Article

j-Dimensional Integral Involving the Logarithmic and Exponential Functions: Derivation and Evaluation

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Abstract: In the fields of science and engineering, tasks involving repeated integrals appear on occasion. The authors' study on repeated integrals of a class of exponential and logarithmic functions is presented in this publication. The paper includes several examples that demonstrate the evaluation of the analytical parts of the multi-dimensional integral derived. All the results in this work are new.

Keywords: multi-dimensional integral; Cauchy integral; Hurwitz–Lerch zeta function; Catalan's constant

MSC: Primary 30E20; 33-01; 33-03; 33-04

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1. Introduction

The idea of multiple or repeated integrals has been around for decades. Work involving multiple definite integrals has been used in the studies of Reynolds et al. [1], Jain et al. [2], Agarwal et al. [3], Sergey et al. [4], Wiener [5], Kiyoshi [6] and Olver [7], to name a few.

Our definition of a multiple definite integral in the present paper is obtained using the contour integral method in [8,9]. In our definition, we are able to evaluate the multiple integral in terms of the Hurwitz–Lerch zeta function and other special functions and fundamental constants. The importance of this work lies in the formulation of a *j*-dimensional generalized integral transform in terms of a special function from which other inetgral transforms can be derived. For example, quadruple, sextuple, double integrals, etc., can be derived from one generalized transform. This, in our opinion, could prove useful for researchers requiring such an integral transform for their research. The generalized integral transform is derived in terms of the Hurwitz–Lerch zeta function, which arises due to the contour integral method in [8].

In Section 2, we define and derive the *j*-dimensional definite integral in terms of a contour integral. In Section 3, we define the Hurwitz–Lerch zeta function and derive a contour integral representation for this special function. In Section 4, we establish a definite *j*-dimensional definite integral in terms of the Hurwitz–Lerch zeta function. Almost all Hurwitz–Lerch zeta functions have an asymmetrical zero distribution [10]. We also derive special cases of this *j*-dimensional definite integral in terms of fundamental constants and dimensionality of the definite integral.

In this paper, we derive the *j*-dimensional definite integral given by

$$\int_0^\infty \cdots \int_0^\infty x_1^{m-1} e^{-b(x_1^n + x_2^n)} x_2^{-m+n-1} \log^k \left(\frac{ax_1 \left(\prod_{l=1}^j x_{2l+1}^{2l-1} x_{2l+2}^{-2l-1} \right)}{x_2} \right) \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2l-1m + \frac{n}{2} - 1} x_{2l+2}^{-2l-1m + \frac{n}{2} - 1} \right) dx_1 \cdots dx_{2j+2}, \quad (1)$$

where the parameters k, a, b, n, m are general complex numbers and $0 < \text{Re}(m) < 1/2$, $\text{Im}(n) \leq 0$, $0 \leq \text{Im}(a) < 2\pi$, $0 \leq \text{Im}(b) < 2\pi$. This definite integral will be used to derive

special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [8]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw, \quad (2)$$

where C is, in general, an open contour in the complex plane; see Figure 1, where the bilinear concomitant has the same value at the end points of the contour and the contour lies on either side of the cut. We then multiply both sides by a function of $x_1 \dots x_{2j+2}$, and then take a definite j -dimensional integral of both sides. This yields a definite integral in terms of a contour integral. Then, we multiply both sides of Equation (2) by another function of y and take the infinite sum of both sides such that the contour integrals of both equations are the same.

Figure 1. Path of the cut.

2. Definite Integral of the Contour Integral

We use the method in [8]. The variable of integration in the contour integral is $r = \frac{w+m}{n}$. The cut and contour are in the first quadrant of the complex r -plane. The cut approaches the origin from the interior of the first quadrant and the contour goes around the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula, we form the j -dimensional integral by replacing y with

$$\log \left(\frac{ax_1 \left(\prod_{l=1}^j x_{2l+1}^{2^{l-1}} x_{2l+2}^{-2^{l-1}} \right)}{x_2} \right),$$

and multiplying by

$$x_1^{m-1} e^{-b(x_1^n + x_2^n)} x_2^{-m+n-1} \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}m + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}m + \frac{n}{2} - 1} \right),$$

and then taking the definite integral with respect to $x_1 \in [0, \infty) \dots x_{2j+2} \in [0, \infty)$ to obtain

$$\begin{aligned}
 & \frac{1}{\Gamma(k+1)} \int_0^\infty \dots \int_0^\infty x_1^{m-1} e^{-b(x_1^n + x_2^n)} x_2^{-m+n-1} \log^k \left(\frac{ax_1 \left(\prod_{l=1}^j x_{2l+1}^{2^{l-1}} x_{2l+2}^{2^{l-1}} \right)}{x_2} \right) \\
 & \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}m + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}m + \frac{n}{2} - 1} \right) dx_1 \dots dx_{2j+2} \\
 & = \frac{1}{2\pi i} \int_0^\infty \dots \int_0^\infty \int_C a^w w^{-k-1} e^{-b(x_1^n + x_2^n)} x_1^{m+w-1} x_2^{-m+n-w-1} \\
 & \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}(m+w) + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}(m+w) + \frac{n}{2} - 1} \right) dw dx_1 \dots dx_{2j+2} \\
 & = \frac{1}{2\pi i} \int_C \int_0^\infty \dots \int_0^\infty a^w w^{-k-1} e^{-b(x_1^n + x_2^n)} x_1^{m+w-1} x_2^{-m+n-w-1} \\
 & \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}(m+w) + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}(m+w) + \frac{n}{2} - 1} \right) dx_1 \dots dx_{2j+2} dw \\
 & = \frac{1}{2\pi i} \int_C 2^j \pi^{j+1} a^w b^{-j-1} n^{-2j-2} w^{-k-1} \csc \left(\frac{\pi 2^j (m+w)}{n} \right) dw,
 \end{aligned} \tag{3}$$

from Equation (3.326.2) in [11], where $\operatorname{Re} \left(\frac{(m+w)}{n} \right) > 0$, $0 < \operatorname{Re}(m) < 1/2$, $\operatorname{Im}(n) \leq 0$, $0 \leq \operatorname{Im}(a) < 2\pi$, $0 \leq \operatorname{Im}(b) < 2\pi$ —see Figure 1, for example—and using the reflection formula (8.334.3) in [11] for the Gamma function $\Gamma(z)$. We are able to switch the order of integration over $x_1 \dots x_{2j+2}$ and r using Fubini's theorem for multiple integrals—see (9.112) in [12]—since the integrand is of bounded measure over the space $\mathbb{C} \times [0, \infty) \dots \times [0, \infty)$.

3. The Hurwitz–Lerch Zeta Function and Infinite Sum of the Contour Integral

In this section, we use Equation (2) in [1] to derive the contour integral representations for the Hurwitz–Lerch zeta function. The significance of this section is to derive a special function equivalent to the definite integral of the contour integral derived in Section 2 in terms of the same contour integral.

3.1. The Hurwitz–Lerch Zeta Function

The Hurwitz–Lerch zeta function (25.14) in [10,13] has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^n}{(n+v)^s}, \tag{4}$$

where $v \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$; $s \in \mathbb{C}$ when $|z| < 1$; $\operatorname{Re}(s) > 1$ when $|z| = 1$. Here, $\mathbb{Z}_{\leq 0}$ denotes the set of non-positive integers and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt, \tag{5}$$

where $\operatorname{Re}(v) > 0$, and either $|z| \leq 1, z \neq 1, \operatorname{Re}(s) > 0$, or $z = 1, \operatorname{Re}(s) > 1$.

3.2. Infinite Sum of the Contour Integral

Using Equation (2) in [1] and replacing y with

$$\log(a) + \frac{i\pi 2^j (2y+1)}{n},$$

and then multiplying both sides by

$$-i(2\pi)^{j+1} b^{-j-1} n^{-2j-2} e^{\frac{i\pi 2^j m(2y+1)}{n}},$$

and then taking the infinite sum over $y \in [0, \infty)$ and simplifying in terms of the Hurwitz–Lerch zeta function, we obtain

$$\begin{aligned}
 & \frac{ib^{-j-1}(2\pi)^{j+k+1}n^{-2(j+1)}\left(\frac{i2^j}{n}\right)^k e^{\frac{i\pi 2^j m}{n}} \Phi\left(e^{\frac{i2^{j+1}m\pi}{n}}, -k, \frac{\pi - i2^{-j}n \log(a)}{2\pi}\right)}{\Gamma(k+1)} \\
 &= -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C i(2\pi)^{j+1} a^w b^{-j-1} n^{-2(j+1)} w^{-k-1} e^{\frac{i\pi 2^j (2y+1)(m+w)}{n}} dw \\
 &= -\frac{1}{2\pi i} \int_C \sum_{y=0}^{\infty} i(2\pi)^{j+1} a^w b^{-j-1} n^{-2(j+1)} w^{-k-1} e^{\frac{i\pi 2^j (2y+1)(m+w)}{n}} dw \\
 &= \frac{1}{2\pi i} \int_C 2^j \pi^{j+1} a^w b^{-j-1} n^{-2(j+1)} w^{-k-1} \csc\left(\frac{\pi 2^j (m+w)}{n}\right) dw,
 \end{aligned} \tag{6}$$

from Equation (1.232.3) in [11], where $\text{Im}\left(\frac{(m+w)}{n}\right) > 0$ in order for the sum to converge.

4. Definite Integral in Terms of the Hurwitz–Lerch Zeta Function

In this section, we evaluate Equation (7) for various values of the parameters to derive special cases in terms of the dimensionality of the integral and special functions and fundamental constants. The special functions and constants used are the polylogarithm function $Li_k(z)$ given in (25.12.10) in [13], the Riemann zeta function $\zeta(s)$ given in (25.2.1) in [13], Apéry’s constant $\zeta(3)$ given in (25.6.9) in [13] and Catalan’s constant K given in (25.11.40) in [13].

Theorem 1. For $k, a \in \mathbb{C}, 0 < \text{Re}(m) < 1/2, \text{Re}(b) > 0, 0 \leq \text{Im}(b) \leq 2\pi, 0 \leq \text{Im}(a) \leq 2\pi, \text{Im}(n) < 0$,

$$\begin{aligned}
 & \int_0^\infty \dots \int_0^\infty x_1^{m-1} e^{-b(x_1^n + x_2^n)} x_2^{-m+n-1} \log^k \left(\frac{ax_1 \left(\prod_{l=1}^j x_{2l+1}^{2^{l-1}} x_{2l+2}^{2^{l-1}} \right)}{x_2} \right) \\
 & \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}m + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}m + \frac{n}{2} - 1} \right) dx_1 \dots dx_{2j+2} \\
 &= -ib^{-j-1}(2\pi)^{j+k+1}n^{-2(j+1)}\left(\frac{i2^j}{n}\right)^k \\
 & e^{\frac{i\pi 2^j m}{n}} \Phi\left(e^{\frac{i2^{j+1}m\pi}{n}}, -k, \frac{\pi - i2^{-j}n \log(a)}{2\pi}\right).
 \end{aligned} \tag{7}$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, their left-hand sides are also identical. The required result is obtained by simplifying with the Gamma function. $\square$

Remark 1. The degenerate case.

$$\begin{aligned}
 & \int_0^\infty \dots \int_0^\infty x_1^{m-1} e^{-b(x_1^n + x_2^n)} x_2^{-m+n-1} \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}m + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}m + \frac{n}{2} - 1} \right) dx_1 \dots dx_{2j+2} \\
 &= 2^j \pi^{j+1} b^{-j-1} n^{-2(j+1)} \csc\left(\frac{\pi 2^j m}{n}\right).
 \end{aligned} \tag{8}$$

Proof. Use Equation (7) and set $k = 0$ and simplify using entry (2) in table below (64:12:7) in [14]. $\square$

Remark 2. A $2j + 2$ -dimensional integral in terms of the logarithmic and trigonometric functions.

$$\begin{aligned} \int_0^\infty \cdots \int_0^\infty \frac{e^{-b(x_1^n + x_2^n)} x_2^{-m+n-q-1}}{x_1 \log \left(\frac{x_1 \left(\prod_{l=1}^j x_{2l+1}^{2^{l-1}} x_{2l+2}^{-2^{l-1}} \right)}{x_2} \right)} \\ \left(x_2^m x_1^q \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}q + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}q + \frac{n}{2} - 1} \right) \right. \\ \left. - x_1^m x_2^q \left(\prod_{l=1}^j e^{-b(x_{2l+1}^n + x_{2l+2}^n)} x_{2l+1}^{2^{l-1}m + \frac{n}{2} - 1} x_{2l+2}^{-2^{l-1}m + \frac{n}{2} - 1} \right) \right) dx_1 \cdots dx_{2j+2} \\ = 2\pi^j b^{-j-1} n^{-2j-1} \log \left(\cot \left(\frac{\pi 2^{j-1} m}{n} \right) \tan \left(\frac{\pi 2^{j-1} q}{n} \right) \right). \quad (9) \end{aligned}$$

Proof. Use Equation (7) and form a second equation by replacing $m \rightarrow q$ and taking the difference, and then set $k = -1, a = 1$ and simplify using entry (3) in the table below (64:12:7) in [14]. $\square$

Remark 3. A quadruple integral in terms of the logarithmic and trigonometric functions.

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\pi(x_1^3 + x_2^3 + x_3^3 + x_4^3)} (12\sqrt{x_1} x_2^{5/3} x_3^{5/6} \sqrt[6]{x_4} - x_2^{7/4} x_3^{3/4} \sqrt[4]{x_4})}{x_1^{3/4} \log \left(\frac{x_1 x_3}{x_2 x_4} \right)} dx_1 dx_2 dx_3 dx_4 \\ = \frac{\log((2+\sqrt{3}) \tan(\frac{\pi}{9}))}{27\pi}. \quad (10) \end{aligned}$$

Proof. Use Equation (9) and set $b = \pi, m = 1/4, q = 1/3, n = 3, j = 1$ and simplify. $\square$

Remark 4. An octuple integral in terms of the constant $\log(3)$.

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp(-\frac{1}{16}\pi(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4))}{x_1^{3/4} \sqrt[3]{x_8} \log \left(\frac{x_1 x_3 x_5^2 x_7^4}{x_2 x_4 x_6^2 x_8} \right)} \\ \left(12\sqrt{x_1} x_2^{8/3} x_3^{4/3} x_4^{2/3} x_5^{5/3} \sqrt[3]{x_6} x_7^{7/3} - x_2^{11/4} x_3^{5/4} x_4^{3/4} x_5^{3/2} \sqrt{x_6} x_7^2 \sqrt[3]{x_8} \right) dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 dx_7 dx_8 \\ = \frac{2\log(3)}{\pi}. \quad (11) \end{aligned}$$

Proof. Use Equation (9) and set $b = \pi/16, m = 1/4, q = 1/3, n = 4, j = 3$ and simplify. $\square$

Remark 5. The polylogarithm function $Li_k(z)$.

$$\begin{aligned} \int_0^\infty \cdots \int_0^\infty x_1^{2^{-j}-1} x_2^{-2^{-j}+2^j-1} e^{-b(x_1^{2^j} + x_2^{2^j})} \left(\prod_{l=1}^j x_{2l+1}^{2^{-j+l-1}+2^j-1} x_{2l+2}^{-2^{-j+l-1}+2^j-1} e^{-b(x_{2l+1}^{2^j} + x_{2l+2}^{2^j})} \right) \\ \log^k \left(-\frac{x_1 \left(\prod_{l=1}^j x_{2l+1}^{2^{l-1}} x_{2l+2}^{-2^{l-1}} \right)}{x_2} \right) dx_1 \cdots dx_{2j+2} \\ = -i(2^j)^{-2(j+1)} e^{i\pi 2^{-j} - i\pi 2^{j-1}} i^k b^{-j-1} (2\pi)^{j+k+1} Li_{-k}(e^{i2^{1-j}\pi}). \quad (12) \end{aligned}$$

Proof. Use Equation (7) and set $a = -1, m = 1/n$ and simplify using Equation (64:12:2) in [14]. $\square$

Remark 6. A quadruple integral in terms of the Riemann zeta function $\zeta(k)$.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\sqrt{x_2} \sqrt{x_3} e^{-b(x_1^2 + x_2^2 + x_3^2 + x_4^2)} \log^k \left(-\frac{x_1 x_3}{x_2 x_4} \right)}{\sqrt{x_1} \sqrt{x_4}} dx_1 dx_2 dx_3 dx_4 = -\frac{i^k 2^{k-2} (2^{k+1} - 1) \pi^{k+2} \zeta(-k)}{b^2}. \quad (13)$$

Proof. Use Equation (12) and set $j = 1$ and simplify using entry (4) in the table below (25:12:5) in [14]. $\square$

Remark 7. A quadruple integral in terms of the constant $\log(2)$.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{2}\sqrt{\frac{\pi}{2}}(x_1^2+x_2^2+x_3^2+x_4^2)} \sqrt{x_2}\sqrt{x_3}}{\sqrt{x_1}\sqrt{x_4} \log\left(-\frac{x_1x_3}{x_2x_4}\right)} dx_1 dx_2 dx_3 dx_4 = -i \log(2). \quad (14)$$

Proof. Use Equation (13), set $b = \sqrt{\frac{\pi}{8}}$ and apply l'Hopital's rule as $k \rightarrow -1$ and simplify using Equation (25.4.1) in [13]. $\square$

Remark 8. A quadruple integral in terms of Apéry's constant $\zeta(3)$.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{8}\sqrt{\frac{3}{2\pi}}(x_1^2+x_2^2+x_3^2+x_4^2)} \sqrt{x_2}\sqrt{x_3}}{\sqrt{x_1}\sqrt{x_4} \log^3\left(-\frac{x_1x_3}{x_2x_4}\right)} dx_1 dx_2 dx_3 dx_4 = i\zeta(3). \quad (15)$$

Proof. Use Equation (13) and set $k = -3, b = \sqrt{\frac{3}{128\pi}}$ and simplify. $\square$

Remark 9. A sextuple integral in terms of Catalan's constant K .

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{32}\sqrt{\frac{\pi}{3}}(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4+x_6^4)} x_2^{11/4} x_3^{5/4} x_4^{3/4} x_5^{3/2} \sqrt{x_6}}{x_1^{3/4} \log^2\left(-\frac{x_1x_3x_5^2}{x_2x_4x_6}\right)} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \\ &= -\sqrt[4]{-1}(\pi^2 - 48iK). \end{aligned} \quad (16)$$

Proof. Use Equation (12) and set $k = -2, b = \left(\frac{\pi}{98304}\right)^{1/3}, j = 2$ and simplify using Equation (2.2.1.2.7) in [15]. $\square$

5. Conclusions

In this paper, we have presented a novel method for deriving a new j -dimensional integral transform along with some interesting definite integrals, using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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References

1. Reynolds, R.; Stauffer, A. A Double Logarithmic Transform Involving the Exponential and Polynomial Functions Expressed in Terms of the Hurwitz–Lerch Zeta Function. *Symmetry* **2021**, *13*, 1983. [CrossRef]
2. Jain, S.; Mehrez, K.; Baleanu, D.; Agarwal, P. Certain Hermite–Hadamard Inequalities for Logarithmically Convex Functions with Applications. *Mathematics* **2019**, *7*, 163. [CrossRef]
3. Agarwal, P.; Agarwal, R.P.; Ruzhansky, M. *Special Functions and Analysis of Differential Equations*, 1st ed.; Chapman and Hall/CRC: Boca Raton, FL, USA, 9 September 2020.

4. Sekatskii, S.K.; Beltraminelli, S.; Merlini, D. On Equalities Involving Integrals of the Logarithm of the Riemann—Function with Exponential Weight Which Are Equivalent to the Riemann Hypothesis. *Int. J. Anal.* **2015**, *2015*, 980728. [[CrossRef](#)]
5. Wiener, N. The homogeneous chaos. *Am. J. Math.* **1938**, *60*, 897–936. [[CrossRef](#)]
6. Kiyosi, I.T.Ô. Multiple Wiener Integral. *J. Math. Soc. Jpn.* **1951**, *3*, 157–169. [[CrossRef](#)]
7. Olver, F.W.J. *Asymptotics and Special Functions*; A. K. Peters: Wellesley, MA, USA, 1997.
8. Reynolds, R.; Stauffer, A. A Method for Evaluating Definite Integrals in Terms of Special Functions with Examples. *Int. Math. Forum* **2020**, *15*, 235–244. [[CrossRef](#)]
9. Reynolds, R.; Stauffer, A. Definite Integral of Exponential Polynomial and Hyperbolic Function in terms of the Incomplete Gamma Function. *Eur. J. Pure Appl. Math.* **2021**, *14*, 1295–1305. [[CrossRef](#)]
10. Laurincikas, A.; Garunkstis, R. *The Lerch Zeta-Function*; Springer Science & Business Media: Dordrecht, The Netherlands, 11 December 2013; 189p.
11. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: Cambridge, MA, USA, 2000.
12. Harrison, M.; Waldron, P. *Mathematics for Economics and Finance*; Taylor & Francis: Abingdon, UK, 2011.
13. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Digital Library of Mathematical Functions*; With 1 CD-ROM (Windows, Macintosh and UNIX), MR 2723248 (2012a:33001); U.S. Department of Commerce, National Institute of Standards and Technology: Washington, DC, USA; Cambridge University Press: Cambridge, UK, 2010.
14. Oldham, K.B.; Myland, J.C.; Spanier, J. *An Atlas of Functions: With Equator, the Atlas Function Calculator*, 2nd ed.; Springer: New York, NY, USA, 2009.
15. Lewin, L. *Polylogarithms and Associated Functions*; North-Holland Publishing Co.: New York, NY, USA, 1981.