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Spectrality of a Class of Self-Affine Measures with Prime Determinant

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Abstract: We study the spectrality of a class of self-affine measures with prime determinant. Spectral measures are connected with fractal geometry that shows some kind of geometrical self-similarity under magnification. To make the self-affine measure becomes a spectral measure with lattice spectrum, we provide two new sufficient conditions related to the elements of digit set and zero set, respectively. The two sufficient conditions are more precise and easier to be verified as compared with the previous research. Moreover, these conditions offer a fresh perspective on a conjecture of Lagarias and Wang.

Keywords: spectral measure; prime number; digit set; lattice spectrum

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1. Introduction

Let $R \in M_n(\mathbb{Z})$ be an $n \times n$ expanding matrix (i.e., all its eigenvalues have modulus strictly greater than 1) with integer entries. Let $B \subset \mathbb{Z}^n$ be a finite digit set of $|B|$ ($|\cdot|$ denotes the cardinality). Hutchinson [1] proved that for the affine iterated function system (IFS) $\{\phi_b(x) = R^{-1}(x+b)\}_{b\in B}$, there exists a unique invariant probability measure $\mu := \mu_{R,B}$ defined by

$$\mu = \frac{1}{|B|}\sum_{b\in B} \mu \circ \phi_b^{-1}.$$

$\mu_{R,B}$ is also called a *self-affine measure*. Such a measure is supported on the *attractor* $T(R,B) \subset \mathbb{R}^n$ which is the unique compact set that satisfies

$$T(R,B) = \bigcup_{b\in B} \phi_b(T(R,B)).$$

The set $T(R,B)$ is also called the *invariant set* of the IFS $\{\phi_b(x)\}_{b\in B}$, it can be described as:

$$T(R,B) = \left\{ \sum_{j=1}^{\infty} R^{-j} b_j : b_j \in B \right\}.$$

In the present paper, we study the spectrality of a class of self-affine measures $\mu_{R,B}$ when $|\det(R)| = |B| = p$ is a prime. For a probability measure μ with compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^n$, such that $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for Hilbert space $L^2(\mu)$. The set Λ is referred to as a *spectrum* for μ. Particularly, if a spectral measure is the Lebesgue measure μ_L restricted on the compact set $\Omega \subset \mathbb{R}^n$, then we call Ω a *spectral set*.

A spectral measure is a natural generalization of a spectral set. Fuglede [2] conjectured that $\Omega \subset \mathbb{R}^n (0 < \mu_L(\Omega) < \infty)$ is a spectral set if and only if Ω tiles $\mathbb{R}^n$ by translation, which is known as spectrum-tiling conjecture or Fuglede conjecture. We say that Ω *tiles* $\mathbb{R}^n$ by translation if there exists a subset $\mathcal{T}$ (called a *tiling set*) so that

$$\mathbb{R}^n = \bigcup_{t\in\mathcal{T}} (\Omega + t) \quad \text{and} \quad \mu_L(\Omega + t_1) \cap \mu_L(\Omega + t_2) = 0 \ \text{ for all } t_1 \neq t_2 \in \mathcal{T}.$$

Tao [3] demonstrated that the Fuglede conjecture is false by constructing a counterexample in $\mathbb{R}^n$ for $n \geq 5$. After that, Matolcsi [4] obtained that the Fuglede conjecture is invalid for $n \geq 4$ by improving Tao's results. Moreover, Kolountzakis and Matolcsi [5] also disproved the Fuglede conjecture for $n = 3$. In the one dimensional case, Pedersen and Wang [6] proved that if Ω tiles the non-negative half line $\mathbb{R}^+$ by translation, then Ω tiles $\mathbb{R}$ by translation and is a spectral set. However, Fuglede conjecture remains unclear in dimension 1 and 2. As can be seen, it is quite difficult to establish the relation between spectra and tiling. Nevertheless, as it points out the direction for the existence of a spectral measure, a considerable amount of literature has been developed around the theme of spectral measure theory, which is increasingly recognized as a significant subject of research in harmonic analysis (see [7–10] and the references cited therein).

In the extensive work on Fuglede conjecture, the innovative and seminal work of Jorgensen and Pedersen [11] first discovered the existence of a singular, non-atomic spectral measure, which is the Hausdorff measure supported on a 1/4-Cantor set ($R = 4$, $B = \{0, 2\}$). They found that the Fourier transform theory can be applied to certain classes of fractals. The important characteristic of fractals is that the parts are similar to the whole, that is, it shows some kind of geometrical self-similarity under magnification [12], such as the outline of leaves and coastline. The researchers found that harmonic analysis based on fractal sets can be applied to image compression and physics [13,14]. Because of Jorgensen and Pedersen's discovery, more attention has been devoted to finding conditions so that a self-affine measure $\mu_{R,B}$ becomes a spectral measure, as well as finding out the corresponding spectrum of a spectral measure [15–20].

The Fourier transform of $\mu_{R,B}$ is defined as usual,

$$\hat{\mu}_{R,B}(\xi) := \int e^{2\pi i \langle x,\xi \rangle} d\mu_{R,B}(x) = \prod_{j=1}^{\infty} m_B(R^{*-j}\xi) \quad (\xi \in \mathbb{R}^n), \tag{1}$$

where R^* denotes the conjugate matrix of R, and

$$m_B(x) = \frac{1}{|B|} \sum_{b \in B} e^{2\pi i \langle b,x \rangle} \quad (x \in \mathbb{R}^n).$$

It is known that $m_B(x)$ is a $\mathbb{Z}^n$-periodic function for $B \subset \mathbb{Z}^n$. Let $Z(\hat{\mu}_{R,B})$ denote the zero set of $\hat{\mu}_{R,B}$, i.e., $Z(\hat{\mu}_{R,B}) = \{\xi \in \mathbb{R}^n : \hat{\mu}_{R,B}(\xi) = 0\}$. For any $\lambda_1 \neq \lambda_2 \in \Lambda$, the orthogonality condition

$$0 = \langle e^{2\pi i \langle \lambda_1, x \rangle}, e^{2\pi i \langle \lambda_2, x \rangle} \rangle_{L^2(\mu_{R,B})} = \int e^{2\pi i \langle \lambda_1 - \lambda_2, x \rangle} d\mu_{R,B} = \hat{\mu}_{R,B}(\lambda_1 - \lambda_2)$$

is directly related to the zero set $Z(\hat{\mu}_{R,B})$. $E(\Lambda)$ is an orthogonal family of $L^2(\mu_{R,B})$ if and only if $(\Lambda - \Lambda) \setminus \{0\} \subset Z(\hat{\mu}_{R,B})$. It follows from (1) that

$$Z(\hat{\mu}_{R,B}) = \{\xi \in \mathbb{R}^n : \exists\, \alpha \in \mathbb{N} = \{1, 2, \ldots\} \text{ such that } m_B(R^{*-\alpha}\xi) = 0\}. \tag{2}$$

It is well known that a compatible pair is of vital importance when dealing with the spectrality of a self-affine measure. Łaba and Wang [21] proved that compatible pairs generate spectral self-affine measures in dimension one. In the higher dimensional case, for a subset $S \subset \mathbb{Z}^n$ with the same cardinality as B, $0 \in S$, Dutkay and Jorgensen [16] conjectured that $\mu_{R,B}$ is a spectral measure if there exists the set S such that $(R^{-1}B, S)$ is a compatible pair (or (R, B, S) is a Hadamard triple). The conjecture is valid with a few additional conditions (see [16,17,22]). Finally, the conjecture was proved by Dutkay, Haussermann, and Lai [18], and they showed that all compatible pairs generate spectral self-affine measures. In most cases, it is hard to find conditions to ensure that $\mu_{R,B}$ is a spectral measure. In addition, many spectral measures that cannot be obtained from a compatible pair occur in higher dimensions (see [20]). In particular, when $|\det(R)| = |B| = p$ is a prime, Li [23,24] obtained a class of spectral measures with lattice spectrum that cannot be generated by the condition of compatible pair, and showed the following results:

Theorem 1. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Assume that one of the following six conditions holds:*

(a) $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$;
(b) $p(\mathbb{Z}^n \setminus R(\mathbb{Z}^n)) \subseteq R(\mathbb{Z}^n \setminus R(\mathbb{Z}^n))$;
(c) $p\mathbb{Z}^2 \neq R^2(\mathbb{Z}^2)$, in the case when $n = 2$;
(d) $p(\mathbb{Z}^n) \not\subseteq R^{*2}(\mathbb{Z}^n)$;
(e) $p(\mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)) \subseteq R^*(\mathbb{Z}^n \setminus R^*(\mathbb{Z}^n))$;
(f) $p\mathbb{Z}^2 \neq R^{*2}(\mathbb{Z}^2)$, in the case when $n = 2$.

Let $B \subset \mathbb{Z}^n$ be a finite digit set with $0 \in B$, $|B| = p$. If $Z(\hat{\mu}_{R,B}) \cap \mathbb{Z}^n \neq \emptyset$, then there exists $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that $B = R^r\tilde{B}$, where $\tilde{B}$ is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$, thus $\mu_{R,B}$ is a spectral measure with lattice spectrum.

The purpose of the current study is to provide another two new sufficient conditions (see Theorem 3) for Theorem 1 so that $\mu_{R,B}$ is a spectral measure with lattice spectrum. It should be noted that these two conditions are sufficient but not necessary, unlike the condition of compatible pair. Furthermore, because the two sufficient conditions are related to the elements of digit set B and zero set $Z(\hat{\mu}_{R,B})$, respectively, they are more precise and easier to be verified than the conditions (a)–(f) of Theorem 1, which may contribute to our applications of integral self-affine tiles. It is obvious that we extend the previous research.

The plan of this paper is as follows. Section 2 presents the main results and their proofs, as well as provides the Hermite normal forms of R^2 and R^{*2} which cannot satisfy the two sufficient conditions. Finally, Section 3 gives a concluding remark about a conjecture of Lagarias and Wang.

2. Main Results

To investigate the spectrality of $\mu_{R,B}$, one can simplify the digit set with the help of the following lemma.

Lemma 1. *([23]) Let $R \in M_n(\mathbb{Z})$ be an expanding matrix and $B \subset \mathbb{Z}^n$ be a finite digit set. Then, there exist $r \in \mathbb{N}_0$ and a finite set $B_1 \subset \mathbb{Z}^n$ such that*

$$B = R^r B_1 \text{ and } B_1 \not\subset R(\mathbb{Z}^n). \tag{3}$$

According to the results in [23] (pp. 399–400), we know that the spectrality of $\mu_{R,B}$ is the same as μ_{R,B_1}. We first obtain the conditions that μ_{R,B_1} is a spectral measure, and then turn to provide two sufficient conditions (see Theorem 3) that lead to $\mu_{R,B}$ is a spectral measure with lattice spectrum under a more general form.

Theorem 2. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Let $B_1 \subset \mathbb{Z}^n$ be a finite digit set with $0 \in B_1$ and $|B_1| = p$. Suppose that there is a $\tilde{b} \in B_1 \setminus \{0\}$ such that $p\tilde{b} \notin R^2(\mathbb{Z}^n)$. If there is a $l \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^n$, then μ_{R,B_1} is a spectral measure with lattice spectrum.*

Proof. Let $B_1 = \{b_0 = 0, b_1, \ldots, b_{p-1}\}$. From $l \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^n$, we have

$$\hat{\mu}_{R,B_1}(l) = \prod_{j=1}^{\infty} m_{B_1}(R^{*-j}l) = 0. \tag{4}$$

From (2), it follows that there exists a $\alpha \in \mathbb{N}$ $(\alpha = \alpha(l))$, such that $m_{B_1}(R^{*-\alpha}l) = 0$. As $\hat{\mu}_{R,B_1}(0) = 1$, we have $l \in \mathbb{Z}^n \setminus \{0\}$. Since $R^*(\mathbb{Z}^n) \subset \mathbb{Z}^n$, we divide l into two cases: $l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$ and $l \in R^*(\mathbb{Z}^n)$.

Case 1: $l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$. We divide the proof into the following three steps. In step 1, we construct a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$. In step 2, we get two required equations. In step 3, we prove that B_1 is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$.

Step 1. We construct a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$.

For $m_{B_1}(R^{*-\alpha}l) = 0$, a result in [23] (p. 401) tells us that

$$\begin{aligned}&\{0, \langle (R^\dagger)^\alpha b_1, l\rangle, \langle (R^\dagger)^\alpha b_2, l\rangle, \ldots, \langle (R^\dagger)^\alpha b_{p-1}, l\rangle\}\\ &\equiv \{0, p^{\alpha-1}, 2p^{\alpha-1}, \ldots, (p-1)p^{\alpha-1}\} \pmod{p^\alpha},\end{aligned} \tag{5}$$

where $R^\dagger = pR^{-1}$ and $R^\dagger \in M_n(\mathbb{Z})$, we intend to prove $\alpha = 1$ in the final step.

As $l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$ and $(R^\dagger)^* = pR^{*-1}$, we get $(R^\dagger)^* l \in (R^\dagger)^*\mathbb{Z}^n \setminus p(\mathbb{Z}^n)$, thus, there exists a $\vartheta \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle R^\dagger \vartheta, l\rangle \not\equiv 0 \pmod{p} \tag{6}$$

(Otherwise, we have $\langle R^\dagger \vartheta, l\rangle \equiv 0 \pmod{p}$ for any $\vartheta \in \mathbb{Z}^n$. Now we take $\vartheta_1 = (1, 0, \ldots, 0)^t, \ldots, \vartheta_n = (0, \ldots, 0, 1)^t$, which implies that $(R^\dagger)^* l \in p(\mathbb{Z}^n)$, a contradiction). From (6), we get

$$\vartheta \notin R(\mathbb{Z}^n) \tag{7}$$

(If not, there would exist a $\tau \in \mathbb{Z}^n$, such that $\vartheta = R\tau$, then $\langle R^\dagger \vartheta, l\rangle = \langle R^\dagger R\tau, l\rangle = p\langle \tau, l\rangle \in p\mathbb{Z}$, which contradicts with (6)).

The condition $p\tilde{b} \notin R^2(\mathbb{Z}^n)$ implies that $\tilde{b} \notin R(\mathbb{Z}^n)$. In fact, since $pR(\mathbb{Z}^n) = R^2R^\dagger(\mathbb{Z}^n) \subseteq R^2(\mathbb{Z}^n)$, it follows from $p\tilde{b} \notin R^2(\mathbb{Z}^n)$ that $p\tilde{b} \notin pR(\mathbb{Z}^n)$, i.e., $\tilde{b} \notin R(\mathbb{Z}^n)$. From $\tilde{b} \notin R(\mathbb{Z}^n)$ and $R^\dagger R = pI_n$, we have $R^\dagger \tilde{b} \notin p(\mathbb{Z}^n)$. Hence, there exists a $\gamma \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle R^\dagger \tilde{b}, \gamma\rangle \not\equiv 0 \pmod{p}.$$

Let $H = \{h_1, h_2, \ldots, h_{p-1}\}$ with $h_j \equiv j \pmod{p}$, $j = 1, 2, \ldots, p-1$. Since $\gcd(p, h_j) = 1$, it follows that

$$\langle R^\dagger h_j \tilde{b}, \gamma\rangle \not\equiv 0 \pmod{p},$$

which yields $h_j\tilde{b} \notin R(\mathbb{Z}^n)$, $j = 1, 2, \ldots, p-1$. Thus, $\{0, h_1\tilde{b}, h_2\tilde{b}, \cdots, h_{p-1}\tilde{b}\}$ is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$, which implies

$$\mathbb{Z}^n = R(\mathbb{Z}^n) \cup (h_1\tilde{b} + R(\mathbb{Z}^n)) \cup \cdots \cup (h_{p-1}\tilde{b} + R(\mathbb{Z}^n)), \tag{8}$$

where the p sets of the right side are mutually disjoint. Consequently, any $z \in \mathbb{Z}^n$ has a unique representation

$$z = j_z\tilde{b} + R\beta_z, \text{ where } j_z \in \{0, h_1, h_2, \ldots, h_{p-1}\} \text{ and } \beta_z \in \mathbb{Z}^n. \tag{9}$$

Step 2. We obtain two required equations, that is, (12) and (13).

Since $R^\dagger \tilde{b} \in \mathbb{Z}^n$, from (9), we have

$$R^\dagger \tilde{b} = j_1\tilde{b} + R\beta, \text{ where } j_1 \in \{0, h_1, h_2, \ldots, h_{p-1}\} \text{ and } \beta \in \mathbb{Z}^n. \tag{10}$$

As $R^\dagger = pR^{-1}$, we find that the condition $p\tilde{b} \notin R^2(\mathbb{Z}^n)$ is equivalent to

$$R^\dagger \tilde{b} \notin R(\mathbb{Z}^n), \tag{11}$$

which yields $j_1 \neq 0$ in (10). For any positive integer n, using (10) and $j_1 \neq 0$, we get

$$(R^{\dagger})^n \tilde{b} = j_1^n \tilde{b} + R\beta_n, \text{ where } j_1 \in H \text{ and } \beta_n \in \mathbb{Z}^n. \tag{12}$$

Since $\vartheta \in \mathbb{Z}^n \setminus 0$, from (9) and taking (7) into consideration, we have

$$\vartheta = j_\vartheta \tilde{b} + R\beta_\vartheta, \text{ where } j_\vartheta \in H \text{ and } \beta_\vartheta \in \mathbb{Z}^n.$$

From this, together with (6), we deduce

$$\begin{aligned} \langle R^{\dagger}\vartheta, l\rangle &\equiv j_\vartheta \langle R^{\dagger}\tilde{b}, l\rangle \pmod p \\ &\not\equiv 0 \pmod p. \end{aligned}$$

Together with $p \nmid j_\vartheta$, yields

$$\langle R^{\dagger}\tilde{b}, l\rangle \not\equiv 0 \pmod p. \tag{13}$$

Step 3. We prove that B_1 is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$. It follows from (12) that

$$\begin{aligned} \langle (R^{\dagger})^n \tilde{b}, l\rangle &= j_1^{n-1}\langle R^{\dagger}\tilde{b}, l\rangle + p\langle \beta_{n-1}, l\rangle \\ &\equiv j_1^{n-1}\langle R^{\dagger}\tilde{b}, l\rangle \pmod p \\ &\not\equiv 0 \pmod p \text{ (by (13))}. \end{aligned} \tag{14}$$

Applying (14) to (5), we obtain that $\alpha = 1$. Hence, (5) becomes

$$\begin{aligned} &\{0, \langle R^{\dagger}b_1, l\rangle, \langle R^{\dagger}b_2, l\rangle, \ldots, \langle R^{\dagger}b_{p-1}, l\rangle\} \\ &\equiv \{0, 1, 2, \ldots, (p-1)\} \pmod p. \end{aligned} \tag{15}$$

It follows that B_1 is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$. If not, there would exist $\zeta \in \mathbb{Z}^n$ and $b_{q_1}, b_{q_2} \in B_1$ such that $b_{q_1} - b_{q_2} = R\zeta$, then $\langle R^{\dagger}(b_{q_1} - b_{q_2}), l\rangle = \langle R^{\dagger}R\zeta, l\rangle = p\langle \zeta, l\rangle \in p\mathbb{Z}$, which contradicts with (15).

Case 2: $l \in R^*(\mathbb{Z}^n)$. In this case, a result in [23] (p. 402) tells us that there exist a $t \in \mathbb{N}_0$ and a $\hat{l} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$l = R^{*t}\hat{l} \text{ and } \hat{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n). \tag{16}$$

Because of $m_B(x) = 1$ for any $x \in \mathbb{Z}^n$, it follows from (4) that

$$\begin{aligned} \hat{\mu}_{R,B_1}(l) &= \prod_{j=1}^{\infty} m_{B_1}(R^{*-j}l) = \prod_{j=1}^{\infty} m_{B_1}(R^{*(t-j)}\hat{l}) = \prod_{j=1}^{\infty} m_{B_1}(R^{*-j}\hat{l}) \\ &= \hat{\mu}_{R,B_1}(\hat{l}) = 0. \end{aligned}$$

The case $\hat{l}$ is similar to the l of Case 1. Therefore, we obtain that B_1 is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$.

Up to now, we have completely showed that B_1 is a complete set of coset representatives of $\mathbb{Z}^n/R(\mathbb{Z}^n)$. It is time to conclude that μ_{R,B_1} is a spectral measure with some lattice spectrum Γ^* (see [23] (p. 403)). This completes the proof of Theorem 2. □

With the same notations above, for the pair (R, B), we now turn to prove that $\mu_{R,B}$ is a spectral measure with lattice spectrum.

Theorem 3. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Let $B \subset \mathbb{Z}^n$ be a finite digit set with $0 \in B$ and $|B| = p$. Suppose that one of the following two conditions (at least one) holds:*

(i) There is a $b_r \in B \setminus \{0\}$ such that $pb_r \notin R^{r+2}(\mathbb{Z}^n)$, where r is given by (3);
(ii) There is a $l \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^n$ such that $pl \notin R^{*t+2}(\mathbb{Z}^n)$, where t is given by (16).

If there is a $\lambda \in Z(\hat{\mu}_{R,B}) \cap R^{*-\kappa}(\mathbb{Z}^n)$ for some κ satisfying $0 \le \kappa \le r+t$, then $\mu_{R,B}$ is a spectral measure with lattice spectrum.

Proof. From Lemma 1, we have $B = R^r B_1$ with $B_1 = \{b_0 = 0, b_1, \ldots, b_{p-1}\} \subset \mathbb{Z}^n$. Since $\lambda \in R^{*-\kappa}(\mathbb{Z}^n)$, we may assume that there exists a $\tilde{l} \in \mathbb{Z}^n \setminus \{0\}$ such that $\lambda = R^{*-\kappa}\tilde{l}$. From $\lambda \in Z(\hat{\mu}_{R,B})$, if $0 \le \kappa \le r$, then

$$\hat{\mu}_{R,B}(R^{*-\kappa}\tilde{l}) = \hat{\mu}_{R,B_1}(R^{*(r-\kappa)}\tilde{l}) = \hat{\mu}_{R,B_1}(\tilde{l}) = 0.$$

Obviously, we have $\tilde{l} \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^n$, take $\tilde{l} = l$, then it is the same as in Theorem 2. If $\tilde{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, we take $0 \le \kappa \le r$. If $\tilde{l} \in R^*(\mathbb{Z}^n)$, we take $0 \le \kappa \le r+t$.

(i) It is enough to show that the condition (i) guarantees $j_1 \neq 0$ in (10).

If $b_r \in B \setminus \{0\}$, as $B = R^r B_1$, then there exists a $\tilde{b}_r \in B_1 \setminus \{0\}$ such that $b_r = R^r \tilde{b}_r$ and $\tilde{b}_r \notin R(\mathbb{Z}^n)$. Hence, $pb_r \notin R^{r+2}(\mathbb{Z}^n)$ is equivalent to

$$R^{\dagger}\tilde{b}_r \notin R(\mathbb{Z}^n), \tag{17}$$

(17) plays the same role as (11), which guarantees $j_1 \neq 0$ in (10). Thus, B_1 is a complete set of coset representatives of $\mathbb{Z}^n / R(\mathbb{Z}^n)$.

(ii) Since $l \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^n$ with $l = R^{*t}\hat{l}$ and $\hat{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, together with $(R^{\dagger})^* = pR^{*-1}$, we obtain that $pl \notin R^{*t+2}\mathbb{Z}^n$ is equivalent to

$$(R^{\dagger})^*\hat{l} \notin R^*(\mathbb{Z}^n). \tag{18}$$

Firstly, we construct a complete set of coset representatives of $\mathbb{Z}^n / R^*(\mathbb{Z}^n)$. Secondly, we give two required equations, that is, (20) and (22). Finally, we prove that B_1 is a complete set of coset representatives of $\mathbb{Z}^n / R(\mathbb{Z}^n)$.

Since $\hat{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, we have $(R^{\dagger})^*\hat{l} \in (R^{\dagger})^*\mathbb{Z}^n \setminus p(\mathbb{Z}^n)$. Hence, there exists a $\sigma \in \mathbb{Z}^n \setminus \{0\}$ such that $\langle (R^{\dagger})^*\hat{l}, \sigma \rangle \not\equiv 0 \pmod{p}$. Furthermore, we have $\langle (R^{\dagger})^* h\hat{l}, \sigma \rangle \not\equiv 0 \pmod{p}$ for any $h \in H$. This implies $h\hat{l} \notin R^*(\mathbb{Z}^n)$ for any $h \in H$. It follows that $\{0, h_1\hat{l}, h_2\hat{l}, \cdots, h_{p-1}\hat{l}\}$ is a complete set of coset representatives of $\mathbb{Z}^n / R^*(\mathbb{Z}^n)$, which indicates:

$$\mathbb{Z}^n = R^*(\mathbb{Z}^n) \cup (h_1\hat{l} + R^*(\mathbb{Z}^n)) \cup \cdots \cup (h_{p-1}\hat{l} + R^*(\mathbb{Z}^n)), \tag{19}$$

where the p sets of the right side are mutually disjoint. Since $(R^{\dagger})^*\hat{l} \in \mathbb{Z}^n$, it follows from (19) that

$$(R^{\dagger})^*\hat{l} = j_2\hat{l} + R^*\eta, \text{ where } j_2 \in \{0, h_1, h_2, \ldots, h_{p-1}\} \text{ and } \eta \in \mathbb{Z}^n. \tag{20}$$

This, together with (18), yields $j_2 \neq 0$.

As $B_1 \not\subset R(\mathbb{Z}^n)$, there exists a $\hat{b}_r \in B_1 \setminus \{0\}$ such that $\hat{b}_r \notin R(\mathbb{Z}^n)$, thus, $R^{\dagger}\hat{b}_r \notin p(\mathbb{Z}^n)$, then there exists a $\omega \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle (R^{\dagger})^*\omega, \hat{b}_r \rangle \not\equiv 0 \pmod{p}, \tag{21}$$

which implies $\omega \notin R^*(\mathbb{Z}^n)$, hence

$$\omega = j_\omega\hat{l} + R^*\eta_\omega, \text{ where } j_\omega \in H \text{ and } \eta_\omega \in \mathbb{Z}^n.$$

This together with (21), we get

$$\begin{aligned} \langle (R^{\dagger})^*\omega, \hat{b}_r \rangle &\equiv j_\omega \langle (R^{\dagger})^*\hat{l}, \hat{b}_r \rangle \pmod{p} \\ &\not\equiv 0 \pmod{p}. \end{aligned}$$

Since $p \nmid j_\omega$, we have

$$\langle (R^\dagger)^* \hat{l}, \hat{b}_r \rangle \not\equiv 0 \ (\mathrm{mod}\ p). \tag{22}$$

From (20) and (22), we deduce $\langle (R^\dagger)^{*n} \hat{l}, \hat{b}_r \rangle \not\equiv 0 \ (\mathrm{mod}\ p)$, that is, (14) holds. Hence, B_1 is a complete set of coset representatives of $\mathbb{Z}^n / R(\mathbb{Z}^n)$.

In conclusion, we have proved that B_1 is a complete set of coset representatives of $\mathbb{Z}^n / R(\mathbb{Z}^n)$. Therefore, we obtain that $\mu_{R,B}$ is a spectral measure with some lattice spectrum $(R^*)^{-r}\Gamma^*$. This completes the proof of Theorem 3. □

Remark 1. *(i) The condition $pb_r \notin R^{r+2}(\mathbb{Z}^n)$ is equivalent to*

$$pb_r \in R^{r+1}(\mathbb{Z}^n \setminus R(\mathbb{Z}^n)). \tag{23}$$

In fact, it is enough to show that (23) guarantees $j_1 \neq 0$ in (10). From $b_r = R^r \tilde{b}_r$, $\tilde{b}_r \notin R(\mathbb{Z}^n)$, and $R^\dagger = pR^{-1}$, we find that (23) is equivalent to $R^\dagger \tilde{b}_r \in \mathbb{Z}^n \setminus R(\mathbb{Z}^n)$, together with (8) yields $j_1 \neq 0$ in (10). If $r = 0$, then $pb_r \notin R^{r+2}(\mathbb{Z}^n)$ is reduced to $p\tilde{b} \notin R^2(\mathbb{Z}^n)$, which is equivalent to $p\tilde{b} \in R(\mathbb{Z}^n \setminus R(\mathbb{Z}^n))$. Note that $p\tilde{b} \notin R^2(\mathbb{Z}^n)$ and $p\tilde{b} \in R(\mathbb{Z}^n \setminus R(\mathbb{Z}^n))$ imply the condition (a) and (b) of Theorem 1, respectively, not vice versa, and they are more precise and easier to be verified than the condition (a) and (b) of Theorem 1.

(ii) The condition $pl \notin R^{*t+2}(\mathbb{Z}^n)$ is equivalent to

$$pl \in R^{*t+1}(\mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)). \tag{24}$$

In fact, it is enough to show that (24) guarantees $j_2 \neq 0$ in (20). In view of $l = R^{*t}\hat{l}$, $\hat{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, and $(R^\dagger)^* = pR^{*-1}$, we obtain that (24) is equivalent to $(R^\dagger)^*\hat{l} \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, it follows from (19) that $j_2 \neq 0$ in (20). If $t = 0$, then $pl \notin R^{*t+2}(\mathbb{Z}^n)$ is reduced to $pl \notin R^{*2}(\mathbb{Z}^n)$, which is equivalent to $pl \in R^*(\mathbb{Z}^n \setminus R^*(\mathbb{Z}^n))$. Note that $pl \notin R^{*2}(\mathbb{Z}^n)$ and $pl \in R^*(\mathbb{Z}^n \setminus R^*(\mathbb{Z}^n))$ imply the condition (d) and (e) of Theorem 1, respectively, not vice versa, and they are more precise and easier to be verified.

(iii) The condition $Z(\hat{\mu}_{R,B}) \cap \mathbb{Z}^n \neq \emptyset$ of Theorem 1 can be substituted by a more general condition $Z(\hat{\mu}_{R,B}) \cap \mathbb{R}^n \neq \emptyset$.

In Theorems 2 and 3, we only consider the case of $|\det(R)| = p$ is a prime, which raises an interesting question: can the method in Theorems 2 and 3 deal with the case of a real symmetric matrix R with $|\det(R)| = pq$ (p, q are distinct primes)?

By Theorem 3, for B and l, we may assume that $B \not\subset R(\mathbb{Z}^n)$ and $l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$, that is, $r = 0$ and $t = 0$. From now on, we always assume $B \not\subset R(\mathbb{Z}^n)$ and $l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n)$.

In dimension one, that is, $n = 1$, we must point out that the conditions (i) and (ii) of Theorem 3 are always hold. In addition, for $n \geq 2$ we find out the Hermite normal forms of R^2 and R^{*2} which cannot satisfy the condition (i) and (ii), respectively. Domich et al. provided the following result (see [25] Theorem 1.2).

Proposition 1. *([25]) Let $A \in M_n(\mathbb{Z})$ be a nonsingular integer matrix. Then there exists a $n \times n$ unimodular matrix U such that $AU = H$, where H is called the Hermite normal form of A, whose entries satisfy*

(1) $h_{ij} = 0$, for any $j > i$;
(2) $h_{ii} > 0$, for any i;
(3) $h_{ij} \leq 0$ and $| h_{ij} | < h_{ii}$, for any $j < i$.

Obviously, we see that H is a lower-triangular matrix. By Proposition 1 and $\det(R^2) = \det(R^{*2}) = p^2$ is a prime power, we find that the Hermite normal forms of R^2 and R^{*2} are the following two cases.

$$(1)\ H_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ h_{i,1} & h_{i,2} & \cdots & h_{i,i-1} & p^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $h_{i,j} \leq 0$ and $\mid h_{i,j} \mid < p^2$, for any $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, i-1$. In particular, $H_1 = \mathrm{diag}(p^2, 1, \ldots, 1)$ is a diagonal matrix for $i = 1$.

$$(2)\ H_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ h_{i,1} & h_{i,2} & \cdots & h_{i,i-1} & p & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ h_{j,1} & h_{j,2} & \cdots & h_{j,i-1} & h_{j,i} & \cdots & h_{j,j-1} & p & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $j > i$, $i, j = 1, 2, \ldots, n$, $h_{i,s} \leq 0$ and $\mid h_{i,s} \mid < p$ for any $s \in \{1, 2, \ldots, i-1\}$. $h_{j,t} \leq 0$ and $\mid h_{j,t} \mid < p$ for any $t \in \{1, 2, \ldots, j-1\}$.

In view of the above two Hermite normal forms of R^2 and R^{*2}, we obtain the forms of $\tilde{b}$ that does not satisfy the condition $p\tilde{b} \notin R^2(\mathbb{Z}^n)$ and l that does not satisfy the condition $pl \notin R^{*2}(\mathbb{Z}^n)$, respectively.

Proposition 2. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Assume that the Hermite normal form of R^2 is H_1. Suppose that $B \subset \mathbb{Z}^n$ is a finite digit set with $0 \in B$. For any nonzero element $\tilde{b} = (b_1, b_2, \ldots, b_n)^t \in B$, if $p \mid b_i - \sum\limits_{s=1}^{i-1} h_{i,s} b_s$, then $p\tilde{b} \in R^2(\mathbb{Z}^n)$, where i is given by H_1.*

Proof. As the Hermite normal form of R^2 is H_1, it follows that there exists a unimodular matrix U_1 such that $R^2 U_1 = H_1$. Since $U_1(\mathbb{Z}^n) = \mathbb{Z}^n$, we have $R^2(\mathbb{Z}^n) = H_1(\mathbb{Z}^n)$. Hence, it suffices to prove $p\tilde{b} \in H_1(\mathbb{Z}^n)$.

Let $k = (k_1, k_2, \ldots, k_n)^t$, where

$$k_1 = pb_1,\ \ldots,\ k_{i-1} = pb_{i-1},\ k_i = \frac{1}{p}(b_i - \sum_{s=1}^{i-1} h_{i,s} b_s), k_{i+1} = pb_{i+1},\ \ldots,\ k_n = pb_n.$$

Since $p \mid b_i - \sum\limits_{s=1}^{i-1} h_{i,s} b_s$, we have $k_i \in \mathbb{Z}$, then $k \in \mathbb{Z}^n$, thus

$$\begin{aligned} H_1 k &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ h_{i,1} & h_{i,2} & \cdots & h_{i,i-1} & p^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} pb_1 \\ pb_2 \\ \vdots \\ pb_{i-1} \\ \frac{1}{p}(b_i - \sum\limits_{s=1}^{i-1} h_{i,s} b_s) \\ \vdots \\ pb_{n-1} \\ pb_n \end{bmatrix} \\ &= (pb_1, pb_2, \ldots, pb_{i-1}, pb_i, \ldots, pb_{n-1}, pb_n)^t \\ &= p\tilde{b}. \end{aligned}$$

Therefore, $p\tilde{b} \in H_1(\mathbb{Z}^n)$. □

Proposition 3. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Assume that the Hermite normal form of R^2 is H_2. Suppose that $B \subset \mathbb{Z}^n$ is a finite digit set with $0 \in B$. For any nonzero element $\tilde{b} = (b_1, b_2, \ldots, b_n)^t \in B$, if $p \mid h_{j,i}(b_i - \sum\limits_{s=1}^{i-1} h_{i,s} b_s)$, then $p\tilde{b} \in R^2(\mathbb{Z}^n)$, where i and j are given by H_2.*

Proof. As the Hermite normal form of R^2 is H_2, it follows that there exists a unimodular matrix U_2 such that $R^2 U_2 = H_2$. Since $U_2(\mathbb{Z}^n) = \mathbb{Z}^n$, we have $R^2(\mathbb{Z}^n) = H_2(\mathbb{Z}^n)$. It suffices to prove $p\tilde{b} \in H_2(\mathbb{Z}^n)$.

Let $k = (k_1, k_2, \ldots, k_n)^t$, where

$$\begin{aligned} &k_1 = pb_1,\ \ldots,\ k_{i-1} = pb_{i-1},\ k_i = b_i - \sum_{s=1}^{i-1} h_{i,s} b_s,\ k_{i+1} = pb_{i+1},\ \ldots, \\ &k_{j-1} = pb_{j-1},\ k_j = b_j - \sum_{s=1}^{i-1} h_{j,s} b_s - \frac{1}{p} h_{j,i}(b_i - \sum_{s=1}^{i-1} h_{i,s} b_s) - \sum_{s=i+1}^{j-1} h_{j,s} b_s, \\ &k_{j+1} = pb_{j+1},\ \ldots,\ k_n = pb_n. \end{aligned}$$

Since $p \mid h_{j,i}(b_i - \sum\limits_{s=1}^{i-1} h_{i,s} b_s)$, we have $k_j \in \mathbb{Z}$, then $k \in \mathbb{Z}^n$. By calculation, we obtain $H_2 k = p\tilde{b}$. Therefore, $p\tilde{b} \in H_2(\mathbb{Z}^n)$. □

Similar to the above $\tilde{b}$, we give the following two propositions about l.

Proposition 4. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Assume that the Hermite normal form of R^{*2} is H_1. Suppose that $B \subset \mathbb{Z}^n$ is a finite digit set with $0 \in B$. For any $l = (l_1, l_2, \ldots, l_n)^t \in Z(\hat{\mu}_{R,B}) \cap \mathbb{Z}^n$, if $p \mid l_i - \sum\limits_{s=1}^{i-1} h_{i,s} l_s$, then $pl \in R^{*2}(\mathbb{Z}^n)$, where i is given by H_1.*

Proposition 5. *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Assume that the Hermite normal form of R^{*2} is H_2. Suppose that $B \subset \mathbb{Z}^n$ is a finite digit set with $0 \in B$. For any $l = (l_1, l_2, \ldots, l_n)^t \in Z(\hat{\mu}_{R,B}) \cap \mathbb{Z}^n$, if $p \mid h_{j,i}(l_i - \sum\limits_{s=1}^{i-1} h_{i,s} l_s)$, then $pl \in R^{*2}(\mathbb{Z}^n)$, where i and j are given by H_2.*

Particularly, if $n = 2$, then the Hermite normal form of R^2 and R^{*2} are the following three cases:

$$H_1 = \begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix}, \ \hat{H}_1 = \begin{bmatrix} 1 & 0 \\ a_{21} & p^2 \end{bmatrix} \text{ and } \tilde{H}_2 = \begin{bmatrix} p & 0 \\ h_{21} & p \end{bmatrix},$$

where $a_{21} \leq 0$ and $\mid a_{21} \mid < p^2$, $h_{21} \leq 0$ and $\mid h_{21} \mid < p$.

Remark 2. *For any $z = (z_1, z_2)^t \in \mathbb{Z}^2 \setminus \{0\}$, it follows from Proposition 2 that $pz \in R^2(\mathbb{Z}^2)$ if $p \mid z_1$ or $p \mid z_2 - a_{21}z_1$. It follows from Proposition 3 that $pz \in R^2(\mathbb{Z}^2)$ if $p \mid h_{21}z_1$. Hence, we have $p\mathbb{Z}^2 \neq R^2(\mathbb{Z}^2)$. Similarly, we obtain $p\mathbb{Z}^2 \neq R^{*2}(\mathbb{Z}^2)$. Therefore, the condition (c) and (f) hold in Theorem 1.*

In summary, we have obtained two sufficient conditions, such that $\mu_{R,B}$ is a spectral measure with lattice spectrum in Theorem 3. Let us illustrate the differences between Theorem 3 and Theorem 1 with the following example.

Example 1. *Let*

$$R = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}. \tag{25}$$

We have $B = R^2 B_1$, where

$$B_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Since $B = R^2 B_1$, take $\tilde{b} = (1,0)^t \in B_1$ (or $b_r = (3,3)^t \in B$), we check that $p\tilde{b} \notin R^2(\mathbb{Z}^2)$ (or $pb_r \notin R^4(\mathbb{Z}^2)$). If $\kappa = 3$, then there exists a $\tilde{l} = (6,3)^t \in \mathbb{Z}^2$ such that $\lambda = (-\frac{2}{9}, \frac{5}{9})^t \in Z(\hat{\mu}_{R,B}) \cap R^{*-3}(\mathbb{Z}^2)$ (i.e., $Z(\hat{\mu}_{R,B}) \cap \mathbb{R}^2 \neq \emptyset$). It follows from Theorem 3 that $\mu_{R,B}$ is a spectral measure with lattice spectrum. By the set $\mathbb{Z}^2 \setminus R(\mathbb{Z}^2)$ in [23] (p. 405), it follows that $p\tilde{b} \in R(\mathbb{Z}^2 \setminus R(\mathbb{Z}^2))$ (or $pb_r \in R^3(\mathbb{Z}^n \setminus R(\mathbb{Z}^n))$).

If $\kappa = 3$, then there is a $l = \tilde{l} = (6,3)^t = R^*(1,2)^t \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^2$, such that $pl \notin R^{*3}(\mathbb{Z}^2)$, and $\lambda = (-\frac{2}{9}, \frac{5}{9})^t \in Z(\hat{\mu}_{R,B}) \cap R^{*-3}(\mathbb{Z}^2)$. From Theorem 3, we find that $\mu_{R,B}$ is a spectral measure with lattice spectrum. For the expanding matrix R^*, we can check that

$$\mathbb{Z}^2 \setminus R^*(\mathbb{Z}^2) = \left\{ \begin{pmatrix} 3k_1 + 1 \\ k_1 + k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \bigcup \left\{ \begin{pmatrix} 3k_1 + 2 \\ k_1 + k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\}.$$

It is straightforward to verify that $pl \in R^{*2}(\mathbb{Z}^2 \setminus R^*(\mathbb{Z}^2))$.

For the pair (R, B) in (25), it should be pointed out that we cannot find out a $S \subset \mathbb{Z}^2$ such that $(R^{-1}B, S)$ is a compatible pair. In fact, Example 1 is an example in [23] (p. 405). For the pair (25), it is not easy to verify the inclusion relation between the sets of the condition (b), as well as the condition (e) of Theorem 1. For the given (R, B), a major advantage of the conditions we have obtained in this paper is that they are more precise and easier to be verified than the conditions of Theorem A. We need only choose an element from the digit set B and zero set $Z(\hat{\mu}_{R,B})$, respectively, such that the corresponding conditions are valid.

The following example demonstrates that the two conditions (i) and (ii) of Theorem 3 are sufficient but not necessary.

Example 2. *Let*

$$R = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

We have $B = RB_1$, where

$$B_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

If $\kappa = 1$, then there exists a $\tilde{l} = (1,1)^t \in \mathbb{Z}^2$, such that $\lambda = (1, \frac{1}{3})^t \in Z(\hat{\mu}_{R,B}) \cap R^{*-1}(\mathbb{Z}^2)$. Since B_1 is a complete set of coset representatives of $\mathbb{Z}^2/R(\mathbb{Z}^2)$, it follows that $\mu_{R,B}$ is a spectral measure with lattice spectrum $(R^*)^{-1}\Gamma^*$. However, for any $b_r \in B$, we have $pb_r \in R^3(\mathbb{Z}^2)$, and we also have $pl \in R^{*2}(\mathbb{Z}^2)$ for any $l \in Z(\hat{\mu}_{R,B_1}) \cap \mathbb{Z}^2$, we see that both of the conditions (i) and (ii) of Theorem 3 fail, thus, the two conditions (i) and (ii) of Theorem 3 are sufficient but not necessary.

3. Concluding Remarks

We observe that the two sufficient conditions (i)–(ii) of Theorem 3 are closely relevant to a conjecture of Lagarias and Wang in [26]. In the end, we give a remark about these two sufficient conditions applied to integral self-affine tiles.

Suppose that $R \in M_n(\mathbb{Z})$ is an expanding matrix, $B \subset \mathbb{Z}^n$ is a finite digit set, $0 \in B$ and $|B| = |\det(R)|$. If $\mu_L(T(R,B)) > 0$, then $T(R,B)$ is defined as an *integer self-affine tile* and the corresponding B is referred to as a *tile digit set* (with respect to R). $\mathbb{Z}[R,B]$ denotes the smallest $R-$ invariant sublattice of $\mathbb{Z}^n$ containing B. Lagarias and Wang [26] provided the following useful fact.

Proposition 6. *([26]) Suppose that the columns of a matrix $P \in M_n(\mathbb{Z})$ are a basis of $\mathbb{Z}[R,B]$, i.e., $\mathbb{Z}[R,B] = P(\mathbb{Z}^n)$, then there exists a matrix $R_0 := P^{-1}RP \in M_n(\mathbb{Z})$ and a digit set $B_0 := P^{-1}B \subset \mathbb{Z}^n$, such that $\mathbb{Z}[R_0,B_0] = \mathbb{Z}^n$, $0 \in B_0$, and $T(R,B) = P(T(R_0,B_0))$.*

If B_0 is a complete set of coset representatives of $\mathbb{Z}^n/R_0(\mathbb{Z}^n)$, then B is called a standard digit set (with respect to R). In this case, for the pair (R_0,B_0), we have $\mu_L(T(R_0,B_0)) > 0$ (see [27]). For a standard digit set, Lagarias and Wang [26] Theorem 4.1 proved the following result.

Theorem 4. *([26]) Let $R \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(R)| = p$ is a prime. Suppose that $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$ and $B \subset \mathbb{Z}^n$ is a digit set with $|B| = p$. Then $\mu_L(T(R,B)) > 0$ if and only if B is a standard digit set.*

The following conjecture was formulated in [26] by Lagarias and Wang.

Conjecture 1. *The condition $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$ in Theorem 4 is redundant.*

In recent decades, considerable interest about Conjecture 1 has developed. A paper [28] by He and Lau showed that $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$ can be substituted by span(B) = $\mathbb{R}^n$. Li [24] proved that $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$ can be substituted by any one of the conditions (a)–(f) of Theorem 1. Since the two sufficient conditions (i) and (ii) of Theorem 3 are supplementary to Theorem 1, we find that $p(\mathbb{Z}^n) \not\subseteq R^2(\mathbb{Z}^n)$ can be substituted by any one of the conditions (i) and (ii) of Theorem 3.

Remark 3. *The two sufficient conditions (i) and (ii) of Theorem 3 shed new light on the Conjecture 1. To further research on it, it suffices to consider the following two cases:*

$$(\tilde{i})\ \ p\tilde{b} \in R^2(\mathbb{Z}^n) \text{ for any } \tilde{b} \in B,$$

$$(\tilde{ii})\ \ pl \in R^{*2}(\mathbb{Z}^n) \text{ for any } l \in \mathbb{Z}^n \setminus R^*(\mathbb{Z}^n).$$

Furthermore, we only need to consider $\tilde{b}$ in Propositions 2 and 3, and l in Propositions 4 and 5.

Obviously, Remark 3 shows clearly the cases to be resolved. The Propositions 2–5 we obtained provide new insights into the Conjecture 1 and generalize the related results.

Finally, we would like to point out that we only consider the spectrality of self-affine measures $\mu_{R,B}$ with $|\det(R)| = p$ is a prime, however, the idea and the method which we used in Section 2 may be also suitable to a real symmetric matrix R with $|\det(R)| = pq$, where p, q are distinct primes. Next, we will further focus on the spectrality of self-affine measures $\mu_{R,B}$ with the real symmetric matrix, as we know, real symmetric matrices play an important role in quantum mechanics and engineering technology.

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