \end{array}, & \text{if } \frac{1}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

Proof. The inequality in Theorem 3 for mapping $\mathcal{F} : (0, \infty) \to \mathbb{R}$, $\mathcal{F}(x) = x^{p+2}$, $p \in (-1, \infty)\{0\}$ leads to this conclusion. □

Proposition 3. *For $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, the following inequality was true:*

$$\begin{aligned}
&\left|2\lambda\mathcal{A}\left(\kappa_1^2 \ln \kappa_1, \kappa_2^2 \ln \kappa_2\right) - (2\lambda - 1)\mathcal{H}^2(\kappa_1, \kappa_2)\ln(\mathcal{H}(\kappa_1, \kappa_2)) - \mathcal{G}^2(\kappa_1, \kappa_2)\ln(\mathcal{I}(\kappa_1, \kappa_2))\right| \\
\leq\ & \kappa_1\kappa_2(\kappa_2 - \kappa_1) \\
&\times \begin{cases}
\begin{array}{l}\{[\Delta_1(\kappa_1, \kappa_2; \lambda) + \Delta_3(\kappa_1, \kappa_2; \lambda) \\ +\Delta_5(\kappa_1, \kappa_2; \lambda) + \Delta_7(\kappa_1, \kappa_2; \lambda)]\kappa_1(\kappa_1 + 2\ln \kappa_1) \\ +[\Delta_2(\kappa_1, \kappa_2; \lambda) + \Delta_4(\kappa_1, \kappa_2; \lambda) \\ +\Delta_6(\kappa_1, \kappa_2; \lambda) + \Delta_8(\kappa_1, \kappa_2; \lambda)]\kappa_2(\kappa_2 + 2\ln \kappa_2)\}\end{array}, & \text{if } 0 \leq \lambda < \frac{1}{2} \\[2ex]
\begin{array}{l}\{[\Delta_9(\kappa_1, \kappa_2; \lambda) + \Delta_{11}(\kappa_1, \kappa_2; \lambda)]\kappa_1(\kappa_1 + 2\ln \kappa_1) \\ +[\Delta_{10}(\kappa_1, \kappa_2; \lambda) + \Delta_{12}(\kappa_1, \kappa_2; \lambda)]\kappa_2(\kappa_2 + 2\ln \kappa_2)\}\end{array}, & \text{if } \frac{1}{2} \leq \lambda \leq 1.
\end{cases}
\end{aligned}$$

Proof. The inequality in Theorem 3 for mapping $\mathcal{F} : (0, \infty) \to \mathbb{R}$, $\mathcal{F}(x) = x^2 \ln x$ leads to this conclusion. □

6. Conclusions

In this research, we proved some new inequalities of a midpoint type, trapezoidal type, and Simpson type for differentiable harmonically convex functions. We also showed that the results proved in this research were the refinements of some existing results in [7,9]. The findings of this study can be utilized in symmetry. The results for the case of symmetric harmonically convex functions can be obtained in future studies. It is an interesting and new problem that upcoming researchers can develop similar inequalities for in their future work, with regards to differentiable coordinated harmonically convex functions.

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