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A Constrained Shepard Type Operator for Modeling and Visualization of Scattered Data †

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Abstract: For solving the problem of modeling and visualization of scattered data that should preserve some constraints, we use a modified Shepard type operator that is required to fulfill some special conditions, highlighting the symmetry with other methods. We illustrate the properties of the obtained operators by some numerical examples.

Keywords: shepard operator; scattered data; constrained interpolation

MSC: 41A29; 41A05; 41A25; 41A35

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1. Introduction

Some of the most important interpolation methods for large scattered data sets are the Shepard type methods. The problem of modeling and visualization of scattered data that should preserve some constraints appears in many scientific areas, e.g., when the data should satisfy lower and upper bounds, due to various constraints (economical, physical, socio-political, chemical, etc. [1]). For example, there are cases when the data have to preserve some constraints, subject to certain physical laws (e.g., the densities, percentage mass concentrations in a chemical reaction, volume and mass, see [2,3]). Such problems require to impose some special conditions to the interpolants (see, e.g., [1–4]).

The purpose of the paper is to impose some constraints to Shepard-Bernoulli operator, introduced in [5], and to enforce it to satisfy them using a symmetrical way with the method described in [1]. First, we recall some results regarding Shepard-Bernoulli interpolation, studied in [5–7].

Consider the function $f \in C^{(m,n)}(X)$, $X = [a,b] \times [c,d]$ and a set of N distinct points $(x_i, y_i) \in X,\ i = 1, ..., N$. The bivariate Shepard operator (introduced in [8]) is given by

$$Sf(x,y) = \sum_{i=1}^{N} A_i(x,y) f(x_i, y_i), \tag{1}$$

where

$$A_i(x,y) = \frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{N} r_j^{\mu}(x,y)}{\sum\limits_{k=1}^{N} \prod\limits_{\substack{j=1 \\ j \neq k}}^{N} r_j^{\mu}(x,y)}, \tag{2}$$

with $\mu > 0$ and $r_i(x,y)$ are the distances between (x,y) and the given points (x_i, y_i), $i = 1, ..., N$. The parameter μ influences the behavior of Sf in the neighborhood of the nodes. If $0 < \mu \leq 1$ then Sf has peaks at the nodes. For $\mu > 1$ then Sf has flat spots and if μ is large enough Sf becomes a step function.

Proposition 1. *The following properties hold:*

1. $A_i(x_k, y_k) = \delta_{ik}$, $i, k = 1, ..., N$.
2. *degree of exactness of* S *is* 0 $(dex(S) = 0)$.

Shepard interpolation leads to flat spots at each data point and the accuracy tends to decrease in the areas where the interpolation nodes are sparse. This can be improved using the local version of Shepard interpolation, introduced by Franke and Nielson in [9] and improved in [10–12]:

$$Sf(x,y) = \frac{\sum\limits_{i=1}^{N} W_i(x,y) f(x_i, y_i)}{\sum\limits_{i=1}^{N} W_i(x,y)}, \tag{3}$$

with

$$W_i(x,y) = \left[\frac{(R_w - r_i(x,y))_+}{R_w r_i(x,y)}\right]^2, \tag{4}$$

where R_w is a radius of influence about the node (x_i, y_i) and it is varying with i. This is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g., [11]).

The Bernoulli polynomials are defined by (see, e.g., [13]):

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x), & n \geq 1, \\ \displaystyle\int_0^1 B_n(x)dx = 0, & n \geq 1. \end{cases} \tag{5}$$

The values of $B_n(x)$ at $x = 0$ are the Bernoulli numbers and they are denoted by B_n. For $f \in C^m[a,b]$, the univariate Bernoulli interpolant is given by

$$B_m f(x) := B_m[f; a, b] = f(a) + \sum_{i=1}^{m} S_i\left(\tfrac{x-a}{h}\right) \tfrac{h^{i-1}}{i!} \Delta_h f^{(i-1)}(a), \tag{6}$$

where $h = b - a$ and

$$\begin{aligned} &S_i\left(\tfrac{x-a}{h}\right) = B_i\left(\tfrac{x-a}{h}\right) - B_i, \qquad i \geq 1, \\ &\Delta_h f^{(i-1)}(a) = f^{(i-1)}(b) - f^{(i-1)}(a), \;\; 1 \leq i \leq m. \end{aligned} \tag{7}$$

Denote $h := b - a$, $k := d - c$ and consider the operators:

$$\begin{aligned} \Delta_{(h,0)} f(x,y) &:= f(x+h, y) - f(x,y), \\ \Delta_{(0,k)} f(x,y) &:= f(x, y+k) - f(x,y), \\ \Delta_{(h,k)} f(x,y) &:= \Delta_{(h,0)} \Delta_{(0,k)} f(x,y) = \Delta_{(0,k)} \Delta_{(h,0)} f(x,y). \end{aligned} \tag{8}$$

For $f \in C^{m,n}(X)$, the Bernoulli interpolant on the rectangle is [13]:

$$\begin{aligned} B_{m,n} f(x,y) :=& f(a,c) + \sum_{i=1}^{m} \Delta_{(h,0)} f^{(i-1,0)}(a,c) \tfrac{h^{i-1}}{i!} S_i\left(\tfrac{x-a}{h}\right) \\ &+ \sum_{j=1}^{n} \Delta_{(0,k)} f^{(0,j-1)}(a,c) \tfrac{k^{j-1}}{j!} S_j\left(\tfrac{y-c}{k}\right) \\ &+ \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta_{(h,k)} f^{(i-1,j-1)}(a,c) \tfrac{h^{i-1} k^{j-1}}{i! j!} S_i\left(\tfrac{x-a}{h}\right) S_j\left(\tfrac{y-c}{k}\right), \end{aligned} \tag{9}$$

where S_k, $k > 1$ are given in (7). The polynomial from (9) satisfies the following interpolation conditions:

$$\begin{aligned}
B_{m,n}f(a,c) &= f(a,c), \\
(\Delta_{(h,0)}B_{m,n}f)^{(i,0)}(a,c) &= \Delta_{(h,0)}f^{(i,0)}(a,c), \quad 0 \le i \le m-1, \\
(\Delta_{(0,k)}B_{m,n}f)^{(0,j)}(a,c) &= \Delta_{(0,k)}f^{(0,j)}(a,c), \quad 0 \le j \le n-1, \\
(\Delta_{(h,k)}B_{m,n}f)^{(i,j)}(a,c) &= \Delta_{(h,k)}f^{(i,j)}(a,c), \quad 0 \le i \le m-1, 0 \le j \le n-1.
\end{aligned} \tag{10}$$

The bivariate Shepard-Bernoulli operator (introduced in [5]) preserves the advantages and improve the reproduction qualities, have better accuracy and computational efficiency:

$$S_B f(x,y) = \sum_{i=1}^{N} A_i(x,y) B^i_{m,n} f(x,y), \quad \mu > 0, \tag{11}$$

where $B^i_{m,n}f$ denotes the Bernoulli interpolant $B_{m,n}[f;(x_i,y_i),(h_i,k_i)]$ in the rectangle with opposite vertices (x_i,y_i), (x_{i+1},y_{i+1}), given by (9), having $h_i = x_{i+1} - x_i$, $k_i = y_{i+1} - y_i$, $i = 1,\dots,N$.

The improved form of the Shepard-Bernoulli operator, based on (3), is (see [5]):

$$S^w_B f(x,y) := \frac{\sum_{i=1}^{N} W_i(x,y) B^i_{m,n} f(x,y)}{\sum_{i=1}^{N} W_i(x,y)}. \tag{12}$$

2. Constraints of the Shepard-Bernoulli Operator

Consider the function $f \in C^{(m,n)}(X)$, $X = [a,b] \times [c,d]$ and a set of N distinct points $(x_i,y_i) \in X$, $i = 1,\dots,N$. The classical Shepard operator S, given in (1) satisfies the following property:

$$\min_{i=1,\dots,N} \{f(x_i,y_i)\} \le Sf(x,y) \le \max_{i=1,\dots,N} \{f(x_i,y_i)\}. \tag{13}$$

A consequence of this property is that a positive interpolant is guaranteed if the data values are positive.

The modified Shepard operator, given in (3), has superior qualities but it does not satisfy the property (13).

We will impose constraints to the operators given in (11) and (12) using the steps of the method described in [1], whose notations will be used.

Let C_U and C_L be the upper and lower bounds in $\mathbb{R}$, a constant K in $(0,1)$ and $p = \frac{1}{K} - 1$. We mention that K is an input parameter which gives us flexibility to use a value suitable for the application. We consider

$$\begin{aligned}
d_U(x_i,y_i) &:= f(x_i,y_i) - C_U, \\
d_L(x_i,y_i) &:= f(x_i,y_i) - C_L, \\
D_U(x_i,y_i) &:= d_U(x_i,y_i) + K[B^i_{m,n}f(x,y) - f(x_i,y_i)], \\
D_L(x_i,y_i) &:= d_L(x_i,y_i) + K[B^i_{m,n}f(x,y) - f(x_i,y_i)], \\
Q(x_i,y_i) &:= f(x_i,y_i) + K[B^i_{m,n}f(x,y) - f(x_i,y_i)],
\end{aligned}$$

and

$$\mu_U(x_i,y_i) := \begin{cases} \left(\frac{D_U(x_i,y_i)}{d_U(x_i,y_i)}\right)^p, & \text{if } f(x_i,y_i) \le B^i_{m,n}f(x_i,y_i) \le C_U, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

$$\mu_L(x_i,y_i) := \begin{cases} \left(\frac{D_L(x_i,y_i)}{d_L(x_i,y_i)}\right)^p, & \text{if } C_L \le B^i_{m,n}f(x_i,y_i) \le f(x_i,y_i) \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Let

$$R(x_i, y_i) = \begin{cases} C_U + \mu_U(x_i, y_i) D_U(x_i, y_i), & \text{if } f(x_i, y_i) \leq B^i_{m,n} f(x_i, y_i), \\ C_L + \mu_L(x_i, y_i) D_L(x_i, y_i), & \text{otherwise.} \end{cases} \tag{16}$$

The constrained Shepard-Bernoulli operators are given by

$$S_{c_1} f(x, y) = \sum_{i=1}^{N} A_i(x, y) R(x_i, y_i), \tag{17}$$

$$S_{c_2} f(x, y) = \frac{\sum\limits_{i=1}^{N} W_i(x,y) R(x_i, y_i)}{\sum\limits_{i=1}^{N} W_i(x,y)}, \tag{18}$$

with $A_i(x, y)$ and $W_i(x, y)$ given by (2) and (4), respectively.

Theorem 1. *For $(x, y) \in X$, it holds*

$$C_L \leq S_{c1} f(x, y) \leq C_U, \tag{19}$$

and

$$C_L \leq S_{c2} f(x, y) \leq C_U. \tag{20}$$

Proof. Replacing (14) and (15) in (16), we get

$$R(x_i, y_i) =$$
$$= \begin{cases} C_U + \frac{(f(x_i,y_i) - C_U + K(B^i_{m,n} f(x,y) - f(x_i,y_i)))^{p+1}}{(f(x_i,y_i) - C_U)^p}, & \text{if } f(x_i, y_i) \leq B^i_{m,n} f(x_i, y_i) \leq C_U \\ C_L + \frac{(f(x_i,y_i) - C_L + K(B^i_{m,n} f(x,y) - f(x_i,y_i)))}{(f(x_i,y_i) - C_L)^p}^{p+1}, & \text{if } C_L \leq B^i_{m,n} f(x_i, y_i) \leq f(x_i, y_i). \end{cases}$$

If $f(x_i, y_i) \leq B^i_{m,n} f(x_i, y_i) \leq C_U$, it holds

$$\begin{aligned} R(x_i, y_i) &\leq C_U + \frac{(f(x_i,y_i) - C_U + B^i_{m,n} f(x,y) - f(x_i,y_i))^{p+1}}{(f(x_i,y_i) - C_U)^p} \\ &\leq C_U + \left(\frac{B^i_{m,n} f(x,y) - C_U}{f(x_i,y_i) - C_U} \right)^p (B^i_{m,n} f(x, y) - C_U) \leq C_U. \end{aligned}$$

If $C_L \leq B^i_{m,n} f(x_i, y_i) \leq f(x_i, y_i)$, it holds

$$R(x_i, y_i) \geq C_L + f(x_i, y_i) - C_L \geq f(x_i, y_i) \geq C_L.$$

Therefore, by (17) and (2), the inequality (19) is proved.

Similarly, taking into account (18) and (4), (20) follows. □

Theorem 2. *For $f \in C^{(m,n)}(X)$, the following interpolation properties hold:*

$$S_{c1} f(x_k, y_k) = f(x_k, y_k),$$

for $1 \leq k \leq N$ and $\mu > m + n - 2$.

Proof. We have

$$S_{c_1} f(x_k, y_k) = \sum_{i=1}^{N} A_i(x_k, y_k) R(x_i, y_i)$$

and by the property $A_i(x_k, y_k) = \delta_{ik}$, see Proposition 1, we get

$$\begin{aligned}
S_{c_1}f(x_k,y_k) &= R(x_k,y_k) \\
&= \begin{cases} C_U + \left(\frac{D_U(x_k,y_k)}{d_U(x_k,y_k)}\right)^p D_U(x_k,y_k), & \text{if } f(x_k,y_k) \le Q(x_k,y_k), \\ C_L + \left(\frac{D_L(x_k,y_k)}{d_L(x_k,y_k)}\right)^p D_L(x_k,y_k), & \text{otherwise.} \end{cases} \\
&= \begin{cases} C_U + \frac{(d_U(x_k,y_k)+K(B^k_{m,n}f(x,y)-f(x_k,y_k)))^{p+1}}{[d_U(x_k,y_k)]^p}, & \text{if } f(x_k,y_k) \le Q(x_k,y_k), \\ C_L + \frac{(d_L(x_k,y_k)+K(B^k_{m,n}f(x,y)-f(x_k,y_k)))^{p+1}}{[d_L(x_k,y_k)]^p}, & \text{otherwise.} \end{cases}
\end{aligned}$$

By the interpolation properties of the Bernoulli operator, we have $B^k_{m,n}f(x_k,y_k) = f(x_k,y_k)$, for $k = 1,\ldots,N$, whence it follows

$$\begin{aligned}
(S_{c_1}f)(x_k,y_k) &= \begin{cases} C_U + d_U(x_k,y_k), & \text{if } f(x_k,y_k) \le Q(x_k,y_k) \\ C_L + d_L(x_k,y_k), & \text{otherwise.} \end{cases} \\
&= f(x_k,y_k), \text{ for } k = 1,\ldots,N.
\end{aligned}$$

□

Theorem 3. *The degree of exactness of the operator S_{c_1} is* 0.

Proof. Considering $e_{k,j}(x,y) = x^k y^j$, with $k \le m$ and $j \le n$, we have

$$\begin{aligned}
S_{c_1}e_{k,j}(x,y) &= \sum_{i=1}^{N} A_i(x,y)R(x_i,y_i) \\
&= \begin{cases} C_U + \sum_{i=1}^{N} A_i(x,y)\left(\frac{D_U(x_i,y_i)}{d_U(x_i,y_i)}\right)^p D_U(x_i,y_i), & \text{if } f(x_i,y_i) \le Q(x_i,y_i), \\ C_L + \sum_{i=1}^{N} A_i(x,y)\left(\frac{D_L(x_i,y_i)}{d_L(x_i,y_i)}\right)^p D_L(x_i,y_i), & \text{otherwise.} \end{cases} \\
&= \begin{cases} C_U + \sum_{i=1}^{N} \frac{A_i(x,y)(d_U(x_i,y_i)+K(B^i_{m,n}e_{k,j}(x,y)-e_{k,j}(x_i,y_i)))^{p+1}}{(d_U(x_i,y_i))^p}, & \text{if } f(x_i,y_i) \le Q(x_i,y_i), \\ C_L + \sum_{i=1}^{N} \frac{A_i(x,y)(d_L(x_i,y_i)+K(B^i_{m,n}e_{k,j}(x,y)-e_{k,j}(x_i,y_i)))^{p+1}}{(d_L(x_i,y_i))^p}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Having degree of exactness of $B^i_{m,n}$ equal to (m,n) (see, e.g., [5,13]), for $k \le m$ and $j \le n$, we get

$$\begin{aligned}
S_{c_1}e_{k,j}(x,y) &= \begin{cases} C_U + \sum_{i=1}^{N} A_i(x,y)d_U(x_i,y_i), & \text{if } f(x_i,y_i) \le Q(x_i,y_i) \\ C_L + \sum_{i=1}^{N} A_i(x,y)d_L(x_i,y_i), & \text{otherwise.} \end{cases} \\
&= \begin{cases} C_U + \sum_{i=1}^{N} A_i(x,y)(e_{k,j}(x_i,y_i) - C_U), & \text{if } f(x_i,y_i) \le Q(x_i,y_i) \\ C_L + \sum_{i=1}^{N} A_i(x,y)(e_{k,j}(x_i,y_i) - C_L), & \text{otherwise.} \end{cases}
\end{aligned}$$

Applying the property that $dex(S) = 0$ (see Proposition 1), we get $S_{c_1}e_{k,j}(x,y) = e_{k,j}(x,y)$ for $k = j = 0$. □

3. Numerical Examples

To illustrate the performance of the proposed constructions, we consider the following test functions ([10–12]):

$$\text{Gentle: } f_1(x,y) = \exp[-\tfrac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3,$$
$$\text{Saddle: } f_2(x,y) = \frac{(1.25+\cos 5.4y)}{6+6(3x-1)^2},$$
$$\text{Franke: } f_3(x,y) = 0.75e^{-\frac{1}{4}[(9x-2)^2+(9y-2)^2]} + 0.75e^{[-\frac{1}{49}(9x+1)^2-\frac{1}{10}(9y+1)]} + 0.5e^{-\frac{1}{4}[(9x-7)^2+(9y-3)^2]} - 0.2e^{[-(9x-4)^2-(9y-7)^2]}.$$

Table 1 shows the minimum and the maximum values of $S_{c_2} f_i$, $i = 1,2,3$, for cases $C_L = 0; C_U = 1$ and $C_L = 0; C_U = 2$, considering 20 random generated nodes, $K = 0.5$ and $N_w = 8$.

Table 1. Minimum and maximum of $S_{c_2} f_i$, $i = 1,2,3$.

	$C_L = 0; C_U = 1$		$C_L = 0; C_U = 2$	
	min	**max**	**min**	**max**
$S_{c_2} f_1$	0.0274	0.3479	0	0.5167
$S_{c_2} f_2$	0.0103	0.6163	6.7525×10^4	0.3590
$S_{c_2} f_3$	0.0346	0.9959	1.0621×10^4	1.7136

In Figures 1–3 we plot the graphs of f_i, $S_B^w f_i$, $S_{c_2} f_i$, for $i = 1,2,3$ (that have better approximation properties than $S_{c_1} f_i$).

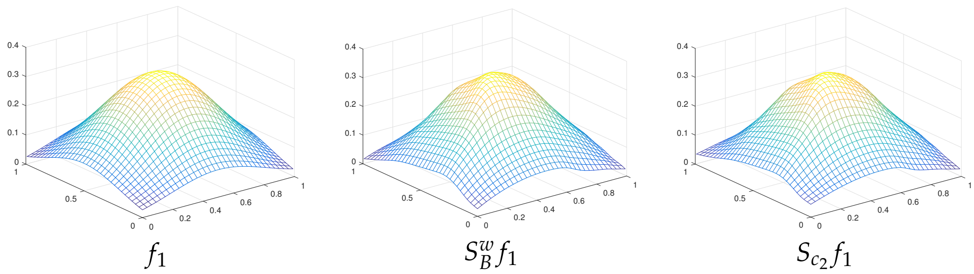

Figure 1. Graphs for f_1.

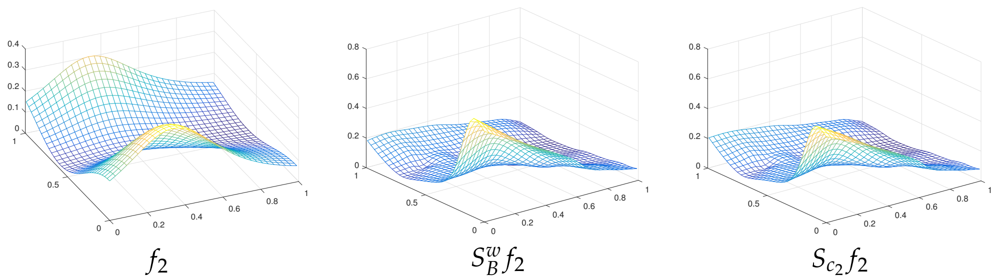

Figure 2. Graphs for f_2.

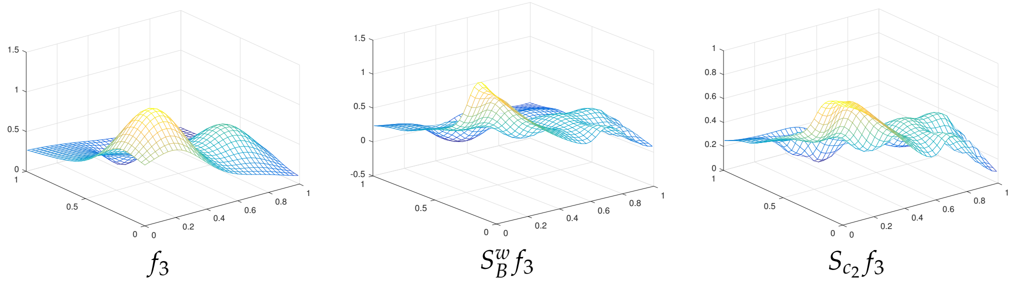

Figure 3. Graphs for f_3.

4. Conclusions

By Table 1, we remark that the values of $S_{c_2} f_i$, $i = 1, 2, 3$ preserve the lower bound of C_L and the upper bound of C_U, as it is theoretically proved in the previous section. Further, by the same table and the figures, we note the good approximation properties of the constructed operators.

By Figures 1–3, it is seen that the behaviour of the operators $S_{c_2} f_i$, $i = 1, 2, 3$, is better than the behaviour of the improved form of the Shepard-Bernoulli operators, $S_B^w f_i$, $i = 1, 2, 3$.

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