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The Strong Equitable Vertex 1-Arboricity of Complete Bipartite Graphs and Balanced Complete k -Partite Graphs

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Abstract: An equitable k -coloring of a graph G is a proper k -coloring of G such that the sizes of any two color classes differ by at most one. An equitable (q, r) -tree-coloring of a graph G is an equitable q -coloring of G such that the subgraph induced by each color class is a forest of maximum degree at most r . Let the strong equitable vertex r -arboricity of a graph G , denoted by $va_r^{\equiv}(G)$, be the minimum p such that G has an equitable (q, r) -tree-coloring for every $q \geq p$. The values of $va_1^{\equiv}(K_{n,n})$ were investigated by Tao and Lin and Wu, Zhang, and Li where exact values of $va_1^{\equiv}(K_{n,n})$ were found in some special cases. In this paper, we extend their results by giving the exact values of $va_1^{\equiv}(K_{n,n})$ for all cases. In the process, we introduce a new function related to an equitable coloring and obtain a more general result by determining the exact value of each $va_1^{\equiv}(K_{m,n})$ and $va_1^{\equiv}(G)$ where G is a balanced complete k -partite graph $K_{n_1, \dots, n_k}$. Both complete bipartite graphs $K_{m,n}$ and balanced complete k -partite graphs $K_{n_1, \dots, n_k}$ are symmetry in several aspects and also studied broadly. For the other aspect of symmetry, by the definition of equitable k -coloring of graphs, in a specific case that the number of colors divides the number of vertices of graph, we can say that the graph is a balanced k -partite graph.

Keywords: equitable coloring; k -tree-coloring; vertex k -arboricity; complete multipartite graph

Citation: Laomala, J.; Nakprasit, K.M.; Nakprasit, K.; Ruksasakchai, W. The Strong Equitable Vertex 1-Arboricity of Complete Bipartite Graphs and Balanced Complete k -Partite Graphs. *Symmetry* **2022**, *14*, 287. <https://doi.org/10.3390/sym14020287>

Academic Editor: Domenico Labbate

Received: 7 January 2022

Accepted: 26 January 2022

Published: 31 January 2022

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1. Introduction

Throughout this paper, all graphs considered are finite and simple. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G respectively. Let $K_{n_1, \dots, n_k}$ be a complete k -partite graph in which partite set X_i has size n_i for $1 \leq i \leq k$. Let $K_{k \times n}$ denote a complete k -partite graph with each partite set has size n , and naturally we always assume that $k \geq 2$. For a real number r , $\lfloor r \rfloor$ is the largest integer number not greater than r , and $\lceil r \rceil$ is the smallest integer number not less than r . A k -set is a set with k elements.

An equitable k -coloring of a graph is a proper vertex k -coloring such that the sizes of every two color classes differ by at most 1. A graph G is *equitably k -colorable* if G admits an equitable k -coloring, equivalently if $V(G)$ can be partitioned into k independent sets where each set is a $\lfloor |V(G)|/k \rfloor$ -set or a $\lceil |V(G)|/k \rceil$ -set. The *equitable chromatic number* of a graph G is the minimum number k such that G is equitably k -colorable. In contrast with ordinary proper coloring, a graph may have an equitable k -coloring but has no equitable $(k+1)$ -coloring. For example, $K_{7,7}$ has an equitable k -coloring for $k = 2, 4, 6$ and $k \geq 8$, but has no equitable k -coloring for $k = 3, 5$, and 7. This leads to the definition of the *equitable chromatic threshold* which is the minimum p such that G is equitably q -colorable for every $q \geq p$.

The topic of equitable coloring, introduced by Meyer [1], was motivated by a problem of municipal garbage collection [2]. To model the problem, the vertices of the graph are used to represent garbage collection routes. If two routes cannot be run in the same day, then two corresponding vertices share the same edge. Thus the problem of assigning routes for k days satisfying the number of routes run on each day are about the same can be represented

by an equitable k -coloring. A similar approach can be applied to scheduling [3–6] and modeling load balance in parallel memory systems [7,8].

In [9], Fan et al introduced an equitable relaxed coloring in which every color class induces a forest with maximum degree at most 1 and the sizes of any two color classes differ by at most 1. On the basis of the aforementioned research, Wu, Zhang, and Li [10] introduced an *equitable (q, r) -tree-coloring* of a graph G which is a q -coloring of vertices of G such that the subgraph induced by each color class is a forest of maximum degree at most r and the sizes of any two color classes differ by at most one. In other words, a graph G has an equitable (q, r) -tree-coloring if $V(G)$ can be partitioned into q sets such that each set is a $\lfloor |V(G)|/q \rfloor$ - or $\lceil |V(G)|/q \rceil$ -set inducing a forest with maximum degree at most r . Let the *strong equitable vertex k -arboricity*, denoted by $va_r^{\equiv}(G)$, be the minimum p such that G has an equitable (q, r) -tree-coloring for every $q \geq p$. The values of $va_1^{\equiv}(K_{n,n})$ were investigated by Tao and Lin [11] and Wu, Zhang, and Li [10] where exact values of $va_1^{\equiv}(K_{n,n})$ were found in some special cases. In this paper, we extend their results by giving the exact values of $va_1^{\equiv}(K_{n,n})$ for all cases. In the process, we introduce a new function related to an equitable coloring and furthermore, obtain a general result by determining the exact value of each $va_1^{\equiv}(K_{m,n})$ and $va_1^{\equiv}(G)$ where G is a balanced complete k -partite graph $K_{n_1, \dots, n_k}$.

2. Helpful Lemmas

To find $va_1^{\equiv}(K_{m,n})$ and $va_1^{\equiv}(K_{k*n})$, we introduce the notion of $p(q : n_1, \dots, n_k)$ which can be computed in linear time.

Definition 1. Assume that $K_{n_1, \dots, n_k}$ has an equitable q -coloring. Define $p(q : n_1, \dots, n_k) = \lceil n_1/d \rceil + \dots + \lceil n_k/d \rceil$ where d is the minimum integer not less than $\lceil (n_1 + \dots + n_k)/q \rceil$ satisfying at least one of the following conditions:

- (i) there exist n_i and n_j ($i \neq j$) not divisible by d ;
- (ii) there exists n_i with $n_i / \lfloor n_i/d \rfloor > d + 1$.

Lemma 1. Assume $G = K_{n_1, \dots, n_k}$ has an equitable q -coloring. If $p(q : n_1, \dots, n_k) \leq r \leq q$, then G has an equitable r -coloring.

Proof. Let $p = p(q : n_1, \dots, n_k)$ and $N = n_1 + \dots + n_k$. We prove by reverse induction that G has an equitable r -coloring when $p \leq r \leq q$. By assumption, G has an equitable q -coloring. Assume G has an equitable r -coloring where $p < r \leq q$. Let $b = \lceil N/r \rceil$. It follows that, for $1 \leq i \leq k$, there are non-negative integers r_i and s_i where $r_i - s_i \geq 0$ such that $n_i = (r_i - s_i)b + s_i(b - 1) = r_i b - s_i$ and $r = r_1 + \dots + r_k$.

Case 1: There exists j with $r_j \neq \lceil n_j/b \rceil$.

Note that $n_j = \lceil n_j/b \rceil b - g_j$ for some g_j satisfying $0 \leq g_j \leq b - 1$. From $r_j b - s_j = \lceil n_j/b \rceil b - g_j$, we obtain $(r_j - \lceil n_j/b \rceil)b = s_j - g_j$. Since $r_j \neq \lceil n_j/b \rceil$, $0 \leq g_j \leq b - 1$, and s_j is non-negative, it follows that $s_j - g_j$ is a positive multiple of b . Rewrite $n_j = (r_j - s_j)b + s_j(b - 1)$ into $n_j = (r_j - s_j + b - 1)b + (s_j - b)(b - 1)$. Since $s_j - g_j$ is a positive multiple of b and g_j is non-negative, it follows that $s_j - b$ is non-negative. Thus we can partition X_j into $r_j - s_j + b - 1$ color classes of size b and $s_j - b$ color classes of size $b - 1$. That is, we can partition X_j into $r_j - 1$ sets of size b or $b - 1$. Since we can partition all other X_i into r_i color classes of size b or $b - 1$ and $(\sum_{i \neq j} r_i) + (r_j - 1) = (\sum_{i=1}^k r_i) - 1 = r - 1$, the graph G has an equitable $(r - 1)$ -coloring.

Case 2: $r_i = \lceil n_i/b \rceil$ for $1 \leq i \leq k$.

In this case, $r = \lceil n_1/b \rceil + \dots + \lceil n_k/b \rceil$. Since $r > p = \lceil n_1/d \rceil + \dots + \lceil n_k/d \rceil$, it follows that $d > b$. Definition 1 implies that b satisfies neither conditions (i) nor (ii) in the definition. Thus we may assume $n_i = r_i b$ for $2 \leq i \leq k$, and $n_j / \lfloor n_j/b \rfloor \leq b + 1$ for $1 \leq j \leq k$.

Subcase 2.1: $n_1 \neq r_1 b$.

Then $b < n_1 / \lfloor n_1/b \rfloor = n_1 / (\lceil n_1/b \rceil - 1) = n_1 / (r_1 - 1)$. Since b violates condition (ii), we have $n_1 / (r_1 - 1) = n_1 / \lfloor n_1/b \rfloor \leq b + 1$. Thus $b < n_1 / (r_1 - 1) \leq b + 1$. Consequently,

we can partition n_1 into $r_1 - 1$ color classes of size b or $b + 1$. Combining with r_i color classes of X_i of size b for $i \geq 2$, we have an equitable $(r - 1)$ -coloring.

Subcase 2.2: $n_1 = r_1 b$.

It follows that $n_i = r_i b$ for $1 \leq i \leq k$. If there is j such that $n_j/(r_j - 1) \leq b + 1$, then we have an equitable $(r - 1)$ -coloring as in subcase 2.1. Thus we may assume that $n_i/(r_i - 1) > b + 1$ for $1 \leq i \leq k$.

We claim that $b + 1 \geq d$ and $r_i = \lceil n_i/b \rceil = \lceil n_i/(b + 1) \rceil$. If the claim holds, we have $r = \sum_{i=1}^k \lceil n_i/b \rceil \leq \sum_{i=1}^k \lceil n_i/d \rceil = p$ which contradicts the fact that $r > p$. Thus this situation is impossible. To prove the claim, first we suppose to the contrary that there is n_i which is divisible by $b + 1$. Since $n_i = r_i b$, we have $r_i = t_i(b + 1)$ for some positive integer t_i . Consequently, $n_i/(r_i - 1) = t_i(b + 1)b/(t_i(b + 1) - 1) = b + b/(t_i(b + 1) - 1) \leq b + 1$ which contradicts the fact that $n_i/(r_i - 1) > b + 1$. Thus n_i is not divisible by $b + 1$ where $1 \leq i \leq k$. By condition (i) in Definition 1, we have $b + 1 \geq d$. Since $n_i = r_i b$ and $n_i/(r_i - 1) > b + 1$ where $1 \leq i \leq k$, it follows that $r_i = n_i/b > n_i/(b + 1) > r_i - 1$. This leads to $r_i = \lceil n_i/b \rceil = \lceil n_i/(b + 1) \rceil$. Thus, we have the claim and this completes the proof. $\square$

Example 1. Let $G = K_{1,1,2,2,3}$. Then G has an equitable 8-coloring. One can compute that $p = p(q : n_1, \dots, n_k) = 6$. We can notice that G has an equitable r -coloring for every $6 \leq r \leq 8$ as in Lemma 1.

Lemma 2. If $G = K_{n_1, \dots, n_k}$ has an equitable q -coloring, then G has no equitable $(p - 1)$ -coloring where $p = p(q : n_1, \dots, n_k)$.

Proof. Recall that $p = \lceil n_1/d \rceil + \dots + \lceil n_k/d \rceil$ where d is as in Definition 1. Suppose to the contrary that G has an equitable $(p - 1)$ -coloring. Then there is a partite set, say X_1 of size n_1 , partitioned into at most $\lceil n_1/d \rceil - 1$ color classes. Consequently, there is a color class containing vertices in X_1 with size at least $d + 1$. If there is X_j partitioned into at least $\lceil n_j/d \rceil + 1$ color classes, then there is a color class containing vertices in X_j with size at most $d - 1$. So we have two color classes with sizes differed by at least 2, a contradiction. Thus we assume that X_1 is partitioned into $\lceil n_1/d \rceil - 1$ color classes, and X_j is partitioned into $\lceil n_j/d \rceil$ color classes for each $j \in \{2, \dots, k\}$. We consider two cases.

Case 1: There exists $j \in \{2, \dots, k\}$ such that n_j is not divisible by d .

It follows that there is a color class containing vertices in X_j with size at most $d - 1$. So we have two color classes with sizes differed by at least 2, a contradiction.

Case 2: X_j is partitioned into exactly $\lceil n_j/d \rceil$ color classes and n_j is divisible by d for each $j \in \{2, \dots, k\}$.

Recall that d is as in Definition 1. It follows that n_1 has $n_1/\lfloor n_1/d \rfloor > d + 1$. Since X_1 is partitioned into $\lceil n_1/d \rceil - 1$ color classes, there is a color class containing vertices in X_1 with size at least $d + 2$. However, each color class containing vertices in X_j ($j \in \{2, \dots, k\}$) has size d , so we have a contradiction. Thus G has no equitable $(p - 1)$ -coloring. $\square$

Example 2. Let $G = K_{2,3,4}$. Then G has an equitable 4-coloring. One can compute that $p = p(q : n_1, \dots, n_k) = 4$. We can notice that G cannot have an equitable 3-coloring as in Lemma 2.

Lemmas 1 and 2 yield the following theorem. Note that $K_{n_1, \dots, n_k}$ has an equitable r -coloring when $r \geq n_1 + \dots + n_k$.

Theorem 1. If $K_{n_1, \dots, n_k}$ has an equitable q -coloring, then $p(q : n_1, \dots, n_k)$ is the minimum p such that G has an equitable r -coloring for each r satisfying $p \leq r \leq q$. In particular, the equitable chromatic threshold of $K_{n_1, \dots, n_k}$ is equal to $p(n_1 + \dots + n_k : n_1, \dots, n_k)$.

We use $p(q : n_1, \dots, n_k)$ to find $va_1^{\equiv}(K_{n_1, \dots, n_k})$ as follows.

Lemma 3. Let $G = K_{n_1, \dots, n_k}$ and $N = n_1 + \dots + n_k$. If G has an equitable q -coloring where $N/(q-1) \geq 3$ and G has an equitable $(r, 1)$ -tree-coloring for each $r \geq q$, then $va_1^{\equiv}(G) = p(q : n_1, \dots, n_k)$.

Proof. Let $p = p(q : n_1, \dots, n_k)$. From the assumption for q and Definition 1, the graph G has an equitable $(r, 1)$ -tree coloring for each $r \geq p$. It remains to show that G has no equitable $(p-1, 1)$ -tree-coloring. Suppose to the contrary that G has an equitable $(p-1, 1)$ -tree-coloring. Since $p-1 \leq q-1$, there is a color class of size at least $N/(p-1) \geq N/(q-1) \geq 3$ that is not independent. Observe that the graph induced by this color class has maximum degree greater than 1, which is a contradiction. This completes the proof. $\square$

3. The Strong Equitable Vertex 1-Arboricity of $K_{m,n}$

Let two partite sets of $K_{m,n}$ be X_1 and X_2 where $X_1 = \{u_1, \dots, u_m\}$ and $X_2 = \{v_1, \dots, v_n\}$.

Lemma 4. If $m = 3b + g$ and $n = 3c + h$ where b and c are non-negative integers and $0 \leq g, h \leq 2$, then $va_1^{\equiv}(K_{m,n}) \leq b + c + 2$.

Proof. Let $N = m + n$. First, consider $q \geq N/2$. We partition $V(K_{m,n})$ into q sets equitably. Each resulting set has size not greater than 2, so it cannot induce a graph with maximum degree more than 1. Thus we have an equitable $(q, 1)$ -tree-coloring.

Now assume $b + c + 2 \leq q < N/2$. It follows that $2 < N/q < 3$. Thus there are positive integers r and s such that $N = 3r + 2s$ and $q = r + s$. Since $q \geq b + c + 2$, we have $r \leq b + c$.

Let X'_1 be a $(3b)$ -subset of X_1 and let X'_2 be a $(3c)$ -subset of X_2 (One of these sets may be empty.) Partition X'_1 into b 3-sets and partition X'_2 into c 3-sets. Since $r \leq b + c$, we choose r 3-sets from these $b + c$ sets to initiate a new partition. Next, we partition set of the remaining $N - 3r$ vertices into s 2-sets. Since each of the resulting 3-sets contains vertices from the same partite set, it is an independent set. Moreover, each of remaining 2-sets cannot induce a graph with maximum degree more than 1. Thus this partition is an equitable $(q, 1)$ -tree-coloring for each $q \geq b + c + 2$. $\square$

Lemma 5. For any positive integer n , $va_1^{\equiv}(K_{1,n}) = \lceil (n+2)/3 \rceil$.

Proof. Let $r = \lceil (n+2)/3 \rceil$, and let $n = 3b + h$ for some integer h where $0 \leq h \leq 2$. Observe that $r = b + 1$ when $h = 0$ or 1 , and $r = b + 2$ when $h = 2$. First, we show that $va_1^{\equiv}(K_{1,n}) \leq r$. From Lemma 4, we know that $K_{1,n}$ has an equitable $(q, 1)$ -tree-coloring for each $q \geq b + 2$. It remains to show that $K_{1,n}$ has an equitable $(b+1, 1)$ -tree-coloring if $h = 0$ or 1 .

From $2 \leq (n+1)/(b+1) = (3b+h+1)/(b+1) < 3$, there are a positive integer r and a non-negative integer s such that $n+1 = 3r + 2s$ and $r + s = b + 1$. Choosing u_1 and another vertex in X_2 to be in one set, and partitioning the set of remaining vertices into b sets equitably, we obtain an equitable $(b+1, 1)$ -tree-coloring.

Next, we show that $K_{1,n}$ has no equitable $(r-1, 1)$ -tree-coloring to complete the proof. Suppose to the contrary that $K_{1,n}$ has an equitable $(r-1, 1)$ -tree-coloring. However, $(n+1)/(r-1) \geq 3$. Consequently, every resulting color class has size at least 3. Then a color class containing u_1 induces a graph with maximum degree greater than 1, a contradiction. This completes the proof. $\square$

Lemma 6. For any positive integer n , $va_1^{\equiv}(K_{2,n}) = \lceil (n+3)/3 \rceil$.

Proof. Let $r = \lceil (n+3)/3 \rceil$, and let $n = 3b + h$ for some integer h with $0 \leq h \leq 2$. Observe that $r = b + 1$ when $h = 0$, and $r = b + 2$ when $h = 1$, or 2 . First, we show that $va_1^{\equiv}(K_{2,n}) \leq r$. From Lemma 4, we know that $K_{2,n}$ has an equitable $(q, 1)$ -tree-coloring

for each $q \geq b + 2$. It remains to show that $K_{2,n}$ has an equitable $(b + 1, 1)$ -tree-coloring if $n = 3b$.

Since $2 \leq (n + 2)/(b + 1) = (3b + 2)/(b + 1) < 3$, there are a positive integer r and a non-negative integer s such that $n + 2 = 3r + 2s$ and $r + s = b + 1$. Choosing u_1 and u_2 to be in one set, and partitioning the set of remaining vertices into b sets equitably, we obtain an equitable $(b + 1, 1)$ -tree-coloring.

Next, we show that $K_{2,n}$ has no equitable $(r - 1, 1)$ -tree-coloring to complete the proof. Suppose to the contrary that $K_{2,n}$ has an equitable $(r - 1, 1)$ -tree-coloring. Since $(2 + n)/(r - 1) \geq 3$, every color class has size at least 3. Thus a color class containing u_1 induces a graph with maximum degree greater than 1, a contradiction. $\square$

Lemma 7. If $m = 3b$ and $n = 3c$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$.

Next, we show that $K_{m,n}$ has an equitable $(b + c + 1, 1)$ -tree-coloring. We initiate a partition by assigning three 2-sets $\{u_1, v_1\}, \{u_2, v_2\}, \{u_3, v_3\}$ and then partition $X_1 - \{u_1, u_2, u_3\}$ into $b - 1$ 3-sets and partition $X_2 - \{v_1, v_2, v_3\}$ into $c - 1$ 3-sets. This partition is equivalent to an equitable $(b + c + 1, 1)$ -tree-coloring as required.

Finally, we obtain an equitable $(b + c)$ -coloring of $K_{m,n}$ by partitioning X_1 into b 3-sets and partition X_2 into c 3-sets. Using Lemma 3, we have $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$. $\square$

Lemma 8. If $m = 3b$ and $n = 3c + 1$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$.

Next, we show that $K_{m,n}$ has an equitable $(b + c + 1)$ -coloring. We initiate a partition by assigning sets $\{v_1, v_2\}, \{v_3, v_4\}$ and then we partition X_1 into b 3-sets, and partition $X_2 - \{v_1, v_2, v_3, v_4\}$ into $c - 1$ 3-sets to obtain an equitable $(b + c + 1)$ -coloring.

Finally, we obtain an equitable $(b + c)$ -coloring of $K_{m,n}$ by partitioning X_1 into b 3-sets, and partition X_2 into $c - 1$ 3-sets and one 4-set. Using Lemma 3, we have $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$. $\square$

Lemma 9. If $m = 3b$ and $n = 3c + 2$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = p(b + c + 1 : m, n)$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$. Moreover, we can obtain an equitable $(b + c + 1)$ -coloring of $K_{m,n}$ by partitioning X_1 into b 3-sets, and partitioning X_2 into c 3-sets and one 2-set. Using Lemma 3, we have $va_1^{\equiv}(K_{m,n}) = p(b + c + 1 : m, n)$. $\square$

Lemma 10. If $m = 3b + 1$ and $n = 3c + 1$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$.

Next, we show that $K_{m,n}$ has an equitable $(b + c + 1, 1)$ -tree-coloring. We initiate a partition by assigning a set $\{u_1, v_1\}$ and then we partition $X_1 - \{u_1\}$ into b 3-sets, and partition $X_2 - \{v_1\}$ into c 3-sets to obtain an equitable $(b + c + 1)$ -coloring.

To obtain an equitable $(b + c)$ -coloring of $K_{m,n}$, we partition X_1 into $b - 1$ 3-sets and one 4-set, and we partition X_2 into $c - 1$ 3-sets and one 4-set. Using Lemma 3, we have $va_1^{\equiv}(K_{m,n}) = p(b + c : m, n)$. $\square$

Lemma 11. If $m = 3b + 1$ and $n = 3c + 2$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = b + c + 2$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$.

It remains to show that $K_{m,n}$ has no equitable $(b + c + 1, 1)$ -tree-coloring. Suppose to the contrary that $K_{m,n}$ has such coloring. Since $(m + n)/(b + c + 1) = 3$, each resulting color class has size 3. It follows that each color class is an independent set. However, X_1 can be partitioned into at most b 3-sets and X_2 can be partitioned into at most c 3-sets, a contradiction. $\square$

Lemma 12. If $m = 3b + 2$ and $n = 3c + 2$ for some positive integers b and c , then $va_1^{\equiv}(K_{m,n}) = b + c + 2$.

Proof. From Lemma 4, the graph $K_{m,n}$ has an equitable $(q, 1)$ -tree-coloring for each integer q such that $q \geq b + c + 2$.

It remains to show that $K_{m,n}$ has no equitable $(b + c + 1, 1)$ -tree-coloring. Suppose to the contrary that $K_{m,n}$ has such coloring. Since $(m + n) = 3(b + c + 1) + 1$, one resulting color class has size 3 and the other one has size 4. It follows that each color class is an independent set. However, either X_1 can be partitioned into at most $(b - 1)$ 3-sets and one 4-set and X_2 can be partitioned into at most c 3-sets or X_1 can be partitioned into at most b 3-sets and X_2 can be partitioned into at most $(c - 1)$ 3-sets and one 4-set, a contradiction. $\square$

From Lemmas 5–12, we obtain the following theorem.

Theorem 2. Each $va_1^{\equiv}(K_{m,n})$ is as in Table 1.

Table 1. $va_1^{\equiv}(K_{m,n})$ in terms of m and n where b and c are positive integers.

m	n	$va_1^{\equiv}(K_{m,n})$
1	n	$\lceil (n + 2)/3 \rceil$
2	n	$\lceil (n + 3)/3 \rceil$
$3b$	$3c$	$p(b + c : m, n)$
$3b$	$3c + 1$	$p(b + c : m, n)$
$3b$	$3c + 2$	$p(b + c + 1 : m, n)$
$3b + 1$	$3c + 1$	$p(b + c : m, n)$
$3b + 1$	$3c + 2$	$b + c + 2$
$3b + 2$	$3c + 2$	$b + c + 2$

4. The Strong Equitable Vertex 1-Arboricity of K_{k*n}

For K_{k*n} , we denote $p(q : n, \dots, n)$ by $p(q : k * n)$.

Lemma 13. If $G = K_{k*1}$ or K_{k*2} and $N = |V(G)|$, then $va_1^{\equiv}(G) = \lceil N/2 \rceil$.

Proof. For $q \geq \lceil N/2 \rceil$, we have $N/q \leq 2$. We obtain an equitable $(q, 1)$ -tree-coloring by partitioning $V(G)$ into q sets equitably. Since each set has size at most 2, we obtain a desired coloring.

It remains to show that G has no equitable $(\lceil N/2 \rceil - 1, 1)$ -tree-coloring. Suppose to the contrary that such coloring exists. Then there is a resulting color class with size at least 3 which induces a graph with maximum degree at least 2, a contradiction. This completes the proof. $\square$

Lemma 14. If $n = 3b$ for some positive integer b , then $va_1^{\equiv}(K_{k*n}) = p(kb : k * n)$.

Proof. For $q \geq 3kb/2$, we partition $V(K_{k*n})$ into q sets equitably. Since each set has size at most 2, this partition leads to an equitable $(q, 1)$ -tree-coloring.

For $kb + 1 \leq q \leq 3kb/2 - 1$, then $2 < 3kb/q < 3$. It follows that there are positive integers r and s such that $3kb = 3r + 2s$ and $r + s = q$. Observe that $r < kb$. Partition X_i for each i into b 3-sets to obtain kb 3-sets. Since $r \leq kb$, we initiate a new partition by choosing r sets from these kb 3-sets. Next, we partition the set of remaining vertices into s sets equitably. Since each of 3-sets contains vertices from the same partite set, it is an independent set. Moreover, each of remaining sets has size 2. Thus we obtain an equitable $(q, 1)$ -tree-coloring for each $q \geq kb + 1$.

One can easily see that G has an equitable kb -coloring. Using Lemma 3, we have $va_1^{\equiv}(K_{k*n}) = p(kb : k * n)$. $\square$

Lemma 15. If $n = 3b + 1$ for some positive integer b , then $va_1^{\equiv}(K_{2*n}) = p(2b : n, n)$ and $va_1^{\equiv}(K_{k*n}) = kb + \lceil k/2 \rceil$ for $k \geq 3$.

Proof. If $k = 2$, the result comes from Lemma 10. Now assume that $k \geq 3$. For $q \geq (3kb + k)/2$, we partition $V(K_{k*n})$ into q sets equitably. Since each set has size at most 2, this partition leads to an equitable $(q, 1)$ -tree-coloring.

For $kb + \lceil k/2 \rceil \leq q \leq (3kb + k)/2 - 1$, we have $2 < (3kb + k)/q < 3$. Thus there are positive integers r and s such that $3kb + k = 3r + 2s$ and $r + s = q$. Consequently, $3kb + k = 3(r + s) - s = 3q - s$. Since $q \geq kb + \lceil k/2 \rceil$, it follows that $s \geq 3\lceil k/2 \rceil - k$. From $3kb + k = 3r + 2s$, we have $r \leq kb$. Choose X'_i to be a $(3b)$ -subset of X_i for each $i \in \{1, \dots, k\}$. Partition X'_i for each i into b 3-sets of size 3. Thus we have kb 3-sets. Since $r \leq kb$, we initiate a new partition by choosing r 3-sets from these kb 3-sets. Next, we partition set of remaining vertices into s sets equitably. Since each of 3-sets contains vertices from the same partite set, it is an independent set. Moreover, each of remaining sets has size 2. Thus we obtain an equitable $(q, 1)$ -tree-coloring for each $q \geq kb + \lceil k/2 \rceil$.

It remains to show that G has no equitable $(kb + \lceil k/2 \rceil - 1, 1)$ -tree-coloring. Suppose to the contrary that G has such coloring. Consider $k = 4$. This yields the graph K_{4*n} has an equitable $(4b + 1, 1)$ -tree-coloring. But $|V(G)| = 12b + 4 = 4 + (4b)3$. The color classes are $4b$ 3-sets and one 4-set. Note that each of these color classes must be an independent set. However, there are at most b color classes of size 3 to contain only vertices in X_i where $i \in \{1, 2, 3, 4\}$. Thus there are at most $4b$ color classes of size 3, a contradiction.

Consider $k = 3$ or $k \geq 5$. Since $3kb + k = 3(kb - 2\lceil k/2 \rceil + k + 2) + 2(3\lceil k/2 \rceil - k - 3)$, an equitable $(kb + \lceil k/2 \rceil - 1, 1)$ -tree-coloring yields $kb - 2\lceil k/2 \rceil + k + 2$ color classes of size 3 and $3\lceil k/2 \rceil - k - 3$ color classes of size 2. Note that a color class of size 3 must be independent. However, there are at most b color classes of size 3 to contain only vertices in X_i where $i \in \{1, \dots, k\}$. Thus there are at most kb color classes of size 3, a contradiction. $\square$

Lemma 16. Let $n = 3b + 2$ for some positive integer b . Then $va_1^{\equiv}(K_{k*n}) = kb + k$.

Proof. If $k = 2$, then we obtain the result from Lemma 12. Now assume $k \geq 3$. For $q \geq (3kb + 2k)/2$, we partition $V(G)$ into q sets equitably. Since each set has size at most 2, the resulting partition leads to an equitable $(q, 1)$ -tree-coloring.

For $kb + k \leq q \leq (3kb + 2k)/2 - 1$, we have $2 < (3kb + 2k)/q < 3$. Thus there are positive integers r and s such that $3kb + 2k = 3r + 2s$ and $r + s = q$. Consequently, $3kb + 2k = 3(r + s) - s = 3q - s$. Since $q \geq kb + k$, we have $s \geq k$. From $3kb + 2k = 3r + 2s$, we have $r \leq kb$. Choose X'_i to be a $(3b)$ -subset of X_i where $i \in \{1, \dots, k\}$. Partition X'_i for each i into b 3-sets to obtain kb 3-sets. Since $r \leq kb$, we can initiate a new partition by choosing r sets from these kb sets. Next, we partition the set of the remaining vertices into s 2-sets. Since each of 3-sets contains vertices from the same partite set, it is an independent set. Moreover, each of remaining sets has size 2. Thus we obtain an equitable $(q, 1)$ -tree-coloring for each $q \geq kb + k$.

It remains to show that G has no equitable $(kb + k - 1, 1)$ -tree-coloring. Suppose to the contrary that G has such coloring. Since $3kb + 2k = 3(kb + 2) + 2(k - 3)$, an equitable

$(kb + k - 1, 1)$ -tree-coloring has $kb + 2$ color classes of size 3 and $k - 3$ color classes of size 2. Note that a color class of size 3 must be independent. However, there are at most b color classes of size 3 containing only vertices in X_i where $i \in \{1, \dots, k\}$. Thus there are at most kb color classes of size 3, a contradiction. $\square$

From Lemmas 13–16, we have the following theorem.

Theorem 3. Each $va_1^{\equiv}(K_{k*n})$ is as in Table 2.

Table 2. $va_1^{\equiv}(K_{k*n})$ in terms of n where b is a positive integer.

n	$va_1^{\equiv}(K_{n,n})$	$va_1^{\equiv}(K_{k*n})$ where $k \geq 3$
1	1	$\lceil k/2 \rceil$
2	2	k
$3b$	$p(2b : n, n)$	$p(kb : k * n)$
$3b + 1$	$p(2b : n, n)$	$kb + \lceil k/2 \rceil$
$3b + 2$	$2b + 2$	$kb + k$

5. Conclusions

In this paper, we extend the results of Wu, Zhang, and Li [10] which give exact values of $va_1^{\equiv}(K_{n,n})$ for some special cases by providing the exact values of $va_1^{\equiv}(K_{n,n})$ for all cases. Moreover, we give a more general result by establishing exact values of $va_1^{\equiv}(K_{m,n})$ in Theorem 2 and $va_1^{\equiv}(K_{k*n})$ in Theorem 3.

Author Contributions: Conceptualization, K.N. and K.M.N.; investigation, J.L.; methodology, J.L. and K.N.; validation, K.M.N. and W.R.; writing—original draft preparation, J.L.; writing—review and editing, K.N. and W.R.; supervision, K.M.N. and K.N.; funding acquisition, J.L., K.N. and W.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research has received funding support from the Research Fund for Supporting Lecturer to Admit High-Potential Students to Study and Research on His Expert Program Year 2021 from the Graduate School Khon Kaen University, Thailand [Grant number 641T104-C], National Research Council of Thailand (NRCT) [Grant number N41A640141] and NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [Grant number B05F640106].

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the reviewers for their suggestions and comments. The first author was supported by the Research Fund for Supporting Lecturer to Admit High Potential Student to Study and Research on His Expert Program Year 2021 from the Graduate School Khon Kaen University, Thailand [Grant number 641T104-C]. The third author was supported by National Research Council of Thailand (NRCT) [Grant number N41A640141]. The corresponding author was supported by NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [Grant number B05F640106]. The authors take this opportunity to thank the funders for the support.

Conflicts of Interest: The authors declare no conflict of interest.

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