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Smiley Theorem for Spectral Operators Whose Radical Part Is Locally Nilpotent

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Abstract: Generalizing bicommutant theorem to the higher-order commutator case is very useful for representation theory of Lie algebras, which plays an important role in symmetry analysis. In this paper, we mainly prove that for any spectral operator A on a complex Hilbert space whose radical part is locally nilpotent, if a bounded operator B lies in the k-centralizer of every bounded linear operator in the l-centralizer of A, where k and l are two arbitrary positive integers satisfying $l \geqslant k$, then B must belong to the von Neumann algebra generated by A and the identity operator. This result generalizes a matrix commutator theorem proved by M. F. Smiley. To this aim, Smiley operators are defined and an example of a non-spectral Smiley operator is given by the unilateral shift, indicating that Smiley-type theorems might also hold for general spectral operators.

Keywords: commutant; s-centralizer; spectral operator; unilateral shift; locally nilpotent

MSC: 47B02; 47B47; 47B40; 47B15

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Citation: Jiang, J.; Chen, X.; Li, X. Smiley Theorem for Spectral Operators Whose Radical Part Is Locally Nilpotent. *Symmetry* **2022**, *14*, 283. https://doi.org/10.3390/sym14020283

Academic Editor: Ioan Raşa

Received: 10 November 2021
Accepted: 9 January 2022
Published: 31 January 2022

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1. Introduction and Preliminaries

Lie algebra is a standard language for continuous symmetry, while operator algebra is a foundational language for quantum physics, and they usually interact with each other. For recent studies relevant to the present article in this respect, please refer to [1,2] and the references therein. In 1960, generalizing von Neumann's bicommutant theorem (which is fundamental to the representation theory of C^*-algebras) in the matrix case, M. F. Smiley [3] proved the following important and interesting fact, to which we may refer as **Classical Smiley Theorem**:

Let $\mathbb{F}$ be an algebraically closed field and the characteristic is 0 or at least n, and let $M_n(\mathbb{F})$ be the ring of all $n \times n$ matrices whose entries are in $\mathbb{F}$. Let $A, B \in M_n(\mathbb{F})$ be such that, for some positive integer s, $\mathrm{ad}_A^s(X) = 0$ for X in $M_n(\mathbb{F})$ implies $\mathrm{ad}_X^s(B) = 0$. Then, B is a polynomial in A with coefficients in $\mathbb{F}$. Here, the notation ad *will be defined below.*

After Smiley, D. W. Robinson proved that the above theorem is also valid in the event that $\mathbb{F}$ is not algebraically closed, and gave a final and complete form of Smiley's theorem for matrix algebras (cf. [4] or [5], pp. 114–115). In this paper, we seek to generalize Smiley's theorem to infinite-dimensional complex Hilbert spaces. This is also partially motivated by the first author's study on the long-standing classification problem for quasi-finite representations of Lie algebras of vector fields (cf. [6]), which plays an important role in symmetry analysis for mathematics and physics (see, e.g., [7–9]).

Let $\mathfrak{H}$ be a Hilbert space over complex number field $\mathbb{C}$, and $B(\mathfrak{H})$ the C^*-algebra of all bounded operators on $\mathfrak{H}$. There is a natural Lie product $[X, Y] := XY - YX$ for any X, $Y \in B(\mathfrak{H})$; thus, $B(\mathfrak{H})$ can be viewed as a Lie algebra. The operator algebra generated by a subset $\mathcal{S} \subseteq B(\mathfrak{H})$, denoted by $\langle \mathcal{S} \rangle$, is the smallest algebra containing $\mathcal{S}$. For any operator A in $B(\mathfrak{H})$, we denote by $\mathrm{Pol}(A) := \langle \{A, \mathrm{id}_{\mathfrak{H}}\} \rangle$ the algebra of all polynomials in A with coefficients in $\mathbb{C}$, where $\mathrm{id}_{\mathfrak{H}}$ is the identity operator on $\mathfrak{H}$. Throughout this paper, we denote the adjoint operator of $A \in B(\mathfrak{H})$ by A^*, the spectrum of A by $\sigma(A)$, the set of natural numbers by $\mathbb{N}$, and the set of positive integers by $\mathbb{N}^+$.

For any operator $Z \in B(\mathfrak{H})$, we can define the corresponding **left** (respectively, **right**) **multiplier** on $B(\mathfrak{H})$ by

$$\mathcal{L}_Z X := ZX \quad (\text{respectively, } \mathcal{R}_Z X := XZ\,) \quad \forall X \in B(\mathfrak{H}).$$

Then, $\mathrm{ad}_Z := \mathcal{L}_Z - \mathcal{R}_Z$ is also a linear operator on $B(\mathfrak{H})$. For any $s \in \mathbb{N}^+$, we denote by ad_A^s the s-multiple composition of the operator ad_A, i.e.,

$$\mathrm{ad}_A^s X := [A, [A, [A, [A, \cdots [A, X] \cdots]]]],$$

where $[A, \cdot]$ is repeated s times. If $\mathcal{B}$ is a subset of $B(\mathfrak{H})$, we denote by $\mathcal{B}'$ the **commutant** of $\mathcal{B}$, i.e.,

$$\mathcal{B}' := \{Y \in B(\mathfrak{H}) \mid XY = YX,\ \forall X \in \mathcal{B}\}.$$

The **bicommutant** $\mathcal{B}''$ of $\mathcal{B}$ is $(\mathcal{B}')'$, and the s-**commutant** $\mathcal{B}^s$ of $\mathcal{B}$ is $(\mathcal{B}^{s-1})'$ for any integer $s \geqslant 3$. Also, for any $s \in \mathbb{N}^+$, we define the s-**centralizer** of $\mathcal{B}$ by

$$C_s(\mathcal{B}) := \{Y \in B(\mathfrak{H}) \mid \mathrm{ad}_X^s Y = 0,\ \forall X \in \mathcal{B}\}.$$

These concepts also appeared in some earlier references, such as Chapter 4 in [10] and [5] p. 113. In particular, when $\mathcal{B}$ is a singleton set, we may abbreviate $C_s(\{Z\})$ as $C_s(Z)$. Note that $C_1(Z) = Z'$.

Definition 1. *An operator $A \in B(\mathfrak{H})$ is called a* (k,l)**-type Smiley operator** *if there exist $k, l \in \mathbb{N}^+$ such that $C_k(C_l(A)) \subseteq VN(A)$, where $VN(A)$ is the von Neumann algebra in $B(\mathfrak{H})$ generated by A and $\mathrm{id}_{\mathfrak{H}}$. In addition, a (k,l)-type Smiley operator A is said to be* **proper** *if $C_k(C_l(A))$ is contained in the subalgebra $\mathrm{Pol}(A)$ of $VN(A)$.*

When $\mathfrak{H}$ is an n-dimensional complex Hilbert space, which is isomorphic to $\mathbb{C}^n$, $B(\mathfrak{H})$ is nothing but the matrix algebra $M_n(\mathbb{C})$. Using the notations above, we may restate the classical Smiley theorem over $\mathbb{C}$ as follows (cf. [5], pp. 113–115):

For any $s \in \mathbb{N}^+$ and $A \in M_n(\mathbb{C})$, one has $C_s(C_s(A)) \subseteq \mathrm{Pol}(A)$.

In other words, *every $n \times n$ matrix over $\mathbb{C}$ is a proper (s,s)-type Smiley operator on $\mathbb{C}^n$ for any $s \in \mathbb{N}^+$*. More interestingly, von Neumann's *bicommutant theorem* (see, e.g., Theorem 4.1.5 in [10]) actually tells us that every $A \in B(\mathfrak{H})$ is a $(1,1)$-type Smiley operator. Thus, the following question arises naturally:

Which operators on a Hilbert space are (k,l)-type Smiley operators for given k, l?

In the present article, we partially answer this question for spectral operators on a complex Hilbert space. Loosely speaking, a spectral operator is an operator admitting a spectral reduction; that is, it can be reduced by a family of spectral projections. These projections are also known as the resolution of the identity or the spectral resolution of the given operator. It has been observed in [11] that the spectral reduction is simply the Jordan canonical form in matrix theory. In other words, every complex matrix is a spectral operator. Furthermore, another famous example of a spectral operator is a normal Hilbert space operator that has spectral measures and spectral resolution (cf. Section 4.3 in [12]). Here, we use an equivalent formulation of spectral operators.

Definition 2 (Theorem 5, Section 4, Chapter XV in [13])**.** *An operator $T \in B(\mathfrak{H})$ is called a* **spectral operator** *if there is a* **canonical decomposition** *of $T = S + N$ into a sum of a bounded* **scalar type operator** *S and a* **quasi-nilpotent operator** *N commuting with S. That is, the* **scalar part** *S has a unique spectral resolution E for which $S = \int_{\sigma(S)} z\,\mathrm{d}E(z)$, and the spectrum $\sigma(N)$ of the* **radical part** *N is simply $\{0\}$. Note that T and S have the same spectrum and the same spectral resolution.*

We refer to the good survey [14] and the famous book [13] for more details on spectral operators. Clearly, any **nilpotent operator**, whose kth-power is the zero operator for some $k \in \mathbb{N}^+$, is a very common quasi-nilpotent operator. Moreover, J. Wermer [15] has shown that

The scalar-type operators on a Hilbert space are those operators similar to normal ones.

In this paper, we consider a special but still large family of spectral operators. Denote by $\mathcal{S}_{\ln}(\mathfrak{H})$ the set of all bounded spectral operators $T = S + N$ whose radical part N is **locally nilpotent**, i.e., for every $v \in \mathfrak{H}$, there exists some $k \in \mathbb{N}^+$ such that $N^k v = 0$. The operators in $\mathcal{S}_{\ln}(\mathfrak{H})$ are a direct generalization of matrices to the infinite-dimensional case, since the Jordan canonical form of a complex matrix is the sum of a diagonal matrix and a nilpotent matrix. This is another reason that we here mainly consider the subclass $\mathcal{S}_{\ln}(\mathfrak{H})$ of spectral operators. The following theorem is our main result, which may be viewed as a generalization of Smiley's theorem to $\mathcal{S}_{\ln}(\mathfrak{H})$.

Theorem 1. *Every bounded operator on a complex Hilbert space $\mathfrak{H}$ is a $(1,1)$-type Smiley operator, and every operator in $\mathcal{S}_{\ln}(\mathfrak{H})$ is also a (s,s)-type Smiley operator for any $s \in \mathbb{N}^+$.*

Since $C_k(C_l(A)) \subseteq C_l(C_l(A))$ and $C_k(C_l(A)) \subseteq C_k(C_k(A))$ hold for every operator $A \in B(\mathfrak{H})$ and any $k, l \in \mathbb{N}^+$ satisfying $l \geqslant k$, a slightly more general result, as a direct consequence of Theorem 1, is immediately obtained.

Corollary 1. *Every operator in $\mathcal{S}_{\ln}(\mathfrak{H})$ is a (k,l)-type Smiley operator for any two integers k and l satisfying $l \geqslant k \geqslant 1$.*

This article is organized as follows. In Section 2, we start with some key lemmas and apply them to prove Theorem 1. In the process, we will see that the condition $l \geqslant k$ in Corollary 1 can be dropped when $l \geqslant 2$. In Section 3, we give an example of a Smiley operator, which is provided by a kind of non-spectral operator. Finally, in Section 4, we outline a plan for settling Smiley-type theorems for general spectral operators in future studies.

2. Proof of the Main Theorem

Before proving our main result, we need two crucial lemmas. The proof of the first lemma follows from similar lines of argument as in Lemma 1 in [3]. For completeness, we present the argument here.

Lemma 1. *If $A \in B(\mathfrak{H})$ is similar to a normal operator, i.e., there exists an operator $P \in B(\mathfrak{H})$ with bounded inverse P^{-1} such that $P^{-1}AP$ is normal, then $C_s(A) = A'$, i.e., for every $X \in B(\mathfrak{H})$, we have $\mathrm{ad}_A^s(X) = 0$ for some $s \in \mathbb{N}^+$ implies $[A, X] = 0$.*

Proof. If we prove this lemma for any normal operator A, then the general case readily follows, observing $\mathrm{ad}^s_{P^{-1}AP}(X) = P^{-1}\mathrm{ad}^s_A(PXP^{-1})P$.

Henceforth, let A be a normal operator. Then, there is a spectral resolution E such that $A = \int_{\sigma(A)} z \, \mathrm{d}E(z)$. Due to $[A, E(\Omega)] = 0$ for all Borel sets Ω of $\sigma(A)$, the Jacobi identity for the Lie product (cf. [16], p. 1) shows that $\mathrm{ad}_A \mathrm{ad}_{E(\Omega)}(X) = \mathrm{ad}_{E(\Omega)}\mathrm{ad}_A(X)$ for all X in $B(\mathfrak{H})$. It is well known that $T \in B(\mathfrak{H})$ commutes with both A and A^* if and only if $TE(\Omega) = E(\Omega)T$ for all Borel sets Ω of $\sigma(A)$ (cf. Chapter II, Theorem 2.5.5 in [10]).

However, the classical Fuglede's theorem (Chapter IV, Theorem 4.10 in [12]) tells us that, *for any $T \in B(\mathfrak{H})$ commuting with the normal operator A, the operator T necessarily commutes with A^**. Therefore, $\mathrm{ad}_A^s(X) = [A, \mathrm{ad}_A^{s-1}(X)] = 0$ gives

$$\mathrm{ad}_A^{s-1}\mathrm{ad}_{E(\Omega)}(X) = [E(\Omega), \mathrm{ad}_A^{s-1}(X)] = 0.$$

It follows by induction that

$$0 = \mathrm{ad}^s_{E(\Omega)}(X) = (\mathcal{L}_{E(\Omega)} - \mathcal{R}_{E(\Omega)})^s(X) = \sum_{k=0}^{s}(-1)^k \binom{s}{k} E(\Omega)^{s-k} X E(\Omega)^k.$$

Note that s can be assumed to be an odd number without loss of any generality, and then the above equations simplify to $[E(\Omega), X] = 0$, and consequently $[A, X] = 0$. □

Lemma 2. *Let $A \in B(\mathfrak{H})$ be normal. For any $X \in C_1(A)$, one has $[\Re X, A] = [\Im X, A] = 0$,*

$$[\Re A, X] = [\Re A, \Re X] = [\Re A, \Im X] = [\Im A, X] = [\Im A, \Re X] = [\Im A, \Im X] = 0.$$

Here, the **real part** $\Re X := \frac{X+X^*}{2}$ *and* **imaginary part** $\Im X := \frac{X-X^*}{2i}$ *are both self-adjoint.*

Proof. If $[X, A] = 0$, then the Fuglede's theorem gives $[X, A^*] = 0$, and equivalently $[A, X^*] = 0$. Now, the lemma clearly follows by linearity. □

Proposition 1. *If $A \in B(\mathfrak{H})$ is similar to a normal operator, then, for any $k, l \in \mathbb{N}^+$, A is a (k, l)-type Smiley operator; more precisely, $C_k(C_l(A)) = VN(A)$.*

Proof. Firstly, assume that A is normal, and let $B \in C_k(C_l(A))$. For any $X \in C_1(A)$, we have $[\Re X, A] = [\Im X, A] = 0$ by Lemma 2. Thus, $\mathrm{ad}^k_{\Re X}(B) = \mathrm{ad}^k_{\Im X}(B) = 0$, by definition. Then, $\mathrm{ad}_{\Re X}(B) = \mathrm{ad}_{\Im X}(B) = 0$ by Lemma 1. Consequently, $[X, B] = 0$. That is to say,

$$A'' \subseteq C_k(A') = C_k(C_l(A)) \subseteq A'' = VN(A).$$

Now, consider the general case. Choose $P \in B(\mathfrak{H})$ such that PAP^{-1} is normal, noting

$$P \cdot C_k(C_l(A)) \cdot P^{-1} = C_k(C_l(PAP^{-1})) = C_1(C_1(PAP^{-1})) = P \cdot A'' \cdot P^{-1},$$

we still obtain $C_k(C_l(A)) = A'' = VN(A)$. □

Now, we are in a position to prove the main result. The proof is divided into two cases, and the main strategy is reducing the second case to the first case.

Proof of Theorem 1. It is divided into $s = 1$ and $s \geqslant 2$ cases.

Case $s = 1$. The bicommutant theorem tells us that every $A \in B(\mathfrak{H})$ is a $(1, 1)$-type Smiley operator, i.e., $C_1(C_1(A)) = VN(A)$.

Case $s \geqslant 2$. Let $A \in \mathcal{S}_{\ln}(\mathfrak{H})$ with the canonical decomposition $A = S + N$. By Proposition 1 and its proof, we may assume that the radical part $N \neq 0$, and the scalar part S is normal. Henceforth, let B be any bounded operator in $C_s(C_s(A))$.

Let $\mathfrak{N}_j = \ker N^j := \{v \in \mathfrak{H} \mid N^j v = 0\}$ $(j = 0, 1, 2, \cdots)$ be a filtration of $\mathfrak{H}$, i.e., $\mathfrak{N}_1 \subseteq \mathfrak{N}_2 \subseteq \cdots \subseteq \mathfrak{H} = \bigcup_{j=1}^{\infty} \mathfrak{N}_j$. Clearly, each closed subspace $\mathfrak{N}_j$ is invariant under S since $[S, N] = 0$. Let $\mathfrak{H}_{j+1}$ be the orthogonal complement of $\mathfrak{N}_j$ in $\mathfrak{N}_{j+1}$, which is also closed and invariant under S, since S is normal. The orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_j$ is denoted by E_j. Note that $\mathfrak{N}_j = \mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_j$ and N maps $\mathfrak{H}_{j+1}$ injectively into $\mathfrak{H}_j$ for any $j \in \mathbb{N}^+$. Clearly, $N\mathfrak{H}_{j+1}$ and its orthogonal complement in $\mathfrak{H}_j$, which will be denoted by $\mathfrak{H}'_j$, are also invariant under S, since S is normal and commutes with N. We denote the orthogonal projection from $\mathfrak{H}_j$ onto $\mathfrak{H}'_j$ by P_j, and let $\bar{P}_j := P_j E_j$ be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}'_j$. Note that $\bar{P}_j N = 0$.

For $j, k \in \mathbb{N}^+$, denote by $E_j(k)$ the orthogonal projection from $\mathfrak{H}$ onto the closure of $\mathfrak{H}'_j + N\mathfrak{H}'_{j+1} + N^2\mathfrak{H}'_{j+2} + \cdots + N^{k-1}\mathfrak{H}'_{j+k-1} \subseteq \mathfrak{H}_j$. Inspired by the Cartan elements in the finite-dimensional irreducible representations of the Lie algebra $\mathfrak{gl}(2)$—see, e.g., the standard book ([16], pp. 31–34)—let us consider two commuting self-adjoint operators, namely the projection $F(k) := \sum_{\ell=1}^{k} E_{k+1-\ell}(\ell)$ and the weight operator $H(k) := \sum_{\ell=1}^{k} \ell E_{k+1-\ell}(\ell)$. Direct computation shows that

$$[F(k), S] = [F(k), N] = [H(k), S] = 0, \qquad [H(k), N] = [H(k), NF(k)] = NF(k).$$

Then, $\mathrm{ad}_A^2 H(k) = 0$, and it follows that $\mathrm{ad}_{H(k)}^s B = 0$ by definition of B. By Lemma 1, we have $[H(k), B] = 0$. Similarly, $[F(k), B] = 0$. In particular, $[E_j(k), B] = [E_j, B] = 0$ for any $j, k \in \mathbb{N}^+$. Further consider $Y(k) := H(k) + NF(k)$, and we have $\mathrm{ad}_A^2 Y(k) = 0$; thus, $\mathrm{ad}_{Y(k)}^s B = 0$. Note that $Y(k)$ is similar to $H(k)$; more precisely, $Y(k) = e^{-NF(k)} H(k) e^{NF(k)}$; thus, we obtain $[Y(k), B] = 0$ by Lemma 1, and then $[N, B]F(k) = 0$. Therefore, $[N, B] = 0$.

Now, for any operator $W \in C_1(S)$, by Lemma 2, we may assume that W is self-adjoint by treating its real and imaginary parts separately. Consider

$$Z(k) := (E_1 - \delta_{1k}\bar{P}_1) W N^{k-1} \bar{P}_k, \quad \text{where } \delta_{1k} = 1 \text{ if } k = 1, \text{ otherwise } \delta_{1k} = 0.$$

Obviously, $[Z(k), S] = [Z(k), N] = 0$, thus $\mathrm{ad}_A^2(H(k) + Z(k)) = 0$, and $\mathrm{ad}_{H(k)+Z(k)}^s B = 0$. Simple computation shows that $[H(k), Z(k)] = \lambda_k Z(k)$, where $\lambda_k = k - 1 - \delta_{1k}$. Hence, $e^{-Z(k)/\lambda_k} H(k) e^{Z(k)/\lambda_k} = H(k) + Z(k)$. Lemma 1 gives $[H(k) + Z(k), B] = [Z(k), B] = 0$. From $[\bar{P}_1 W \bar{P}_1, A] = 0$, we routinely obtain $\bar{P}_1 [W, B] \bar{P}_1 = 0$. Thus, $E_1 [W, B] N^{k-1} \bar{P}_k = 0$ for every $k \in \mathbb{N}^+$, which means $E_1 [W, B] E_1 = 0$. Denote $S_1 := S|_{\mathfrak{H}_1}$, $B_1 := B|_{\mathfrak{H}_1}$, and let $J_j : \mathfrak{H}_j \to \mathfrak{H}$ be the inclusion map of $\mathfrak{H}_j$ into $\mathfrak{H}$. Then, for any $W_1 \in C_1(S_1)$, clearly, we have $J_1 W_1 E_1 \in C_1(S)$, and the above argument proves that $[B_1, W_1] = 0$. That is, $B_1 \in C_1(C_1(S_1)) = VN(S_1)$.

Then, we may approximate B_1 in the strong operator topology, by polynomials $p_\lambda(S_1, S_1^*)$ in S_1 and S_1^*. In brief, $p_\lambda(S_1, S_1^*) \to B_1$. By induction on $k \in \mathbb{N}^+$, using the injectivity of N from $\mathfrak{H}_{k+1}$ to $\mathfrak{H}_k$, we iteratively see that

$$p_\lambda(S, S^*)F(k) \to BF(k), \;\; N p_\lambda(S, S^*) E_{k+1} = p_\lambda(S, S^*) N E_{k+1} \to BNE_{k+1} = NBE_{k+1}.$$

Therefore, $p_\lambda(S, S^*) \to B$, so $B \in VN(S)$. What we have actually proven is the following, roughly finding that the double higher-order centralizer kills the radical part. □

Proposition 2. *If $A \in \mathcal{S}_{\mathrm{ln}}(\mathfrak{H})$ has scalar part S, then $C_k(C_l(A)) \subseteq VN(S)$ for any integers $k \geqslant 1$ and $l \geqslant 2$.*

Finally, we show $VN(S) = C_1(C_1(S)) \subseteq C_1(C_1(A)) = VN(A)$ to finish the proof of Theorem 1. In fact, if a bounded operator X belongs to $C_1(A)$, then X must commute with every spectral resolution for A (cf. [14] p. 226 or Lemma 3, Section 3, Chapter XVI in [13]). Then, $X \in C_1(S)$ since A and S have the same spectral resolution. This means that $C_1(A) \subseteq C_1(S)$, and consequently $C_1(C_1(A)) \supseteq C_1(C_1(S))$; now, we are finished.

Corollary 2. *If $A \in \mathcal{S}_{\mathrm{ln}}(\mathfrak{H})$ has scalar part S, then $C_k(C_l(A)) = VN(S) \subseteq VN(A)$ for any integers $l \geqslant 2$ and $k \geqslant 1$.*

Proof. We already know that $C_1(A) \subseteq C_1(S)$, and thus $C_s(A) \subseteq C_s(S)$ follows inductively. In fact, for every $X \in C_{s+1}(A)$, one has $\mathrm{ad}_A X \in C_s(A)$. By induction hypothesis, we obtain $0 = \mathrm{ad}_S^s \mathrm{ad}_A X = \mathrm{ad}_A \mathrm{ad}_S^s X$, and then $\mathrm{ad}_S^{s+1} X = 0$ follows. Now, the corollary follows from the obvious fact $C_k(C_l(A)) \supseteq C_k(C_l(S))$ and Propositions 1 and 2. □

We would like to point out that the above corollary is quite useful. For example, consider a nilpotent Lie algebra (cf. [16], pp. 11–12) $\mathfrak{N} \subseteq \mathcal{S}_{\mathrm{ln}}(\mathfrak{H})$. There exists an integer $n \geqslant 2$ such that $\mathrm{ad}_A^n B = 0$ for any $A, B \in \mathfrak{N}$, thus $C_n(A) \supseteq \mathfrak{N}$, and $C_1(C_n(A)) \subseteq \mathfrak{N}'$. If $\mathfrak{H}$ is separable and irreducible under the action of $\mathfrak{N}$, then a version of Schur's Lemma (cf. [16], p. 26) states that $\mathfrak{N}' = \mathbb{C}\mathrm{id}_{\mathfrak{H}}$. Therefore, the scalar part of every operator in $\mathfrak{N}$ is simply a scalar. On the other hand, there is a natural question posed by W. Wojtynski:

Let $\mathfrak{X}$ be a Banach space and $\mathfrak{L}$ a Banach Lie subalgebra of $B(\mathfrak{X})$ consisting of quasi-nilpotent operators; does the associative algebra $A(\mathfrak{L})$ generated by $\mathfrak{L}$ also consist of quasi-nilpotent operators?

V. S. Shulman and Y. V. Turovskii [17] have given an affirmative answer under a compactness assumption. They also proved that $A(\mathfrak{L})$ is commutative modulo its Jacobson radical. In [1], the assumption is weakened, i.e., claiming that it only needs to be essentially nilpotent, and some necessary and sufficient conditions for an essentially nilpotent Lie algebra of quasi-nilpotent operators to generate the closed algebra of quasi-nilpotent operators are given. From these results, we see that the irreducible module $\mathfrak{H}$ over the nilpotent Lie algebra $\mathfrak{N}$ is 1-dimensional, which is closely linked with Lie's Theorem in Lie theory (cf. [16], p. 15).

Remark 1. *Let A be a bounded normal operator on $\mathfrak{H}$. By Theorem 1, we know that every operator $B \in C_s(C_s(A))$ for some $s \in \mathbb{N}^+$ lies in $VN(A)$. In other words, the operator B is determined by A. More precisely, by Borel functional calculus (cf. [10], p. 72), we can further see that there exists $f \in B_\infty(\sigma(A))$ such that $B = f(A)$, where $B_\infty(\sigma(A))$ is the C^*-algebra of all bounded Borel measurable complex-valued functions on $\sigma(A)$.*

In particular, if A is a compact self-adjoint operator on $\mathfrak{H}$, then A has the canonical spectral decomposition (also known as diagonalization) $A = \sum_{i=1}^{\infty} \lambda_i P_i$ (see Theorem 5.1, Chapter II in [18]). Applying the classical Smiley theorem to the finite-dimensional range space of every P_i, we see that $B = \sum_{i=1}^{\infty} g_i(A)\chi_i(A)$, where g_i is some polynomial and χ_i is the characteristic function of the singleton set $\{\lambda_i\}$.

3. Non-Spectral Example of Smiley Operator

In this section, as an example of a non-spectral Smiley operator, we prove that every infinite-dimensional unilateral shift operator is a proper $(k, 2)$-type Smiley operator when $k \in \{1, 2\}$.

Definition 3 (Proposition 2.10, Chapter II in [18])**.** *Let $\ell^2(\mathbb{N}^+)$ be the separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, which consists of all complex number sequences $(x_1, x_2, \cdots)$ satisfying $\sum_{i=1}^{\infty} |x_i|^2 < +\infty$. The unilateral shift operator A on $\ell^2(\mathbb{N}^+)$ is defined by $A(e_n) = e_{n+1}$ for all $n \in \mathbb{N}^+$, or equivalently, $A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ for any sequence $(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}^+)$. Note that $\sigma(A) = \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the closed unit disk in the complex plane (cf. Proposition 6.5, Chapter VII in [18]). Thus, the unilateral shift operator on $\ell^2(\mathbb{N}^+)$ is obviously not quasi-nilpotent.*

Proposition 3 (Corollary of Problem 147 in [19])**.** *The unilateral shift operator on $\ell^2(\mathbb{N}^+)$ does not have any non-trivial reducing subspace, and so is not a spectral operator.*

Proposition 4. *The unilateral shift operator A on $\ell^2(\mathbb{N}^+)$ is a proper $(k, 2)$-type Smiley operator when $k \in \{1, 2\}$.*

Proof. It suffices to prove $C_2(C_2(A)) \subseteq \mathrm{Pol}(A)$. Taking an arbitrary sequence $\xi = (\xi_1, \xi_2, \cdots)^T \in \ell^2(\mathbb{N}^+)$ and an infinite-dimensional complex matrix $X = (x_{ij})_{i,j \in \mathbb{N}^+}$, then

$$\begin{aligned} \mathrm{ad}_A(X)\xi &= (AX - XA)\xi \\ &= \left(0, \sum_{j=1}^{\infty} x_{1j}\xi_j, \sum_{j=1}^{\infty} x_{2j}\xi_j, \cdots\right)^T - \left(\sum_{j=1}^{\infty} x_{1,j+1}\xi_j, \sum_{j=1}^{\infty} x_{2,j+1}\xi_j, \cdots\right)^T \\ &= \left(-\sum_{j=1}^{\infty} x_{1,j+1}\xi_j, \sum_{j=1}^{\infty} (x_{1j} - x_{2,j+1})\xi_j, \cdots, \sum_{j=1}^{\infty} (x_{kj} - x_{k+1,j+1})\xi_j, \cdots\right)^T, \end{aligned} \tag{1}$$

$$
\begin{aligned}
\mathrm{ad}_A^2(X)\xi &= [A(AX-XA)-(AX-XA)A]\xi \\
&= \left(0, -\sum_{j=1}^{\infty} x_{1,j+1}\xi_j, \sum_{j=1}^{\infty}(x_{1j}-x_{2,j+1})\xi_j, \cdots, \sum_{j=1}^{\infty}(x_{kj}-x_{k+1,j+1})\xi_j, \cdots\right)^T \\
&- \left(-\sum_{j=1}^{\infty} x_{1,j+2}\xi_j, \sum_{j=1}^{\infty}(x_{1,j+1}-x_{2,j+2})\xi_j, \cdots, \sum_{j=1}^{\infty}(x_{k-1,j+1}-x_{k,j+2})\xi_j, \cdots\right)^T \\
&= \left(\sum_{j=1}^{\infty} x_{1,j+2}\xi_j, \sum_{j=1}^{\infty}(-2x_{1,j+1}+x_{2,j+2})\xi_j, \sum_{j=1}^{\infty}(x_{1j}-2x_{2,j+1}+x_{3,j+2})\xi_j,\right. \\
&\left.\sum_{j=1}^{\infty}(x_{2j}-2x_{3,j+1}+x_{4,j+2})\xi_j, \cdots, \sum_{j=1}^{\infty}(x_{k-2,j}-2x_{k-1,j+1}+x_{k,j+2})\xi_j, \cdots\right)^T.
\end{aligned}
\tag{2}
$$

Letting $\mathrm{ad}_A^2(X)\xi = 0$, by the arbitrariness of ξ, we know that every matrix $X \in C_2(A)$ has the form of

$$
\begin{pmatrix}
x_{11} & x_{12} & 0 & 0 & 0 & \cdots & 0 & \cdots \\
x_{21} & x_{22} & 2x_{12} & 0 & 0 & \cdots & 0 & \cdots \\
x_{31} & x_{32} & 2x_{22}-x_{11} & 3x_{12} & 0 & \cdots & 0 & \cdots \\
x_{41} & x_{42} & 2x_{32}-x_{21} & 3x_{22}-2x_{11} & 4x_{12} & \cdots & 0 & \cdots \\
x_{51} & x_{52} & 2x_{42}-x_{31} & 3x_{32}-2x_{21} & 4x_{22}-3x_{11} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{k,1} & x_{k,2} & 2x_{k-1,2}-x_{k-2,1} & 3x_{k-2,2}-2x_{k-3,1} & 4x_{k-3,2}-3x_{k-4,1} & \cdots & kx_{12} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.
$$

In the following, we compute $C_2(C_2(A))$. Since

$$(X\xi)(1) = x_{11}\xi_1 + x_{12}\xi_2,$$

$$(X\xi)(k) = x_{k1}\xi_1 + x_{k2}\xi_2 + \sum_{j=1}^{k-2}[(j+1)x_{k-j,2} - jx_{k-j-1,1}]\xi_{j+2} + kx_{12}\xi_{k+1} \ (k \geqslant 2),$$

setting $(B\psi)(k) = \sum_{j=1}^{\infty} \beta_{kj}\psi_j$ for all $\psi = \{\psi_j\}_{j=1}^{\infty}$ in $\ell^2(\mathbb{N}^+)$, we obtain

$$
(BX\xi)(k) = \sum_{l=1}^{\infty} \beta_{kl}\left\{x_{l1}\xi_1 + x_{l2}\xi_2 + \sum_{j=1}^{l-2}[(j+1)x_{l-j,2} - jx_{l-j-1,1}]\xi_{j+2} + lx_{12}\xi_{l+1}\right\}, \tag{3}
$$

$$
\begin{aligned}
(XB\xi)(k) &= x_{k1}(B\xi)(1) + x_{k2}(B\xi)(2) + \sum_{l=1}^{k-2}[(l+1)x_{k-l,2} - lx_{k-l-1,1}](B\xi)(l+2) \\
&+ kx_{12}(B\xi)(k+1) \\
&= x_{k1}\left(\sum_{j=1}^{\infty}\beta_{1j}\xi_j\right) + x_{k2}\left(\sum_{j=1}^{\infty}\beta_{2j}\xi_j\right) \\
&+ \sum_{l=1}^{k-2}[(l+1)x_{k-l,2} - lx_{k-l-1,1}]\left(\sum_{j=1}^{\infty}\beta_{l+2,j}\xi_j\right) + kx_{12}\left(\sum_{j=1}^{\infty}\beta_{k+1,j}\xi_j\right) \\
&= \sum_{j=1}^{\infty}\left\{x_{k1}\beta_{1j} + x_{k2}\beta_{2j} + \sum_{l=3}^{k}[(l-1)x_{k-l+2,2} - (l-2)x_{k-l+1,1}]\beta_{l,j} + kx_{12}\beta_{k+1,j}\right\}\xi_j.
\end{aligned}
\tag{4}
$$

It follows that

$$
\begin{aligned}
(XB-BX)\xi(k) = &\left\{ x_{k1}\beta_{11} + x_{k2}\beta_{21} + \sum_{l=3}^{k} [(l-1)x_{k-l+2,2} - (l-2)x_{k-l+1,1}]\beta_{l,1} \right. \\
&\left. + kx_{12}\beta_{k+1,1} - \sum_{l=1}^{\infty} \beta_{kl}x_{l1} \right\}\xi_1 + \left\{ x_{k1}\beta_{12} + x_{k2}\beta_{22} \right. \\
&\left. + \sum_{l=3}^{k} [(l-1)x_{k-l+2,2} - (l-2)x_{k-l+1,1}]\beta_{l,2} + kx_{12}\beta_{k+1,2} - \sum_{l=1}^{\infty} \beta_{kl}x_{l2} \right\}\xi_2 \\
&+ \sum_{j=3}^{\infty} \left\{ x_{k1}\beta_{1j} + x_{k2}\beta_{2j} + \sum_{l=3}^{k} [(l-1)x_{k-l+2,2} - (l-2)x_{k-l+1,1}]\beta_{l,j} \right. \\
&\left. + kx_{12}\beta_{k+1,j} - (j-1)\beta_{k,j-1}x_{l2} + \sum_{l=j}^{\infty} \beta_{kl}[(j-1)x_{l-j+2,2} - (j-2)x_{l-j+1,1}] \right\}\xi_j.
\end{aligned}
\tag{5}
$$

Considering a special case when $x_{11} = 1, x_{ij} = 0\ (i \neq 1, j \neq 1)$, we have

$$
X\psi = (\psi_1, 0, -\psi_3, -2\psi_4, \cdots, -(k-2)\psi_k, \cdots)^T, \tag{6}
$$

$$
\begin{aligned}
(XB-BX)\xi = &\left(\beta_{12}\xi_2 + \sum_{j=3}^{\infty} (j-1)\beta_{1,j}\xi_j, -\beta_{21}\xi_1 + \sum_{j=3}^{\infty} (j-2)\beta_{2,j}\xi_j, \right. \\
&\left. -2\beta_{31}\xi_1 - \beta_{32}\xi_2, \cdots, (1-k)\beta_{k1}\xi_1 - (k-2)\beta_{k2}\xi_2, \cdots \right)^T.
\end{aligned}
\tag{7}
$$

Hence

$$
\begin{aligned}
(X(XB-BX))\xi = &\left(\beta_{12}\xi_2 + \sum_{j=3}^{\infty} (j-1)\beta_{1,j}\xi_j, \right. \\
&\left. 0, 2\beta_{31}\xi_1 + \beta_{32}\xi_2, \cdots, (1-k)(2-k)\beta_{k1}\xi_1 + (k-2)^2\beta_{k2}\xi_2, \cdots \right)^T,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
((XB-BX)X)\xi = &\left(\sum_{j=3}^{\infty} (j-1)(2-j)\beta_{1,j}\xi_j, \right. \\
&\left. -\beta_{21}\xi_1 - \sum_{j=3}^{\infty} (j-2)^2\beta_{2,j}\xi_j, -2\beta_{31}\xi_1, \cdots, (1-k)\beta_{k1}\xi_1, \cdots \right)^T.
\end{aligned}
\tag{9}
$$

If $X(XB-BX) = (XB-BX)X$ holds, then, by (8) and (9), we have

$$
\beta_{1k} = 0\ (k \neq 1),\ \beta_{2k} = 0\ (k \neq 2)\ ,\beta_{k1} = 0\ (k \neq 1),\ \beta_{k2} = 0\ (k \neq 2).
$$

Next, we consider another case in which $x_{12} = 1, x_{ij} = 0\ (i \neq 1, j \neq 2)$; from similar procedures as above, we can obtain

$$
\beta_{kj} = 0\ (k \neq j), \quad \beta_{k+1,k+1} - \beta_{k,k} = \beta_{k,k} - \beta_{k-1,k-1}\ (k \geqslant 2).
$$

That is, B is an infinite-dimensional bounded diagonal operator with the matrix form $\mathrm{diag}(\beta_{11}, \beta_{22}, \cdots)$ satisfying $\beta_{k+1,k+1} - \beta_{k,k} = \beta_{k,k} - \beta_{k-1,k-1}\ (k \geqslant 2)$. Since all diagonal entries in a bounded diagonal operator are uniformly bounded (see Problem 61 in [19] or Exercise 8, Section II.1 in [18]), it is immediately known that

$$
\beta_{k+1,k+1} - \beta_{k,k} = \beta_{k,k} - \beta_{k-1,k-1} = 0\ (k \geqslant 2).
$$

Thus, we can set $\beta_{k,k} = \lambda$, where λ is an arbitrary complex number, then $B = \lambda \cdot \mathrm{id}_{\ell^2(\mathbb{N}^+)}$, which states that

$$C_2(C_2(A)) = \mathbb{C}\mathrm{id} \subseteq \mathrm{Pol}(A).$$

The proof is completed now. $\square$

4. Concluding Remarks

In this article, we mainly prove that for any bounded spectral operator $A = S + N$ on a complex Hilbert space $\mathfrak{H}$, if the radical part N is locally nilpotent, i.e., if

$$\mathfrak{H}_0 := \{v \in \mathfrak{H} \mid N^k v = 0 \text{ for some } k \in \mathbb{N}^+\} = \mathfrak{H},$$

then $C_s(C_s(A)) \subseteq VN(A)$ for every $s \in \mathbb{N}^+$. Now, consider the general case—namely, the radical part N is only known to be quasi-nilpotent. Without loss of generality, we may suppose that the scalar part S is normal. Decompose $\mathfrak{H}$ into the direct sum of $\overline{\mathfrak{H}_0}$ (the closure of $\mathfrak{H}_0$) and $\mathfrak{H}_0^\perp$, and then $\overline{\mathfrak{H}_0}$ is clearly invariant under S, N since $SN = NS$, and thus $\mathfrak{H}_0^\perp$ is invariant under S. For convenience, denote by E_0, E_1 the orthogonal projections from $\mathfrak{H}$ onto $\overline{\mathfrak{H}_0}, \mathfrak{H}_0^\perp$, respectively.

Let $B \in C_s(C_s(A))$ and $s \geqslant 2$. For any $W \in C_1(S)$, by similar arguments as in the proof of Theorem 1, it seems that if $C_s(C_s(E_1 A E_1)) \subseteq VN(E_1 S E_1)$ holds, then one might be able to show that $C_s(C_s(A)) \subseteq VN(S)$. Note that $E_1 A E_1$ is a spectral operator on $\mathfrak{H}_0^\perp$ whose radical part is injective, and injective quasi-nilpotent operators behave as weighted shifts, at least for compact ones on separable Hilbert spaces. However, in Section 3, we have shown that the unilateral shifts are Smiley operators. This strongly indicates that our Smiley-type theorem can be further generalized, possibly to general spectral operators. Such generalizations and applications to infinite-dimensional representations of Lie algebras will be carried out in the future.

Author Contributions: Funding acquisition, J.J. and X.C.; investigation, J.J. and X.C.; project administration, J.J. and X.C.; writing—original draft preparation, J.J. and X.C.; investigation, X.L.; writing—original draft preparation, X.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by NSFC (Grants 11801290), and partially supported by NSFC (Grants 11701327), as well as the Natural Science Foundation of Fujian Province of China (No. 2019J05122).

Acknowledgments: The authors would like to thank the reviewers for their suggestions, which helped to improve this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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