symmetry

MDPI

Article

New "Conticrete" Hermite–Hadamard–Jensen–Mercer Fractional Inequalities

Shah Faisal [1], Muhammad Adil Khan [1,*], Tahir Ullah Khan [1], Tareq Saeed [2], Ahmed Mohammed Alshehri [2] and Eze R. Nwaeze [3]

[1] Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan; sfaisaluop@gmail.com (S.F.); tahirullahkhanstd@uop.edu.pk (T.U.K.)
[2] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; tsalmalki@kau.edu.sa (T.S.); amalshehre@kau.edu.sa (A.M.A.)
[3] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA; enwaeze@alasu.edu
* Correspondence: madilkhan@uop.edu.pk

Abstract: The theory of symmetry has a significant influence in many research areas of mathematics. The class of symmetric functions has wide connections with other classes of functions. Among these, one is the class of convex functions, which has deep relations with the concept of symmetry. In recent years, the Schur convexity, convex geometry, probability theory on convex sets, and Schur geometric and harmonic convexities of various symmetric functions have been extensively studied topics of research in inequalities. The present attempt provides novel portmanteauHermite–Hadamard–Jensen–Mercer-type inequalities for convex functions that unify continuous and discrete versions into single forms. They come as a result of using Riemann–Liouville fractional operators with the joint implementations of the notions of majorization theory and convex functions. The obtained inequalities are in compact forms, containing both weighted and unweighted results, where by fixing the parameters, new and old versions of the discrete and continuous inequalities are obtained. Moreover, some new identities are discovered, upon employing which, the bounds for the absolute difference of the two left-most and right-most sides of the main results are established.

Keywords: Jensen's inequality; Mercer's inequality; Hermite–Hadamard inequality; Hölder inequality; majorization

MSC: 26D15; 26A51; 26A33; 26A42

check for updates

Citation: Faisal, S.; Adil Khan, M.; Khan, T.U.; Saeed, T.; Alshehri, A.M.; Nwaeze, E.R. New "Conticrete" Hermite–Hadamard–Jensen–Mercer Fractional Inequalities. *Symmetry* **2022**, *14*, 294. https://doi.org/10.3390/sym14020294

Academic Editor: Paolo Emilio Ricci

Received: 9 January 2022
Accepted: 26 January 2022
Published: 1 February 2022

1. Introduction

Mathematical inequalities have successfully extended their influence to various fields of science and engineering, and they are now accepted and taught as some of the most applicable disciplines of mathematics. Their fruitful applications can be found in, but not limited to, areas such as information theory, economics, engineering, and biology [1,2]. On the basis of such applicability, inequalities and their associated theory have been developed rapidly, where various new and generalized forms of them have come to the surface. For instance, the Hermite–Hadamard inequality [3], Jensen's inequality [4], the Jensen–Mercer inequality [5], the Ostrowski inequality [6], and the Fejér inequality [7] are some names that are immensely popular with researchers. In the present age, researchers are particularly taking interest in generalized inequalities containing various of the above-mentioned versions in one form. In this regard, the Fejér inequality, the Jensen–Mercer inequality, and the Hermite–Jensen–Mercer inequality are commonly known. We selected the most generalized and latest one, that is the Hermite–Jensen–Mercer inequality. This double inequality has recently attracted researchers' attention because it unifies the remarkable

Hermite–Hadamard, Jensen, and Mercer inequalities. The fractional Hermite–Jensen–Mercer inequality is stated as follows: Let f be the real-valued convex function defined on the interval $[\delta_1, \delta_2]$ of real numbers and $[x_1, y_1] \subset [\delta_1, \delta_2]$, with $\alpha > 0$. The Hermite–Jensen–Mercer inequality is given as [8]:

$$\begin{aligned} f\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(y_1-x_1)^\alpha}\Big\{J^\alpha_{(\delta_1+\delta_2-y_1)^+}f(\delta_1+\delta_2-x_1) \\ &\quad +J^\alpha_{(\delta_1+\delta_2-x_1)^-}f(\delta_1+\delta_2-y_1)\Big\} \\ &\leq \frac{f(\delta_1+\delta_2-x_1)+f(\delta_1+\delta_2-y_1)}{2} \\ &\leq f(\delta_1)+f(\delta_2)-\frac{f(x_1)+f(y_1)}{2}. \end{aligned} \tag{1}$$

where $J^\alpha_{x_1^+}$ and $J^\alpha_{y_1^-}$ respectively represent the left- and right-sided Riemann–Liouville integrals of fractional order α defined as follows:

$$J^\alpha_{x_1^+}f(z) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{z}(z-u)^{\alpha-1}f(u)du,\ z > x_1,$$

$$J^\alpha_{y_1^-}f(z) = \frac{1}{\Gamma(\alpha)}\int_{z}^{y_1}(u-z)^{\alpha-1}f(u)du,\ z < y_1.$$

In the present study, one of the reasons for the selection of Riemann–Liouville operators is that these operators have some advantages as compared to other fractional operators. For example, the Riemann–Liouville fractional operators do not need the function to be continuous or differentiable at the origin. In addition to this, these operators can be used for the best descriptions and modeling of phenomena having power-law behaviors because they contain a power function as a kernel in their integral transforms. However, the related research can also be conducted for other fractional operators, such as that of Caputo's, Hadamard's, Katugampola, or generalized k-fractional operators.

The above inequality in (1) is the generalization of the Hermite–Hadamard, Jensen, and Mercer inequalities. The following fractional Hermite–Hadamard inequality can be obtained from (1) when $x_1 = \delta_1$ and $y_1 = \delta_2$ [9]:

$$f\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^\alpha}\left[J^\alpha_{\delta_1+}f(\delta_2)+J^\alpha_{\delta_2-}f(\delta_1)\right] \leq \frac{f(\delta_1)+f(\delta_2)}{2}. \tag{2}$$

Some more results related to fractional Hermite–Hadamard–Mercer-type inequalities were given in [10–14].

Now, we state the definition of majorization, in terms of which we want to present our results:

Definition 1 ([15])**.** *Let* $\mathbf{a} = (a_1, \ldots, a_l)$ *and* $\mathbf{b} = (b_1, \ldots, b_l)$ *be two l-tuples of real numbers with their order arrangements* $a_{[l]} \leq a_{[l-1]} \leq \cdots \leq a_{[1]}$, $b_{[l]} \leq b_{[l-1]} \leq \cdots \leq b_{[1]}$, *then* $\mathbf{a}$ *is said to majorize* $\mathbf{b}$ *(or* $\mathbf{b}$ *is to be majorized by* $\mathbf{a}$*, symbolically* $\mathbf{b} \prec \mathbf{a}$*), if:*

$$\sum_{s=1}^{k} b_{[s]} \leq \sum_{s=1}^{k} a_{[s]} \quad \text{for } k = 1, 2, \ldots, l-1,$$

and:

$$\sum_{s=1}^{l} a_s = \sum_{s=1}^{l} b_s.$$

Niezgoda [16] used the concept of majorization and extended the Jensen–Mercer inequality given as follows:

Theorem 1. *Let (x_{is}) be an $n \times l$ real matrix and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l)$ be an l-tuple such that $\delta_s, x_{is} \in I$ for all $i = 1, 2, \ldots, n,\ s \in \{1, \cdots, l\}$ and f be a convex function defined on I. Furthermore, let $\sigma_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum\limits_{i=1}^{n} \sigma_i = 1$. If δ majorizes every row of (x_{is}), then:*

$$f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \sigma_i x_{is}\right) \leq \sum_{s=1}^{l} f(\delta_s) - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \sigma_i f(x_{is}).$$

The following lemmas will help us to prove our next results [17].

Lemma 1. *Let (x_{is}) be an $n \times l$ real matrix and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l)$, $\mathbf{p} = (p_1, \ldots, p_l)$ be two l-tuples such that $\delta_s, x_{is} \in I$, $p_s \geq 0$ with $p_l \neq 0$, $\eta = \frac{1}{p_l}$ for all $i = 1, 2, \ldots, n,\ s \in \{1, \cdots, l\}$ and f be a convex function defined on I. Furthermore, let $\sigma_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum\limits_{i=1}^{n} \sigma_i = 1$. If for each $i = 1, 2, \ldots, n$, $(x_{i1}, \ldots, x_{il})$ is a decreasing l-tuple and satisfying:*

$$\sum_{s=1}^{k} p_s x_{is} \leq \sum_{s=1}^{k} p_s \delta_s \ \text{for } k = 1, 2, \ldots, l-1,\ \sum_{s=1}^{l} p_s \delta_s = \sum_{s=1}^{l} p_s x_{is},$$

then:

$$f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \eta \sigma_i p_s x_{is}\right) \leq \sum_{s=1}^{l} \eta p_s f(\delta_s) - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \eta \sigma_i p_s f(x_{is}).$$

Lemma 2. *Let (x_{is}) be an $n \times l$ real matrix and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l)$, $\mathbf{p} = (p_1, \ldots, p_l)$ be two $l-tuples$ such that $\delta_s, x_{is} \in I$, $p_s \geq 0$ with $p_l \neq 0$, $\eta = \frac{1}{p_l}$ for all $i = 1, 2, \ldots, n,\ s \in \{1, \cdots, l\}$ and f be a convex function defined on I. Furthermore, let $\sigma_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum\limits_{i=1}^{n} \sigma_i = 1$. If for each $i = 1, \ldots, n$, $(\delta_s - x_{is})$ and x_{is} are monotonic in the same sense and:*

$$\sum_{s=1}^{l} p_s \delta_s = \sum_{s=1}^{l} p_s x_{is},$$

then:

$$f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \eta \sigma_i p_s x_{is}\right) \leq \sum_{s=1}^{l} \eta p_s f(\delta_s) - \sum_{s=1}^{l-1} \sum_{i=1}^{n} \eta \sigma_i p_s f(x_{is}).$$

The theory of majorization has been receiving considerable attention from researchers working in different fields. It helps in the conversion of complicated optimization problems into simple problems that can then easily be solved [18,19]. Some present-day applications of majorization theory in signal processing and communication can be traced to [20,21]. For more successive works carried out via the concept of majorization, one can see [22,23] and the references therein.

As mentioned above, there is a growing trend among researchers to combine different research fields into one. In this regard, it is better to develop such ideas that bring researchers of related fields together. In the field of inequalities, up to now, there are two main concepts (which are continuous and discrete) where mathematicians are conducting research independently. In both cases, researchers have been developing generalized or unified inequalities using (sometimes) generalized integral operators and sometimes a generalized type of convexity, or sometimes, they use both [24]. As a result, they provide a unique platform to researchers working with different integrals or convex functions. In this stage, there is a necessary notion whose applications can lead us to the inequalities that are a mixture or combination of both discrete and continuous versions. The theory of majorization is one of these that fulfills these criteria. The present attempt may be considered as one of the fruitful endeavors in this direction.

Moreover, the name "conticrete" is assigned on the basis of one of the English language rules for "blending or coining of words", according to which "brunch" is used for the meal taken in between "breakfast" and "lunch". Similarly the word "smog" has been created by blending the two words "smoke" and "fog". Here in our case, the word conticrete means mixture of continuous and discrete inequalities.

The main results of the present paper are organized as follows: In Theorem 2, the generalized fractional portmanteauform of the Hermite–Hadamard–Jensen–Mercer-type inequalities is obtained using Riemann–Liouville fractional integrals. Remarks 1 and 2 show that these inequalities cover those previously presented versions of fractional Hermite–Hadamard–Jensen–Mercer-type inequalities, and they also unify continuous and discrete inequalities of the Hermite–Hadamard-, Jensen-, and Mercer-types. In Theorem 3, another form of the Hermite–Hadamard–Jensen–Mercer-type inequality for fractional integrals is developed. Theorems 4 and 5 present weighted versions of the obtained results. Lemmas 3 and 4 contain new identities associated with the right side of Theorem 2 and with the left side of Theorem 3, respectively. Theorems 6 and 7 are proven on the basis of Lemma 3, and they present various bounds for the absolute difference of the two right-most terms in Theorem 2. Theorems 8–10 are proven on the basis of Lemma 4, and they present various bounds for the absolute difference of the two left-most terms in Theorem 3. Corollaries 1–5 provide information about the classical integral forms of the main obtained results. At the end, the conclusion of the whole research work is presented.

2. Main Results

In the underlying theorem, we deduce the Hermite–Hadamard inequality of the Jensen–Mercer-type for fractional integrals.

Theorem 2. *Let $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, and $\mathbf{y} = (y_1, \ldots, y_l)$ be three l-tuples such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \cdots, l\}$, $x_l > y_l$, $\alpha > 0$ and f be a convex function defined on I. If δ majorizes both $\mathbf{x}$ and $\mathbf{y}$, then:*

$$\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) &\leq \frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)^{+}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right. \\
&\quad \left. + J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)^{-}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} \\
&\leq \frac{f\left(\sum\limits_{s=1}^{l}\delta_s - \sum\limits_{s=1}^{l-1}y_s\right) + f\left(\sum\limits_{s=1}^{l}\delta_s - \sum\limits_{s=1}^{l-1}x_s\right)}{2} \\
&\leq \sum_{s=1}^{l} f(\delta_s) - \frac{\sum\limits_{s=1}^{l-1}f(x_s) + \sum\limits_{s=1}^{l-1}f(y_s)}{2}. \qquad (3)
\end{aligned}$$

Proof. We may write:

$$
\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) &= f\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s+\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right\}\right\}\\
&= f\left\{\frac{1}{2}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right.\right.\\
&\quad\left.\left.+t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\right\}.
\end{aligned}
\tag{4}
$$

Using the convexity of f in (4), we have:

$$
\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) \le \frac{1}{2}&\left\{f\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right.\\
&\left.+f\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\right\}.
\end{aligned}
\tag{5}
$$

By multiplication of $t^{\alpha-1}$ with (5) on both sides and taking integration with respect to t, we obtain:

$$
\begin{aligned}
&\frac{1}{\alpha}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\\
&\le \frac{1}{2}\left\{\int_0^1 t^{\alpha-1}f\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}dt\right.\\
&\quad\left.+\int_0^1 t^{\alpha-1}f\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}dt\right\}\\
&= \frac{1}{2\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s}\left(u-\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right)^{\alpha-1}f(u)du\right.\\
&\quad\left.+\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s}\left(\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)-u\right)^{\alpha-1}f(u)du\right\}.
\end{aligned}
\tag{6}
$$

To apply the definition of the fractional integral in (6), first we show that:

$$
\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s<\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s.
$$

As:

$$
\begin{aligned}
&x_l>y_l\\
&\Rightarrow x_l-y_l>0.
\end{aligned}
\tag{7}
$$

Furthermore, $\mathbf{x}\prec\delta$ and $\mathbf{y}\prec\delta$. Then, we may write:

$$\sum_{s=1}^{l-1} y_s + y_l = \sum_{s=1}^{l-1} x_s + x_l$$
$$\Rightarrow \sum_{s=1}^{l-1} y_s - \sum_{s=1}^{l-1} x_s = x_l - y_l. \tag{8}$$

Using (7) in (8), we obtain:

$$\sum_{s=1}^{l-1} y_s - \sum_{s=1}^{l-1} x_s > 0$$
$$\Rightarrow -\sum_{s=1}^{l-1} y_s < -\sum_{s=1}^{l-1} x_s. \tag{9}$$

Adding $\sum\limits_{s=1}^{l} \delta_s$ to both sides of (9), we deduce:

$$\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s < \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s.$$

Now, (6) implies:

$$\frac{1}{\alpha} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) \leq \frac{\Gamma(\alpha)}{2\left(\sum\limits_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}}$$
$$\times \left\{ J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} x_s\right)^-} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) + J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} y_s\right)^+} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) \right\},$$

and so:

$$f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}}$$
$$\times \left\{ J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} x_s\right)^-} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) + J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} y_s\right)^+} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) \right\}. \tag{10}$$

Thus, the first inequality of (3) is complete. To achieve the second inequality, we utilize the convexity of f as follows:

$$f\left(t\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + (1-t)\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right)\right) \leq$$
$$t f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + (1-t) f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right). \tag{11}$$

$$f\left(t\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) + (1-t)\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right)\right) \leq$$
$$t f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) + (1-t) f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right). \tag{12}$$

Adding (11) and (12), then applying Theorem 1 for $n = 1$ and $\sigma_1 = 1$, we obtain:

$$
\begin{aligned}
& f\left(t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right) \\
& +f\left(t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right) \\
& \qquad \leq f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right) \\
& \qquad \leq 2\sum_{s=1}^{l}f(\delta_s)-\left\{\sum_{s=1}^{l-1}f(x_s)+\sum_{s=1}^{l-1}f(y_s)\right\}.
\end{aligned} \tag{13}
$$

By multiplication of $t^{\alpha-1}$ with (13) on both sides and taking integration with respect to t, we acquire the second and third inequality in (3). □

Remark 1. *Taking the same hypothesis, Theorem 2 gives the underlying inequality for the case of* $\alpha = 1$.

$$
\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) & \leq \frac{1}{\sum_{s=1}^{l-1}(y_s-x_s)}\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s} f(u)du \\
& \leq \frac{f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2} \\
& \leq \sum_{s=1}^{l}f(\delta_s)-\frac{1}{2}\left\{\sum_{s=1}^{l-1}f(x_s)+\sum_{s=1}^{l-1}f(y_s)\right\}.
\end{aligned}
$$

Remark 2. *Theorem 2 gives the following inequality for $l = 2$, which was proven by Öağülmüş and Sarikaya in [8].*

$$
\begin{aligned}
f\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right) & \leq \frac{\Gamma(\alpha+1)}{2(y_1-x_1)^\alpha}\left\{J^\alpha_{(\delta_1+\delta_2-y_1)^+}f(\delta_1+\delta_2-x_1)\right. \\
& \left. +J^\alpha_{(\delta_1+\delta_2-x_1)^-}f(\delta_1+\delta_2-y_1)\right\} \\
& \leq \frac{f(\delta_1+\delta_2-y_1)+f(\delta_1+\delta_2-x_1)}{2} \\
& \leq f(\delta_1)+f(\delta_2)-\frac{f(x_1)+f(y_1)}{2}.
\end{aligned}
$$

Moreover, for $l = 2$ and $\alpha = 1$, we obtain the result of Kian and Muslehain [25].

Adopting the same procedure, we give another Hermite–Hadamard inequality of the Jensen–Mercer-type for fractional integrals as follows.

Theorem 3. *Let all conditions in the hypothesis of Theorem 2 hold true, then:*

$$
\begin{aligned}
&f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\\
&\qquad\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right.\\
&\qquad\qquad\left.+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\\
&\qquad\leq\sum_{s=1}^{l}f(\delta_s)-\frac{\sum\limits_{s=1}^{l-1}f(x_s)+\sum\limits_{s=1}^{l-1}f(y_s)}{2}.
\end{aligned}
\tag{14}
$$

Proof. For $t\in[0,1]$, it may be written:

$$
\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&=f\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s+\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right\}\right\}\\
&=f\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right.\right.\\
&\qquad\left.\left.+\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right\}\right\}.
\end{aligned}
\tag{15}
$$

Using the convexity of f in (15), we have:

$$
\begin{aligned}
f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&\leq\frac{1}{2}\left\{f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right.\\
&\qquad\left.+f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right\}.
\end{aligned}
\tag{16}
$$

By multiplication of $t^{\alpha-1}$ with (16) on both sides and taking integration with respect to t, we obtain:

$$
\begin{aligned}
&\frac{1}{\alpha}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\\
&\quad\leq\frac{1}{2}\left\{\int_0^1t^{\alpha-1}f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right.\\
&\qquad\qquad\left.+\int_0^1t^{\alpha-1}f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}\\
&\quad=\frac{1}{2\left(\sum\limits_{s=1}^{l-1}\left(\frac{y_s-x_s}{2}\right)\right)^{\alpha}}\left\{\int_{\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s}^{\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)}\left(u-\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right)^{\alpha-1}f(u)du\right.\\
&\qquad\qquad\left.+\int_{\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)}^{\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s}\left(\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)-u\right)^{\alpha-1}\right\}.
\end{aligned}
\tag{17}
$$

In a similar manner as adopted in Theorem 2, we can show that:

$$\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)<\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s \ \ and \ \ \sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)>\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s$$

Now, (17) implies:

$$\begin{aligned}\frac{1}{\alpha}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&=\frac{2^{\alpha-1}\Gamma(\alpha)}{\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\\&\times\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}.\end{aligned}$$

Therefore, we have:

$$\begin{aligned}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\\&\times\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}.\end{aligned}$$

This proves the first inequality in (14).

With the purpose of proving the second inequality of (14), we use Theorem 1 for $n=2$, $\sigma_1=\frac{t}{2}$ and $\sigma_2=\frac{2-t}{2}$ as follows:

$$\begin{aligned}&f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\leq\\&\qquad\sum_{s=1}^{l}f(\delta_s)-\left(\frac{t}{2}\sum_{s=1}^{l-1}f(x_s)+\frac{2-t}{2}\sum_{s=1}^{l-1}f(y_s)\right),\end{aligned}\tag{18}$$

and:

$$\begin{aligned}&f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\leq\\&\qquad\sum_{s=1}^{l}f(\delta_s)-\left(\frac{t}{2}\sum_{s=1}^{l-1}f(y_s)+\frac{2-t}{2}\sum_{s=1}^{l-1}f(x_s)\right).\end{aligned}\tag{19}$$

Adding (18) and (19), we obtain:

$$\begin{aligned}&f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\\&+f\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\leq2\sum_{s=1}^{l}f(\delta_s)-\left(\sum_{s=1}^{l-1}f(x_s)+\sum_{s=1}^{l-1}f(y_s)\right).\end{aligned}\tag{20}$$

By multiplication of $t^{\alpha-1}$ with (20) on both sides and taking integration with respect to t, we acquire the second inequality of (14). $\square$

Remark 3. *For the case of $l = 2$, the inequality (14) reduces to the following inequality proven by Öağülmüş and Sarikaya in [8].*

$$f\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_1-x_1)^\alpha}\left\{J^\alpha_{\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right)^-}f(\delta_1+\delta_2-y_1)\right.$$
$$\left.+J^\alpha_{\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right)^+}f(\delta_1+\delta_2-x_1)\right\}$$
$$\leq f(\delta_1)+f(\delta_2)-\frac{f(x_1)+f(y_1)}{2}.$$

Remark 4. *For the case of $l = 2$ and $\alpha = 1$, the inequality (14) reduces to the inequality (2.2) given in [25].*

The underlying theorem includes a result based on Lemma 1.

Theorem 4. *Let $\delta = (\delta_1,\ldots,\delta_l)$, $\mathbf{x} = (x_1,\ldots,x_l)$, $\mathbf{y} = (y_1,\ldots,y_l)$, and $\mathbf{p} = (p_1,\ldots,p_l)$ be four l-tuples such that $\delta_s, x_s, y_s \in I$, $p_s \geq 0$ with $p_l \neq 0$ for all $s \in \{1,\ldots,l\}$, $\eta = \frac{1}{p_l}$, $x_l > y_l$, $\alpha > 0$ and f be a convex function defined on I. If $\mathbf{x}$ and $\mathbf{y}$ are decreasing l-tuples and:*

$$\sum_{s=1}^{k} p_s x_s \leq \sum_{s=1}^{k} p_s\delta_s,\ \sum_{s=1}^{k} p_s y_s \leq \sum_{s=1}^{k} p_s\delta_s \text{ for } k=1,\ldots,l-1,$$

$$\sum_{s=1}^{l} p_s\delta_s = \sum_{s=1}^{l} p_s x_s,\ \sum_{s=1}^{l} p_s\delta_s = \sum_{s=1}^{l} p_s y_s,$$

then:

$$f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}\left(\frac{p_sx_s+p_sy_s}{2}\right)\right)$$
$$\leq \frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(\eta p_s y_s-\eta p_s x_s)\right)^\alpha}\left\{J^\alpha_{\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s y_s\right)^+}f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right.$$
$$\left.+J^\alpha_{\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s x_s\right)^-}f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}$$
$$\leq \frac{f\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s y_s\right)+f\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s x_s\right)}{2}$$
$$\leq \sum_{s=1}^{l}\eta p_s f(\delta_s)-\frac{\sum\limits_{s=1}^{l-1}\eta p_s f(x_s)+\sum\limits_{s=1}^{l-1}\eta p_s f(y_s)}{2}. \tag{21}$$

Proof. We may write:

$$f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)$$
$$= f\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s+\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right\}\right\}$$

$$= f\left\{\frac{1}{2}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right.\right.$$
$$\left.\left.+t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}\right\}. \quad (22)$$

Using the convexity of f in (22), we have:

$$f\left(\sum_{s=1}^{l}\eta p_s\delta_s - \eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)$$
$$\leq \frac{1}{2}\left\{f\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}\right.$$
$$\left.+f\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}\right\}. \quad (23)$$

By multiplication of $t^{\alpha-1}$ with (23) on both sides and taking integration with respect to t, we obtain:

$$\frac{1}{\alpha}f\left(\sum_{s=1}^{l}\eta p_s\delta_s - \eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)$$
$$\leq \frac{1}{2}\left\{\int_0^1 t^{\alpha-1}f\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}dt\right.$$
$$\left.+\int_0^1 t^{\alpha-1}f\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right) + (1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}dt\right\}$$

$$= \frac{1}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s - \eta p_s x_s)\right)^{\alpha}}\left\{\int_{\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s}^{\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s}\left(u - \left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right)^{\alpha-1}f(u)du\right.$$
$$\left.+\int_{\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s}^{\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s}\left(\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right) - u\right)^{\alpha-1}f(u)du\right\}. \quad (24)$$

In order to apply the definition of the fractional integral in (24), first we show that:

$$\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s < \sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s.$$

As:

$$\begin{aligned} x_l &> y_l \\ \Rightarrow p_l x_l &> p_l y_l \\ \Rightarrow p_l x_l - p_l y_l &> 0. \end{aligned} \quad (25)$$

Furthermore,

$$\sum_{s=1}^{l}p_s\delta_s = \sum_{s=1}^{l}p_s x_s \ \ and \ \ \sum_{s=1}^{l}p_s\delta_s = \sum_{s=1}^{l}p_s y_s.$$

Therefore, we have:

$$\begin{aligned} \sum_{s=1}^{l} p_s y_s &= \sum_{s=1}^{l} p_s x_s \\ \Rightarrow \sum_{s=1}^{l-1} p_s y_s + p_l y_l &= \sum_{s=1}^{l-1} p_s x_s + p_l x_l \\ \Rightarrow \sum_{s=1}^{l-1} p_s y_s - \sum_{s=1}^{l-1} p_s x_s &= p_l x_l - p_l y_l \end{aligned} \tag{26}$$

Using (25) in (26), we obtain:

$$\begin{aligned} &\sum_{s=1}^{l-1} p_s y_s - \sum_{s=1}^{l-1} p_s x_s > 0. \\ &\Rightarrow -\sum_{s=1}^{l-1} \eta p_s y_s < -\sum_{s=1}^{l-1} \eta p_s x_s \end{aligned} \tag{27}$$

Adding $\sum_{s=1}^{l} \eta p_s \delta_s$ to both sides of (27), we deduce:

$$\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s y_s < \sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s x_s.$$

Now, (24) implies:

$$\begin{aligned} &\frac{1}{\alpha} f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \eta \sum_{s=1}^{l-1} p_s \left(\frac{x_s + y_s}{2}\right)\right) \\ &\leq \frac{\Gamma(\alpha)}{2\left(\sum_{s=1}^{l-1} (\eta p_s y_s - \eta p_s x_s)\right)^{\alpha}} \left\{ J^{\alpha}_{\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s x_s\right)^-} f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s y_s\right) \right. \\ &\qquad \left. + J^{\alpha}_{\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s y_s\right)^+} f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s x_s\right) \right\}, \end{aligned}$$

and so:

$$\begin{aligned} &f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \eta \sum_{s=1}^{l-1} p_s \left(\frac{x_s + y_s}{2}\right)\right) \\ &\leq \frac{\Gamma(\alpha+1)}{2\left(\sum_{s=1}^{l-1} (\eta p_s y_s - \eta p_s x_s)\right)^{\alpha}} \left\{ J^{\alpha}_{\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s x_s\right)^-} f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s y_s\right) \right. \\ &\qquad \left. + J^{\alpha}_{\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s y_s\right)^+} f\left(\sum_{s=1}^{l} \eta p_s \delta_s - \sum_{s=1}^{l-1} \eta p_s x_s\right) \right\}. \end{aligned} \tag{28}$$

Thus, we achieve the first part of (21). To achieve the second part, from the convexity of f, we may write:

$$f\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right)\leq tf\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right), \quad (29)$$

and:

$$f\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right)\leq tf\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right). \quad (30)$$

Adding (29) and (30) and then using Lemma 1 for $n=2$, $\sigma_1=t$, and $\sigma_2=1-t$, we obtain:

$$\begin{aligned}
&f\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right)\\
&+f\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right)\\
&\leq f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\\
&\leq 2\sum_{s=1}^{l}\eta p_s f(\delta_s)-\left\{\sum_{s=1}^{l-1}\eta p_s f(x_s)+\sum_{s=1}^{l-1}\eta p_s f(y_s)\right\}. \quad (31)
\end{aligned}$$

By multiplication of $t^{\alpha-1}$ with (31) on both sides and taking integration with respect to t, we obtain the second and third part of (21). □

The underlying theorem includes a result based on Lemma 2.

Theorem 5. *Let $\delta=(\delta_1,\ldots,\delta_l)$, $\mathbf{x}=(x_1,\ldots,x_l)$, $\mathbf{y}=(y_1,\ldots,y_l)$, and $\mathbf{p}=(p_1,\ldots,p_l)$ be four l-tuples such that $\delta_s,x_s,y_s\in I$, $p_s\geq 0$ with $p_l\neq 0$ for all $s\in\{1,\ldots,l\}$, $\eta=\frac{1}{p_l}$, $x_l>y_l$, $\alpha>0$ and f be a convex function defined on I. If $\delta-\mathbf{x}$, $\mathbf{x}$, $\delta-\mathbf{y}$, and $\mathbf{y}$ are monotonic in the same sense and:*

$$\sum_{s=1}^{l}p_s\delta_s=\sum_{s=1}^{l}p_s x_s,\quad \sum_{s=1}^{l}p_s\delta_s=\sum_{s=1}^{l}p_s y_s,$$

then:

$$\begin{aligned}
&f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}\left(\frac{p_s x_s+p_s y_s}{2}\right)\right)\\
&\leq\frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(\eta p_s y_s-\eta p_s x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s y_s\right)^{+}}f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right.\\
&\qquad\left.+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\eta p_s\delta_s-\sum\limits_{s=1}^{l-1}\eta p_s x_s\right)^{-}}f\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}
\end{aligned}$$

$$
\le \frac{f\left(\sum\limits_{s=1}^{l} \eta p_s \delta_s - \sum\limits_{s=1}^{l-1} \eta p_s y_s\right) + f\left(\sum\limits_{s=1}^{l} \eta p_s \delta_s - \sum\limits_{s=1}^{l-1} \eta p_s x_s\right)}{2}
$$
$$
\le \sum_{s=1}^{l} \eta p_s f(\delta_s) - \frac{\sum\limits_{s=1}^{l-1} \eta p_s f(x_s) + \sum\limits_{s=1}^{l-1} \eta p_s f(y_s)}{2}. \quad (32)
$$

Proof. In similar manner as adopted in Theorem 4, we can easily obtain (32) using Lemma 2. □

Remark 5. *Theorems 4 and 5 represent weighted versions of Theorem 2.*

3. Bounds Associated with the Main Results

In this section, first, we discover two new identities associated with the right and left sides of the main results. Then, utilizing these identities, we establish bounds for the absolute difference of the two right-most and left-most terms of the main results.

Lemma 3. *Let $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, and $\mathbf{y} = (y_1, \ldots, y_l)$ be three l-tuples such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \cdots, l\}$, $\alpha > 0$, $t \in [0,1]$ and f be a differentiable function defined on I. If $f' \in L(I)$, then:*

$$
\frac{f\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} y_s\right) + f\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} x_s\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}}
$$
$$
\times \left\{ J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} y_s\right)^+} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + J^{\alpha}_{\left(\sum\limits_{s=1}^{l} \delta_s - \sum\limits_{s=1}^{l-1} x_s\right)^-} f\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\}
$$
$$
= \frac{\sum\limits_{s=1}^{l-1}(y_s - x_s)}{2} \int_0^1 \left(t^{\alpha} - (1-t)^{\alpha}\right) f'\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(t x_s + (1-t) y_s\right)\right) dt. \quad (33)
$$

Proof. To prove our required result, we consider that:

$$
\begin{aligned}
I &= \int_0^1 \left(t^{\alpha} - (1-t)^{\alpha}\right) f'\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(t x_s + (1-t) y_s\right)\right) dt \\
&= \int_0^1 t^{\alpha} f'\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(t x_s + (1-t) y_s\right)\right) dt \\
&\quad - \int_0^1 (1-t)^{\alpha} f'\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1}\left(t x_s + (1-t) y_s\right)\right) dt \\
&= I_1 - I_2.
\end{aligned}
$$

Assume that $\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s<\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s$, and using integration by parts formula, we obtain:

$$I_1=\int_0^1 t^\alpha f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)dt$$

$$=\frac{t^\alpha f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}\Bigg|_0^1-\frac{\alpha}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}$$
$$\times\int_0^1 t^{\alpha-1}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)dt$$

$$=\frac{f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}-\frac{\Gamma(\alpha+1)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)^{\alpha+1}}J^\alpha_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right).$$

Likewise,

$$I_2=\int_0^1 (1-t)^\alpha f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)dt$$

$$=\frac{(1-t)^\alpha f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}\Bigg|_0^1+\frac{\alpha}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}$$
$$\times\int_0^1 (1-t)^{\alpha-1}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\Big(tx_s+(1-t)y_s\Big)\right)dt$$

$$=-\frac{f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}+\frac{\Gamma(\alpha+1)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)^{\alpha+1}}J^\alpha_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right).$$

Now, we have:

$$I=\frac{f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)+f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)}-\frac{\Gamma(\alpha+1)}{\sum\limits_{s=1}^{l-1}(y_s-x_s)^{\alpha+1}}$$
$$\times\left\{J^\alpha_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+J^\alpha_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}.$$

Multiplying both sides by $\frac{\sum\limits_{s=1}^{l-1}(y_s-x_s)}{2}$, we obtain (33). □

Corollary 1. *If $\alpha = 1$ and $l = 2$, in Lemma 3, then we have the following equality:*

$$\begin{aligned}&\frac{f(\delta_1+\delta_2-x_1)+f(\delta_1+\delta_2-y_1)}{2}-\frac{1}{y_1-x_1}\int_{\delta_1+\delta_2-y_1}^{\delta_1+\delta_2-x_1}f(u)du\\&=\frac{y_1-x_1}{2}\int_0^1(2t-1)f'(\delta_1+\delta_2-(tx_1+(1-t)y_1))dt.\end{aligned}\tag{34}$$

Remark 6. *If we take $x_1 = \delta_1$ and $y_1 = \delta_2$ in Corollary 1, then the equality (34) gives:*

$$\frac{f(\delta_1)+f(\delta_2)}{2}-\frac{1}{y_1-x_1}\int_{\delta_1}^{\delta_2}f(u)du=\frac{y_1-x_1}{2}\int_0^1(2t-1)f'(t\delta_2+(1-t)\delta_1)dt.\tag{35}$$

The equality (35) was proven by Dragomir and Agarwal [3].

On the basis of Lemma 3, we give the following results.

Theorem 6. *Let $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, and $\mathbf{y} = (y_1, \ldots, y_l)$ be three l-tuples such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \cdots, l\}$, $x_l > y_l$, $\alpha > 0$ and f be a differentiable function defined on I. If δ majorizes $\mathbf{x}$, $\mathbf{y}$ and $|f'|$ is convex on I, then:*

$$\begin{aligned}&\Bigg|\frac{f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\\&\quad\times\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\Bigg|\\&\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left\{\sum_{s=1}^{l}|f'(\delta_s)|-\frac{\sum_{s=1}^{l-1}|f'(x_s)|+\sum_{s=1}^{l-1}|f'(y_s)|}{2}\right\}.\end{aligned}\tag{36}$$

Proof. Using Lemma 3, we may write:

$$\begin{aligned}&\Bigg|\frac{f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\\&\quad\times\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\Bigg|\\&=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{2}\int_0^1\left(t^{\alpha}-(1-t)^{\alpha}\right)f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)dt\right|\\&\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\int_0^1\left|\left(t^{\alpha}-(1-t)^{\alpha}\right)\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)\right|dt.\end{aligned}\tag{37}$$

Using Theorem 1 for $n = 2$, $\sigma_1 = t$, and $\sigma_2 = 1 - t$ in (37), as a consequence of the convexity of $|f'|$, we obtain:

$$\leq \frac{\sum\limits_{s=1}^{l-1}|y_s - x_s|}{2}\int_0^1 \left|\left(t^\alpha - (1-t)^\alpha\right)\right| \times\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \left(t\sum_{s=1}^{l-1}|f'(x_s)| + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|\right)\right\}dt.$$

$$= \frac{\sum\limits_{s=1}^{l-1}|y_s - x_s|}{2} \times\left[\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \left(t\sum_{s=1}^{l-1}|f'(x_s)| + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|\right)\right)\right\}dt + \int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha\right)\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \left(t\sum_{s=1}^{l-1}|f'(x_s)| + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|\right)\right)\right\}dt\right].$$

$$= \frac{\sum\limits_{s=1}^{l-1}|y_s - x_s|}{2}\left(C_1 + C_2\right). \tag{38}$$

Now, finding C_1 and C_2, we have:

$$C_1 = \int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \left(t\sum_{s=1}^{l-1}|f'(x_s)| + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|\right)\right)\right\}dt$$

$$= \left(\sum_{s=1}^{l}|f'(\delta_s)|\right)\left(\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)dt\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\int_0^{\frac{1}{2}} t\left((1-t)^\alpha - t^\alpha\right)dt + \sum_{s=1}^{l-1}|f'(y_s)|\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)(1-t)dt\right\}$$

$$= \sum_{s=1}^{l}|f'(\delta_s)|\left(\frac{1}{\alpha+1} - \frac{\frac{1}{2^\alpha}}{\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\left(\int_0^{\frac{1}{2}} t(1-t)^\alpha dt - \int_0^{\frac{1}{2}} t^{\alpha+1}dt\right) + \sum_{s=1}^{l-1}|f'(y_s)|\left(\int_0^{\frac{1}{2}}(1-t)^{\alpha+1}dt - \int_0^{\frac{1}{2}}(1-t)t^\alpha dt\right)\right\}$$

$$= \sum_{s=1}^{l}|f'(\delta_s)|\left(\frac{1}{\alpha+1} - \frac{\frac{1}{2^\alpha}}{\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right) + \sum_{s=1}^{l-1}|f'(y_s)|\left(\frac{1}{\alpha+2} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right)\right\},$$

and:

$$C_2 = \int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha\right)\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \left(t\sum_{s=1}^{l-1}|f'(x_s)| + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|\right)\right)\right\}dt$$

$$= \left(\sum_{s=1}^{l}|f'(\delta_s)|\right)\left(\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha})dt\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\int_{\frac{1}{2}}^{1}t(t^{\alpha}-(1-t)^{\alpha})dt + \sum_{s=1}^{l-1}|f'(y_s)|\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha})(1-t)dt\right\}$$

$$= \sum_{s=1}^{l}|f'(\delta_s)|\left(\frac{1}{\alpha+1}-\frac{\frac{1}{2^{\alpha}}}{\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\left(\int_{\frac{1}{2}}^{1}t^{\alpha+1}dt-\int_{\frac{1}{2}}^{1}t(1-t)^{\alpha}dt\right) + \sum_{s=1}^{l-1}|f'(y_s)|\left(\int_{\frac{1}{2}}^{1}(1-t)t^{\alpha}dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+1}dt\right)\right\}$$

$$= \sum_{s=1}^{l}|f'(\delta_s)|\left(\frac{1}{\alpha+1}-\frac{\frac{1}{2^{\alpha}}}{\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}|f'(x_s)|\left(\frac{1}{\alpha+2}-\frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right) + \sum_{s=1}^{l-1}|f'(y_s)|\left(\frac{1}{(\alpha+1)(\alpha+2)}-\frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right)\right\}.$$

Adding C_1 and C_2, we obtain:

$$C_1 + C_2 = \sum_{s=1}^{l}|f'(\delta_s)|\left(\frac{1-\frac{1}{2^{\alpha}}}{\alpha+1}\right) - \left(\frac{1-\frac{1}{2^{\alpha}}}{\alpha+1}\right)\left(\sum_{s=1}^{l-1}|f'(x_s)| + \sum_{s=1}^{l-1}|f'(y_s)|\right). \tag{39}$$

Inserting (39) in (38), we achieve (36). □

Remark 7. *If we take $l = 2$, then the inequality (36) reduces to the inequality (3.4) given in [8].*

Remark 8. *If we take $l = 2$, $x_1 = \delta_1$, and $y_1 = \delta_2$, then the inequality (36) reduces to the inequality (3.5) given in [9].*

Theorem 7. *Let $\delta = (\delta_1, \dots, \delta_l)$, $\mathbf{x} = (x_1, \dots, x_l)$, and $\mathbf{y} = (y_1, \dots, y_l)$ be three l-tuples and a differentiable function f defined on I where $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \cdots, l\}$, $x_l > y_l$, and $\alpha > 0$. If $q > 1$, δ majorizes $\mathbf{x}$, $\mathbf{y}$ and $|f'|^q$ is convex on I, then:*

$$\left|\frac{f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right) + f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}} \times \left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)^{+}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right) + J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)^{-}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\}\right|$$
$$\leq \frac{\left(1-\frac{1}{2^{\alpha}}\right)\sum_{s=1}^{l-1}|y_s - x_s|}{\alpha+1}\left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{\sum_{s=1}^{l-1}|f'(x_s)|^q + \sum_{s=1}^{l-1}|f'(y_s)|^q}{2}\right\}. \tag{40}$$

Proof. Using Lemma 3, we have:

$$\left|\frac{f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)+f\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\right.$$

$$\left.\times\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}y_s\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}x_s\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right|$$

$$=\left|\frac{\sum\limits_{s=1}^{l-1}(y_s-x_s)}{2}\int_0^1\left(t^{\alpha}-(1-t)^{\alpha}\right)f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)dt\right|$$

$$\leq\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)\right|dt.$$

By applying the power mean inequality to the above integral, we obtain:

$$\leq\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{2}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\right.$$

$$\left.\times\left|f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{2}\left(\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})dt+\int_{\frac{1}{2}}^1(t^{\alpha}-(1-t)^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}. \quad (41)$$

Since $|f'|^q$ is convex, therefore, using Theorem 1 for $n=2$, $\sigma_1=t$, and $\sigma_2=1-t$ in (41), we obtain:

$$=\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{2}\left(\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})dt+\int_{\frac{1}{2}}^1(t^{\alpha}-(1-t)^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left(\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(t\sum_{s=1}^{l-1}|f'(x_s)|^q+(1-t)\sum_{s=1}^{l-1}|f'(y_s)|^q\right)dt\right)^{\frac{1}{q}}$$

$$
= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{2}\left(\int_0^{\frac{1}{2}}((1-t)^\alpha - t^\alpha)dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)dt\right)^{1-\frac{1}{q}}
$$

$$
\times\left\{\int_0^{\frac{1}{2}}((1-t)^\alpha - t^\alpha)\left(\sum_{s=1}^{l}|f'(\delta_s)|^q - \left(t\sum_{s=1}^{l-1}|f'(x_s)|^q + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right)dt\right.
$$

$$
\left.+\int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)\left(\sum_{s=1}^{l}|f'(\delta_s)|^q - \left(t\sum_{s=1}^{l-1}|f'(x_s)|^q + (1-t)\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right)dt\right\}^{\frac{1}{q}}
$$

By calculating these simple integrals, we obtain (40). □

For some further results, we establish another lemma as follows.

Lemma 4. *Let all conditions in the hypothesis of Lemma 3 hold true, then:*

$$
\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^\alpha}\left\{J^\alpha_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.
$$

$$
\left.+J^\alpha_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)
$$

$$
= \frac{\sum_{s=1}^{l-1}(y_s - x_s)}{4}\int_0^1 t^\alpha\left\{f'\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right.
$$

$$
\left.-f'\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right\}dt. \tag{42}
$$

Proof. Adopting the same procedure as given in the proof of Lemma 3, it can be easily proven. □

Remark 9. *For the selection of $l = 2$, the equality (42) reduces to the equality (3.3), which was proven in [8].*

Now, we give some results on the basis of Lemma 4, as:

Theorem 8. *Let all conditions in the hypothesis of Theorem 6 hold true, then:*

$$
\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^\alpha}\left\{J^\alpha_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.
$$

$$
\left.\left.+J^\alpha_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|
$$

$$
\leq \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{2(\alpha+1)}\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \frac{\sum_{s=1}^{l-1}|f'(x_s)| + \sum_{s=1}^{l-1}|f'(y_s)|}{2}\right\}. \tag{43}
$$

Proof. Using Lemma 4, we have:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum\limits_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left.+J^{\alpha}_{\left(\sum\limits_{s=1}^{l}\delta_s-\sum\limits_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$

$$=\left|\frac{\sum\limits_{s=1}^{l-1}(y_s-x_s)}{4}\left\{\int_0^1 t^{\alpha}f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right.\right.$$
$$\left.\left.-\int_0^1 t^{\alpha}f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}\right|.$$

$$\le\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1\left|t^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|t^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}$$

$$=\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}. \tag{44}$$

By utilizing Theorem 1 for $n=2$, $\sigma_1=\frac{2-t}{2}$, and $\sigma_2=\frac{t}{2}$ in (44), we obtain:

$$\le\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1 t^{\alpha}\left(\sum_{s=1}^{l}|f'(\delta_s)|-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(x_s)|+\frac{t}{2}\sum_{s=1}^{l-1}|f'(y_s)|\right)\right)dt\right.$$
$$\left.+\int_0^1 t^{\alpha}\left(\sum_{s=1}^{l}|f'(\delta_s)|-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(y_s)|+\frac{t}{2}\sum_{s=1}^{l-1}|f'(x_s)|\right)\right)\right|dt\right\}$$

$$=\frac{\sum\limits_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\frac{\sum\limits_{s=1}^{l}|f'(\delta_s)|}{\alpha+1}-\frac{\sum\limits_{s=1}^{l-1}|f'(x_s)|}{\alpha+1}+\frac{\sum\limits_{s=1}^{l-1}|f'(x_s)|}{2(\alpha+2)}-\frac{\sum\limits_{s=1}^{l-1}|f'(y_s)|}{2(\alpha+2)}\right.$$
$$\left.+\frac{\sum\limits_{s=1}^{l}|f'(\delta_s)|}{\alpha+1}-\frac{\sum\limits_{s=1}^{l-1}|f'(y_s)|}{\alpha+1}+\frac{\sum\limits_{s=1}^{l-1}|f'(y_s)|}{2(\alpha+2)}-\frac{\sum\limits_{s=1}^{l-1}|f'(x_s)|}{2(\alpha+2)}\right\}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{2(\alpha+1)}\left\{\sum_{s=1}^{l}|f'(\delta_s)| - \frac{\sum_{s=1}^{l-1}|f'(x_s)| + \sum_{s=1}^{l-1}|f'(y_s)|}{2}\right\}.$$

Hence, the proof is accomplished. □

Corollary 2. *Considering $\alpha = 1$ and $l = 2$, then Theorem 8 gives the following inequality:*

$$\left|\frac{1}{y_1 - x_1}\int_{\delta_1+\delta_2-y_1}^{\delta_1+\delta_2-x_1} - f\left(\delta_1+\delta_2 - \frac{x_1+y_1}{2}\right)\right| \leq$$
$$\frac{|y_1 - x_1|}{4}\left\{|f'(\delta_1)| + |f'(\delta_2)| - \frac{|f'(x_1)| + |f'(y_1)|}{2}\right\}.$$

Theorem 9. *Let $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, and $\mathbf{y} = (y_1, \ldots, y_l)$ be three l-tuples such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \cdots, l\}$, $x_l > y_l$, $\alpha > 0$ and f be a differentiable function defined on I. If $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, δ majorizes $\mathbf{x}$, $\mathbf{y}$ and $|f'|^q$ is convex on I, then:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^{+}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left. + J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^{-}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$

$$\leq \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4(\alpha p+1)^{\frac{1}{p}}}\left\{\left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(x_s)|^q + \sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(y_s)|^q + \sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right\}^{\frac{1}{q}}\right\}. \quad (45)$$

Proof. We write from Lemma 4 that:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^{+}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left. + J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^{-}} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$

$$= \left|\frac{\sum_{s=1}^{l-1}(y_s - x_s)}{4}\left\{\int_0^1 t^{\alpha} f'\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right.\right.$$
$$\left.\left. - \int_0^1 t^{\alpha} f'\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}\right|$$

$$\leq \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1\left|t^\alpha f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|t^\alpha f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}$$

$$\leq \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1\left|t^\alpha\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|t^\alpha\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}.$$

By applying Hölder's inequality to the above integral, we have:

$$\leq \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\left(\int_0^1\left|t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (46)$$

Since $|f'|^q$ is convex, therefore, using Theorem 1 for $n=2$, $\sigma_1=\frac{2-t}{2}$, and $\sigma_2=\frac{t}{2}$ in (46), we obtain:

$$= \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left(\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(x_s)|^q+\frac{t}{2}\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(y_s)|^q+\frac{t}{2}\sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right)dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4(\alpha p+1)^{\frac{1}{p}}}\left\{\left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(x_s)|^q + \sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(y_s)|^q + \sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right\}^{\frac{1}{q}}\right\}.$$

Hence, the proof is acquired. □

Corollary 3. *For the selection of $l = 2$, Theorem 9 gives the inequality:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left((y_1-x_1)\right)^{\alpha}}\left\{J^{\alpha}_{\left(a_1+a_2-\left(\frac{x_1+y_1}{2}\right)\right)^+} f\left(a_1+a_2-x_1\right)\right.\right.$$
$$\left.\left. + J^{\alpha}_{a_1+a_2-\left(\frac{x_1+y_1}{2}\right))^-} f\left(a_1+a_2-y_1\right)\right\} - f\left(a_1+a_2-\frac{x_1+y_1}{2}\right)\right|$$
$$\leq \frac{|y_1-x_1|}{4(\alpha p+1)^{\frac{1}{p}}}\left\{\left[|f'(a_1)|^q + |f'(a_2)|^q - \frac{3|f'(x_1)|^q + |f'(y_1)|^q}{4}\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[|f'(a_1)|^q + |f'(a_1)|^q - \frac{3|f'(y_1)|^q + |f'(x_1)|^q}{4}\right]^{\frac{1}{q}}\right\}.$$

Corollary 4. *For the selection of $\alpha = 1$, Theorem 9 gives the following inequality:*

$$\left|\frac{1}{\sum_{s=1}^{l-1}(y_s-x_s)}\int_{\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s} f(u)du - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$
$$\leq \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4(p+1)^{\frac{1}{p}}}\left\{\left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(x_s)|^q + \sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\sum_{s=1}^{l}|f'(\delta_s)|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}|f'(y_s)|^q + \sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right\}^{\frac{1}{q}}\right\}.$$

Remark 10. *For $l = 2$, $x_1 = \delta_1$, and $y_1 = \delta_2$, (45) reduces to (3.6) in [26].*

Theorem 10. *Let all conditions in the hypothesis of Theorem 7 hold true, then:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left. + J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-} f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - f\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$

$$\leq \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4(\alpha+1)^{1-\frac{1}{q}}}\left\{\left\{\frac{1}{\alpha+1}\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\sum_{s=1}^{l-1}|f'(x_s)|^q\right.\right.\right.$$
$$\left.\left.+\frac{1}{2(\alpha+2)}\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right\}^{\frac{1}{q}}+\left\{\frac{1}{\alpha+1}\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\sum_{s=1}^{l-1}|f'(y_s)|^q\right.\right.$$
$$\left.\left.\left.+\frac{1}{2(\alpha+2)}\sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right\}^{\frac{1}{q}}\right\}. \tag{47}$$

Proof. Using Lemma 4, we have:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^+}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left.+J^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)^-}f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-f\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$

$$=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{4}\left\{\int_0^1 t^{\alpha}f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right.\right.$$
$$\left.\left.-\int_0^1 t^{\alpha}f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}\right|$$

$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1\left|t^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|t^{\alpha}\right|\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}.$$

By applying the power mean inequality to the above integral, we obtain:

$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{\alpha}\left|f'\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}. \tag{48}$$

Due to the convexity of $|f'|^q$, using Theorem 1 for $n=2$, $\sigma_1=\frac{2-t}{2}$, and $\sigma_2=\frac{t}{2}$ in (48), we have:

$$= \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4(\alpha+1)^{1-\frac{1}{q}}}\left\{\left(\int_0^1 t^\alpha\left(\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(x_s)|^q+\frac{t}{2}\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^\alpha\left(\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}|f'(y_s)|^q+\frac{t}{2}\sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right)dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4(\alpha+1)^{1-\frac{1}{q}}}\left\{\left\{\frac{1}{\alpha+1}\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\sum_{s=1}^{l-1}|f'(x_s)|^q\right.\right.\right.$$
$$\left.\left.+\frac{1}{2(\alpha+2)}\sum_{s=1}^{l-1}|f'(y_s)|^q\right)\right\}^{\frac{1}{q}}+\left\{\frac{1}{\alpha+1}\sum_{s=1}^{l}|f'(\delta_s)|^q-\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\sum_{s=1}^{l-1}|f'(y_s)|^q\right.\right.$$
$$\left.\left.\left.+\frac{1}{2(\alpha+2)}\sum_{s=1}^{l-1}|f'(x_s)|^q\right)\right\}^{\frac{1}{q}}\right\}.$$

□

Hence, the proof is acquired.

Corollary 5. *Taking $\alpha=1$ and $l=2$, Theorem 10 gives the following inequality:*

$$\left|\frac{1}{y_1-x_1}\int_{\delta_1+\delta_2-y_1}^{\delta_1+\delta_2-x_1}f(u)du-f\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right)\right|$$
$$\leq\frac{|y_1-x_1|}{2^{\frac{3q-1}{q}}}\left\{\left(\frac{|f'(\delta_1)|^q+|f'(\delta_2)|^q}{2}-\left(\frac{2|f'(x_1)|^q+|f'(y_1)|^q}{6}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|f'(\delta_1)|^q+|f'(\delta_2)|^q}{2}-\left(\frac{2|f'(y_1)|^q+|f'(x_1)|^q}{6}\right)\right)^{\frac{1}{q}}\right\}.$$

Example 1. *Let $a,b,c\in\mathbb{R}$ such that $a>b>c$. Suppose that $\alpha>0$ and $\delta=(2a,2b,2c)$, $\mathbf{x}=(a+\frac{b}{2}+\frac{c}{2},b+\frac{a}{2}+\frac{c}{2},\frac{a}{2}+\frac{b}{2}+c)$, and $\mathbf{y}=(a+b,c+a,b+c)$ are three tuples. First, we show that $\frac{a}{2}+\frac{b}{2}+c>b+c$ and $\mathbf{x}\prec\delta$. Clearly, $a+\frac{b}{2}+\frac{c}{2}>b+\frac{a}{2}+\frac{c}{2}>\frac{a}{2}+\frac{b}{2}+c$. As $a>b$ and $b>c$, therefore, we have:*

$$2b<a+b\Rightarrow 2b+2c<a+b+2c\Rightarrow b+c<\frac{a}{2}+\frac{b}{2}+c.$$

Now,

$$b+a<a+a\Rightarrow b+c<2a\Rightarrow 2a+b+c<4a\Rightarrow a+\frac{b}{2}+\frac{c}{2}<2a,$$

and

$$2c<b+a\Rightarrow 2c+a+b<2b+2a\Rightarrow a+\frac{b}{2}+\frac{c}{2}+b+\frac{a}{2}+\frac{c}{2}<2a+2b.$$

$$\textit{Also,} \quad a + \frac{b}{2} + \frac{c}{2} + b + \frac{a}{2} + \frac{c}{2} + \frac{a}{2} + \frac{b}{2} + c = 2a + 2b + 2c.$$

Hence, $\mathbf{x} \prec \delta$. *Similarly, we can show that* $\mathbf{y} \prec \delta$.
Now, applying Theorem 2, for these tuples, we obtain:

$$\begin{aligned}
f\left(\frac{a+3b+4c}{4}\right) &\leq \frac{\Gamma(\alpha+1)}{2\left(\frac{a-b}{2}\right)^{\alpha}}\left\{J^{\alpha}_{(b+c)^{+}} f\left(\frac{a+b+2c}{2}\right) + J^{\alpha}_{\left(\frac{a+b+2c}{2}\right)^{-}} f(b+c)\right\} \\
&\leq \frac{f(b+c) + f\left(\frac{a+b+2c}{2}\right)}{2} \\
&\leq f(2a) + f(2b) + f(2c) \\
&\quad - \frac{f(a+\frac{b}{2}+\frac{c}{2}) + f(b+\frac{a}{2}+\frac{c}{2}) + f(a+b) + f(c+a)}{2}.
\end{aligned}$$

Remark 11. *The results presented in this manuscript may also be given for other fractional operators such as Caputo's, Hadamard's, Katugampola's, and generalized k-fractional operators.*

4. Conclusions and Future Research Work

New portmanteauinequalities containing both continuous and discrete versions were successfully introduced for the first time in the present field of research. This work was carried out using the joint notions of convexity and majorization theory. As a result, new unified fractional portmanteauversions of the Hermite–Hadamard–Jensen–Mercer-type inequality emerged. Firstly, majorized Hermite–Hadamard inequalities of the Jensen–Mercer-type for fractional integrals were established. It was noted that these inequalities cover those previously presented results, as well as unifying continuous and discrete inequalities of the Hermite–Hadamard-, Jensen-, and Mercer-types into a single form. Secondly, another Hermite–Hadamard inequality of the Jensen–Mercer-type for fractional integrals, where integrals appear with another combination of limits, was developed. Weighted versions of these results were also established. It was observed that the results adopt an interesting look when an additional condition of strict monotonicity is applied to the tuples. New identities were discovered based on which various bounds for absolute differences of the left and right sides of the obtained results were obtained. The present work may also be considered as a direct application of majorization theory, which with the combination of convex theory gives rise to the new mixture of inequalities. Moreover, as there is an extensive literature devoted to Hermite–Hadamard and related inequalities, in a future study, we will focus on real-world applications of these inequalities.

Author Contributions: M.A.K. gave the main idea of the main results and supervised. S.F. and T.U.K. wrote Section 2. T.S., A.M.A., and E.R.N. wrote Section 3. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The authors are very much grateful to the Reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare that there are no conflict of interest regarding the publication of this paper.

References

1. Cloud, M.J.; Drachman, B.C.; Lebedev, L.P. *Inequalities with Applications to Engineering*; Springer: Cham, Switzerland; Heidelberg, Germany; New York, NY, USA; Dordrecht, The Netherlands; London, UK, 2014.
2. Lin, Q. Jensen inequality for superlinear expectations. *Stat. Probabil. Lett.* **2019**, *151*, 79–83. [CrossRef]

3. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
4. Adil Khan, M.; Khan, S.; Chu, Y.-M. New estimates for the Jensen gap using s-convexity with applications. *Front. Phys.* **2020**, *8*, 313. [CrossRef]
5. Mercer, A.M. A variant of Jensen's inequality. *J. Ineq. Pure Appl. Math.* **2003**, *4*, 73.
6. Kiris, M.E.; Eyüp, M.; Sarikaya, M.Z. On Ostrowski type inequalities and Čebyšev type inequalities with applications. *Filomat* **2015**, *29*, 1695–1713. [CrossRef]
7. Sarikaya, M.Z. On new Hermite–Hadamard Fejér type integral inequalities. *Stud. Univ. Babes-Bolyai Math.* **2012**, *57*, 377–386.
8. Öağülmüş, H.; Sarikaya, M.Z. Hermite–Hadamard–Mercer type inequalities for fractional integrals. *Filomat* **2021**, *35*, 2425–2436. [CrossRef]
9. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
10. You, X.; Ali, M.A.; Budak, H.; Reunsumrit, J.; Sitthiwirattham, T. Hermite–Hadamard–Mercer-Type inequalities for harmonically convex mappings. *Mathematics* **2021**, *9*, 2556. [CrossRef]
11. Butt, S.I.; Umar, M.; Rashid, S.; Akdemir, A.O.; Chu, Y.-M. New Hermite–Jensen–Mercer-Type inequalities via k-fractional integrals. *Adv. Differ. Equ.* **2020**, *2020*, 635. [CrossRef]
12. Rashid, S.; Noor, M.A.; Noor, K.I.; Safdar, F.; Chu, Y.-M. Hermite–Hadamard type inequalities for the class of convex functions on time scale. *Mathematics* **2019**, *7*, 956. [CrossRef]
13. Cortez, M.V.; Ali, M.A.; Budak, H.; Kalsoom, H.; Agarwal, P. Some new Hermite–Hadamard and related inequalities for convex functions via (p,q)-integral. *Entropy* **2021**, *23*, 828. [CrossRef] [PubMed]
14. Valdés, J.E.N.; Rodríguez, J.M.; Sigarreta, J.M. New Hermite–Hadamard type inequalities involving non-conformable integral operators. *Symmetry* **2019**, *11*, 1108. [CrossRef]
15. Wang, B.-Y. *Foundations of Majorization Inequalities*; Beijing Normal University Press: Beijing, China, 1990.
16. Niezgoda, M. A generalization of Mercer's result on convex functions. *Nonlinear Anal.* **2009**, *71*, 2771–2779. [CrossRef]
17. Faisal, S.; Adil Khan, M.; Iqbal, S. Generalized Hermite–Hadamard–Mercer type inequalities via majorization. *Filomat* **2022**, (accepted).
18. Bhatia, R. *Matrix Analysis*; Springer: New York, NY, USA, 1997.
19. Horn, R.A.; Johnson, C.R. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1990.
20. Jorswieck, E.A.; Boche, H. Majorization and matrix-monotone functions in wireless communications. *Found. Trends Commun. Inf. Theory* **2006**, *3*, 553–701. [CrossRef]
21. Palomar, D.P.; Jiang, Y. MIMO transciver design via majorization theory. *Found. Trends Commun. Inf. Theory* **2006**, *3*, 331–551. [CrossRef]
22. Siddique, N.; Imran, M.; Khan, K.A.; Pećarixcx, J. Majorization inequalities via Green functions and Fink's identity with applications to Shannon entropy. *J. Inequal. Appl.* **2020**, *2020*, 192. [CrossRef]
23. Siddique, N.; Imran, M.; Khan, K.A.; Pećarixcx, J. Difference equations related to majorization theorems via Montgomery identity and Green's functions with application to the Shannon entropy. *Adv. Differ. Equ.* **2020**, *2020*, 430. [CrossRef]
24. Khan, T.U.; Khan, M.A. Hermite–Hadamard inequality for new generalized conformable fractional operators. *AIMS Math.* **2020**, *6*, 3–38.
25. Kian, M.; Moslehian, M.S. Refinements of the operator Jensen–Mercer inequality. *Electron. J. Linear Algebra* **2013**, *26*, 742–753. [CrossRef]
26. Sarikaya, M.Z.; Yildirim, H. On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]