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Lie Symmetries and Conservation Laws of Fokas–Lenells Equation and Two Coupled Fokas–Lenells Equations by the Symmetry/Adjoint Symmetry Pair Method

Lihua Zhang *, Gangwei Wang, Qianqian Zhao and Lingshu Wang

School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China; wanggangwei@bit.edu.cn (G.W.); zqyly@sjtu.edu.cn (Q.Z.); wanglingshu@126.com (L.W.)

* Correspondence: zzlh100@163.com

Abstract: The Fokas–Lenells equation and its multi-component coupled forms have attracted the attention of many mathematical physicists. The Fokas–Lenells equation and two coupled Fokas–Lenells equations are investigated from the perspective of Lie symmetries and conservation laws. The three systems have been turned into real multi-component coupled systems by appropriate transformations. By procedures of symmetry analysis, Lie symmetries of the three real systems are obtained. Explicit conservation laws are constructed using the symmetry/adjoint symmetry pair method, which depends on Lie symmetries and adjoint symmetries. The relationships between the multiplier and the adjoint symmetry are investigated.

Keywords: Fokas–Lenells equation; symmetries; adjoint symmetries; conservation laws; multiplier; the SA method; multi-component

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1. Introduction

For nonlinear differential equations [1–12], differential-difference equations [13–15], and fractional differential equations [16–20], it is important to admit Lie symmetries [1,2,10,15]. By means of Lie symmetries, one can reduce the order of nonlinear systems, get group-invariant solutions or perform Lie symmetries classifications [1,2]. Based on the original work of Noether, one symmetry corresponds to a conservation law for partial differential equations (PDEs) with classical Lagrangians [3]. However, there are many important PDEs without classical Lagrangians. Researchers have made many generalizations of Noether's work, so that conservation laws can be investigated whether the equations possess classical Lagrangians or not [4–9]. Among these, the symmetry/adjoint symmetry pair (SA) method [5,6,13] is one of the most efficient methods, and it relies on the Lie symmetries and adjoint symmetries of nonlinear systems. In this paper, the SA method will be taken to investigate conservation laws of three Fokas–Lenells equations. We will show that the SA method is very effective in finding conservation laws of multi-component coupled systems.

The Fokas–Lenells (FL) equation:

$$IU_t - \alpha U_{tx} + \gamma U_{xx} + \sigma |U|^2(U + I\alpha U_x) = 0, \quad \sigma = \pm 1 \quad (1)$$

was proposed by Fokas [21], where U is a complex function dependent on x and t , and α and γ are real constants ($\alpha \neq 0, \gamma \neq 0$). The optical application of Equation (1) as well as the relationships between Equation (1) and the nonlinear Schrödinger equation have been stated [22]. The integrability properties, asymptotic behavior of the solutions, and analytical solutions have been studied in [23–29].

The coupled FL equations have the expression:

$$\begin{aligned}U_{xt} + \alpha\beta^2 U - 2I\alpha\beta U_x - \alpha U_{xx} + I\alpha\beta^2 UVU_x &= 0, \\V_{xt} + \alpha\beta^2 V + 2I\alpha\beta V_x - \alpha V_{xx} - I\alpha\beta^2 UVV_x &= 0,\end{aligned}\quad (2)$$

where U and V are both complex functions dependent on x and t , α and β are real numbers, and $\alpha \neq 0, \beta \neq 0$ [30]. Analytical solutions of Equation (2) have been obtained by Darboux transformation [31]. Based on the determinant representation of the Darboux transformation, higher-order soliton solutions of Equation (2) have been derived in [32]. When $\alpha = \beta = 1$, rogue wave and n -order rogue waves have been studied in [33,34].

Another coupled FL equations have the form:

$$\begin{aligned}IU_{xt} - 2IU_{xx} + 4U_x - (2|U|^2 + |V|^2)U_x - UV^*V_x + 2IU &= 0, \\IV_{xt} - 2IV_{xx} + 4V_x - (2|V|^2 + |U|^2)V_x - VU^*U_x + 2IV &= 0,\end{aligned}\quad (3)$$

where U and V are both complex functions dependent on x and t , and the asterisk represents a complex conjugation. It was shown that Equation (3) has a multi-Hamiltonian integrability [35]. One-soliton solutions, higher-order-soliton solutions and other solutions have been studied in [36–38].

To the best of our knowledge, the Lie symmetries and explicit conservation laws of Equations (1)–(3) possess much mathematical and physical significance, and they have not been reported. The framework of the rest is shown in what follows. In Section 2, we first transform the FL equation and two coupled FL equations into real equations, then search for their Lie symmetries by the classical Lie symmetry method. In Section 3, conservation laws related to the three systems will be found by making use of the SA method. In Section 4, some closing words and discussions are presented.

2. Lie Symmetry Analysis of Equations (1)–(3)

Suppose that:

$$U = u + Iv,$$

with u and v being real functions dependent on x and t , then the FL Equation (1) is changed to:

$$\begin{aligned}\Delta_1 &\equiv u_t - \alpha v_{xt} + \gamma v_{xx} + \sigma u^2 v + \sigma \alpha u^2 u_x + \sigma v^3 + \sigma \alpha v^2 u_x = 0, \\ \Delta_2 &\equiv -v_t - \alpha u_{xt} + \gamma u_{xx} + \sigma u^3 - \sigma \alpha u^2 v_x + \sigma v^2 u - \sigma \alpha v^2 v_x = 0.\end{aligned}\quad (4)$$

By taking:

$$U = u + Iv, V = p + Iq,$$

the coupled FL Equation (2) are changed to:

$$\begin{aligned}v_{xt} - \alpha v_{xx} - \alpha\beta^2(vp + uq)v_x + (-\alpha\beta^2 vq - 2\alpha\beta + \alpha\beta^2 up)u_x + \alpha\beta^2 v &= 0, \\u_{xt} - \alpha u_{xx} - \alpha\beta^2(vp + uq)u_x + (\alpha\beta^2 vq + 2\alpha\beta - \alpha\beta^2 up)v_x + \alpha\beta^2 u &= 0, \\q_{xt} - \alpha q_{xx} + \alpha\beta^2(vp + uq)q_x + (\alpha\beta^2 vq + 2\alpha\beta - \alpha\beta^2 up)p_x + \alpha\beta^2 q &= 0, \\p_{xt} - \alpha p_{xx} + \alpha\beta^2(vp + uq)p_x + (-\alpha\beta^2 vq - 2\alpha\beta + \alpha\beta^2 up)q_x + \alpha\beta^2 p &= 0,\end{aligned}\quad (5)$$

where u, v, p and q are all real functions with respect to x and t . By the same transformation with Equation (2), the coupled FL Equation (3) can be transformed into:

$$\begin{aligned}p_{xt} - 2p_{xx} + (vp - uq)u_x - (qv + up)v_x + (4 - u^2 - v^2 - 2p^2 - 2q^2)q_x + 2p &= 0, \\-q_{xt} + 2q_{xx} + (-vp + uq)v_x - (qv + up)u_x + (4 - u^2 - v^2 - 2p^2 - 2q^2)p_x - 2q &= 0, \\u_{xt} - 2u_{xx} + (-vp + uq)p_x - (qv + up)q_x + (4 - 2u^2 - 2v^2 - p^2 - q^2)v_x + 2u &= 0, \\-v_{xt} + 2v_{xx} + (vp - uq)q_x - (qv + up)p_x + (4 - 2u^2 - 2v^2 - p^2 - q^2)u_x - 2v &= 0.\end{aligned}\quad (6)$$

Next, we will seek the Lie symmetries of Equations (4)–(6) by the classical Lie symmetry method. Suppose the vector field of Equation (4) is:

$$V = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u} + \psi \frac{\partial}{\partial v}, \quad (7)$$

where ξ, τ, ϕ and ψ are undetermined functions with respect to x, t, u and v . Equation (7) is also called a Lie symmetry of Equation (4).

By the procedure of the classical Lie symmetry method, Equation (7) should satisfy the following two conditions:

$$\begin{cases} pr^{(2)}V(\Delta_1) \Big|_{\Delta_1=0, \Delta_2=0} = 0, \\ pr^{(2)}V(\Delta_2) \Big|_{\Delta_1=0, \Delta_2=0} = 0, \end{cases} \quad (8)$$

where $\Delta_1 = u_t - \alpha v_{xt} + \gamma v_{xx} + \sigma(u^2 + v^2)(v + \alpha u_x)$, $\Delta_2 = -v_t - \alpha u_{xt} + \gamma u_{xx} + \sigma(u^2 + v^2)(u - \alpha v_x)$, and $pr^{(2)}V$ denotes the second prolongation of V and:

$$pr^{(2)}V = \phi \frac{\partial}{\partial u} + \psi \frac{\partial}{\partial v} + \phi^x \frac{\partial}{\partial u_x} + \psi^x \frac{\partial}{\partial v_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \psi^{xx} \frac{\partial}{\partial v_{xx}} + \psi^{xt} \frac{\partial}{\partial v_{xt}}, \quad (9)$$

The above coefficient functions are given by:

$$\begin{aligned} \phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, & \phi^{xx} &= D_{xx}(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx}, \\ \phi^{xt} &= D_{xt}(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt}, & \psi^x &= D_x(\psi - \xi v_x - \tau v_t) + \xi v_{xx} + \tau v_{xt}, \\ \psi^{xx} &= D_{xx}(\psi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt}, & \psi^{xt} &= D_{xt}(\psi - \xi v_x - \tau v_t) + \xi v_{xxt} + \tau v_{xtt}, \end{aligned} \quad (10)$$

Here, D_x and D_t are total differential operators. The exact solutions of Equation (8) can be found with the aid of Maple or Mathematica, and then we can obtain Lie symmetries for Equation (4) as follows:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, & V_2 &= \frac{\partial}{\partial t}, & V_3 &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}, \\ V_4 &= (-x - \frac{2\gamma}{\alpha}t) \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + (-\frac{1}{2}u + \frac{xv}{\alpha}) \frac{\partial}{\partial u} + (-\frac{1}{2}v - \frac{xu}{\alpha}) \frac{\partial}{\partial v}. \end{aligned} \quad (11)$$

From the definition of the Lie bracket, i.e., $[V_i, V_j] = V_i V_j - V_j V_i$, the commutation relations of V_1, V_2, V_3 and V_4 are shown in the following:

$$\begin{aligned} [V_r, V_s] &= 0, \quad r, s = 1, 2, 3 & [V_r, V_s] &= 0, \quad r, s = 3, 4 \\ [V_1, V_4] &= -[V_4, V_1] = -V_1 + \frac{1}{\alpha}V_3, & [V_2, V_4] &= -[V_4, V_2] = V_2 - \frac{2\gamma}{\alpha}V_1. \end{aligned}$$

Therefore, V_i ($i = 1, \dots, 4$) are closed under the Lie bracket, and they form a four-dimensional Lie algebra.

Similarly, we can gain the Lie symmetries of Equation (5):

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, & V_2 &= \frac{\partial}{\partial t}, & V_3 &= -v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} + q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}, & V_4 &= -u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}, \\ V_5 &= (-x - 2\alpha t) \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + (-u - 2\alpha \beta t v) \frac{\partial}{\partial u} + (-v + 2\alpha \beta t u) \frac{\partial}{\partial v} + 2\alpha \beta t q \frac{\partial}{\partial p} - 2\alpha \beta t p \frac{\partial}{\partial q}. \end{aligned} \quad (12)$$

The commutation relations of Equation (12) are listed in the following:

$$\begin{aligned} [V_r, V_s] &= 0, \quad r, s = 1, 2, 3, 4 & [V_k, V_5] &= -[V_5, V_k] = 0, \quad k = 3, 4, 5 \\ [V_1, V_5] &= -[V_5, V_1] = -V_1, & [V_2, V_5] &= -[V_5, V_2] = -2\alpha V_1 + V_2 + 2\alpha \beta V_3. \end{aligned}$$

Therefore, V_i ($i = 1, \dots, 5$) are closed under the Lie bracket, and they form a five-dimensional Lie algebra.

By a similar method as with the systems Equations (4) and (5), we can get the Lie symmetries of Equation (6):

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = -q \frac{\partial}{\partial p} + p \frac{\partial}{\partial q}, \quad V_4 = -v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}, \\ V_5 &= -p \frac{\partial}{\partial u} - q \frac{\partial}{\partial v} + u \frac{\partial}{\partial p} + v \frac{\partial}{\partial q}, \quad V_6 = -q \frac{\partial}{\partial u} + p \frac{\partial}{\partial v} - v \frac{\partial}{\partial p} + u \frac{\partial}{\partial q}, \\ V_7 &= (-x - 4t) \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + (-4tv - \frac{u}{2}) \frac{\partial}{\partial u} + (4tu - \frac{v}{2}) \frac{\partial}{\partial v} + (-4tq - \frac{p}{2}) \frac{\partial}{\partial p} + (4tp - \frac{q}{2}) \frac{\partial}{\partial q}. \end{aligned} \quad (13)$$

The commutation relations of Equation (13) are listed in the following:

$$\begin{aligned} [V_r, V_s] &= 0, \quad r, s = 1, 2, 3, 4 & [V_k, V_5] &= -[V_5, V_k] = 0, \quad k = 1, 2, 5, 7 \\ [V_r, V_6] &= -[V_6, V_r] = 0, \quad r = 1, 2, 6, 7 & [V_s, V_7] &= -[V_7, V_s] = 0, \quad s = 3, 4, 7 \\ [V_1, V_7] &= -[V_7, V_1] = -V_1, & [V_2, V_6] &= -[V_6, V_2] = -4V_1 + V_2 + 4V_6, \\ [V_3, V_5] &= -[V_5, V_3] = -V_6, & [V_3, V_6] &= -[V_6, V_3] = V_5, \\ [V_4, V_5] &= -[V_5, V_4] = V_6, & [V_4, V_6] &= -[V_6, V_4] = -V_5, \\ [V_5, V_6] &= -[V_6, V_5] = 2V_4 - 2V_3. \end{aligned}$$

Therefore, $V_i (i = 1, 2, \dots, 7)$ are closed under the Lie bracket, and they form a seven-dimensional Lie algebra.

3. Conservation Laws of Equations (4)–(6) by the SA Method

3.1. Conservation Laws of Equation (4)

Based on the obtained Lie symmetries in Section 2, we can construct conservation laws for Equations (4)–(6) by the steps of the SA method [5,6].

From the definition of the Fréchet derivative and adjoint Fréchet derivative, one obtains the linearizing operator of Equation (4):

$$L = \begin{bmatrix} (2\sigma\alpha uu_x + 2\sigma uv) + \sigma\alpha(u^2 + v^2)D_x + D_t & \sigma(u^2 + 3v^2 + 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} \\ \sigma(3u^2 + v^2 - 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} & \sigma(2uv - 2\alpha vv_x) - \sigma\alpha(u^2 + v^2)D_x - D_t \end{bmatrix}, \quad (14)$$

and the adjoint linearizing operator of Equation (4):

$$L^* = \begin{bmatrix} 2\sigma(uv - \alpha vv_x) - \sigma\alpha(u^2 + v^2)D_x - D_t & \sigma(3u^2 + v^2 - 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} \\ \sigma(u^2 + 3v^2 + 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} & 2\sigma(uv + \alpha uu_x) + \sigma\alpha(u^2 + v^2)D_x + D_t \end{bmatrix}. \quad (15)$$

Using the Lie symmetries Equation (11), the symmetry components $\hat{\eta}^p$ of Equation (4) can be expressed by:

$$\begin{aligned} \hat{\eta}^1 &= (\hat{\eta}_1^1, \hat{\eta}_2^1) = (-u_x, -v_x), \quad \hat{\eta}^2 = (\hat{\eta}_1^2, \hat{\eta}_2^2) = (-u_t, -v_t), \quad \hat{\eta}^3 = (\hat{\eta}_1^3, \hat{\eta}_2^3) = (v, -u), \\ \hat{\eta}^4 &= (\hat{\eta}_1^4, \hat{\eta}_2^4) = \left(-\frac{1}{2}u + \frac{xv}{\alpha} + (x + \frac{2\gamma}{\alpha}t)u_x - tu_t, -\frac{1}{2}v - \frac{xu}{\alpha} + (x + \frac{2\gamma}{\alpha}t)v_x - tv_t\right). \end{aligned} \quad (16)$$

From Equation (15), one obtains the following adjoint linearizing system:

$$L^* \omega_\sigma = \begin{bmatrix} 2\sigma(uv - \alpha vv_x) - \sigma\alpha(u^2 + v^2)D_x - D_t & \sigma(3u^2 + v^2 - 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} \\ \sigma(u^2 + 3v^2 + 2\alpha uv_x) + \gamma D_{xx} - \alpha D_{xt} & 2\sigma(uv + \alpha uu_x) + \sigma\alpha(u^2 + v^2)D_x + D_t \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (17)$$

with ω^1 and ω^2 being functions dependent on x, t, u and v . After calculation, we find that the only solution to Equation (17) is:

$$(\omega^1, \omega^2) = (Cu, -Cv), \quad (18)$$

with C being a constant. This is the adjoint symmetry of Equations (4), and we take $C = 1$ in the following for simplicity.

Substituting $\hat{\eta}^1 = (\hat{\eta}_1^1, \hat{\eta}_2^1)$ in Equation (16) and (ω^1, ω^2) in Equation (18) into the conservation laws identity of Theorem 1 in [6], one can obtain the following conservation laws of Equation (4) with respect to V_1 :

$$\begin{aligned} X_1 &= -\sigma\alpha uu_x(u^2 + v^2) - \sigma\alpha vv_x(u^2 + v^2) + \gamma(u_{xx}v - v_{xx}u) + \alpha(u_xv_t - v_xu_t), \\ T_1 &= -uu_x - vv_x + \alpha(uv_{xx} - vu_{xx}). \end{aligned} \quad (19)$$

Using the other pairs $\hat{\eta}^i$ ($i = 2, 3, 4$) in Equation (16) and (ω^1, ω^2) in Equation (18), one can obtain the other three conservation laws of Equation (4):

$$\begin{aligned} X_2 &= -\sigma\alpha uu_t(u^2 + v^2) - \sigma\alpha vv_t(u^2 + v^2) + \gamma(u_{xt}v - v_{xt}u) + \gamma(u_xv_t - v_xu_t), \\ T_2 &= -uu_t - vv_t + \alpha(uv_{xt} - vu_{xt}). \end{aligned} \quad (20)$$

$$X_3 = -\alpha(vv_t + uu_t), \quad T_3 = \alpha(vv_x + uu_x). \quad (21)$$

$$\begin{aligned} X_4 &= \sigma\alpha xuv(u_xv + uv_x) + 2\sigma\gamma t(u^3u_x + v^3v_x) - \sigma\alpha tuv(u_tv + uv_t) - \frac{\sigma\alpha}{2}(u^2 + v^2)^2 + \gamma t(v_xu_t - v_tu_x) \\ &\quad - x(vv_t + uu_t) + \alpha x(v_xu_t - v_tu_x) + \gamma x(uv_{xx} - vu_{xx}) + \gamma t(u_{xt}v - v_{xt}u) - \frac{\gamma}{\alpha}(u^2 + v^2) \\ &\quad + \frac{\alpha}{2}(uv_t - vu_t) + \sigma\alpha x(u^3u_x + v^3v_x) - \sigma\alpha t(u^3u_t + v^3v_t) + \frac{2\gamma^2t}{\alpha}(uv_{xx} - vu_{xx}) + 2\sigma\gamma tuv(uv_x + vu_x), \\ T_4 &= \frac{1}{2}(u^2 + v^2) + (2x + \frac{2\gamma t}{\alpha})(uu_x + vv_x) - t(uu_t + vv_t) + \frac{\alpha}{2}(vu_x - uv_x) \\ &\quad + (\alpha x + 2\gamma t)(vu_{xx} - uv_{xx}) + \alpha t(uv_{xt} - vu_{xt}). \end{aligned}$$

Remark 1. Taking conservation laws Equation (19) as an example, we show that adjoint symmetries are also multipliers corresponding to conservation laws. After some calculation, Equation (19) will reduce to:

$$\bar{X}_1 = \gamma(uv_x - vu_x) + \frac{1}{4}\sigma\alpha(u^2 + v^2)^2 - \alpha uv_t, \quad \bar{T}_1 = \frac{1}{2}(u^2 + v^2) + \alpha u_xv.$$

and:

$$D_x\bar{X}_1 + D_t\bar{T}_1 = u\Delta_1 + (-v)\Delta_2 = 0. \quad (22)$$

Based on the definition of multipliers [4], Equation (22) means that $(u, -v)$ is the multiplier corresponding to (X_1, T_1) . In fact, $(u, -v)$ are also multipliers corresponding to (X_2, T_2) (X_3, T_3) and (X_4, T_4) .

Remark 2. As to the conservation law (X_3, T_3) , since $D_xX_3 + D_tT_3 \equiv 0$, it is not relevant to the solutions of Equation (4), and it is trivial. The other conservation laws (X_1, T_1) , (X_2, T_2) and (X_4, T_4) are nontrivial and can be proved. For example, $D_xX_2 + D_tT_2 = -u\Delta_1 + v\Delta_2 = 0$. The accuracy of the above conservation laws has been verified.

3.2. Conservation Laws of Equation (5)

From the definition of the Fréchet derivative and adjoint Fréchet derivative, one obtains the linearizing operator of Equation (5):

$$L = \begin{bmatrix} \alpha\beta^2(pu_x - qv_x) + \alpha\beta(\beta up - \beta vq - 2)D_x & \alpha\beta^2(-pv_x - qu_x + 1) - \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & \alpha\beta^2(uu_x - vv_x) & -\alpha\beta^2(vu_x + uv_x) \\ \alpha\beta^2(-pv_x - qu_x + 1) - \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & -\alpha\beta^2(pu_x - qv_x) - \alpha\beta(\beta up - \beta vq - 2)D_x & -\alpha\beta^2(uv_x + vu_x) & -\alpha\beta^2(uu_x - vv_x) \\ -\alpha\beta^2(pp_x - qq_x) & \alpha\beta^2(pq_x + qp_x) & \alpha\beta^2(vq_x - up_x) - \alpha\beta(\beta up - \beta vq - 2)D_x & \alpha\beta^2(vp_x + uq_x + 1) + \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} \\ \alpha\beta^2(pq_x + qp_x) & \alpha\beta^2(pp_x - qq_x) & \alpha\beta^2(vp_x + uq_x + 1) + \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & -\alpha\beta^2(vq_x - up_x) + \alpha\beta(\beta up - \beta vq - 2)D_x \end{bmatrix},$$

and the adjoint linearizing operator of Equation (5):

$$L^* = \begin{bmatrix} \alpha\beta^2(vq_x - up_x) - \alpha\beta(\beta up - \beta vq - 2)D_x & \alpha\beta^2(vp_x + uq_x + 1) + \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & -\alpha\beta^2(pp_x - qq_x) & \alpha\beta^2(pq_x + qp_x) \\ \alpha\beta^2(vp_x + uq_x + 1) + \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & -\alpha\beta^2(vq_x - up_x) + \alpha\beta(\beta up - \beta vq - 2)D_x & \alpha\beta^2(pq_x + qp_x) & \alpha\beta^2(pp_x - qq_x) \\ \alpha\beta^2(uu_x - vv_x) & -\alpha\beta^2(vu_x + uv_x) & \alpha\beta^2(pu_x - qv_x) + \alpha\beta(\beta up - \beta vq - 2)D_x & \alpha\beta^2(-pv_x - qu_x + 1) - \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} \\ -\alpha\beta^2(vu_x + uv_x) & -\alpha\beta^2(uu_x - vv_x) & \alpha\beta^2(-pv_x - qu_x + 1) - \alpha\beta^2(vp + uq)D_x - \alpha D_{xx} + D_{xt} & -\alpha\beta^2(pu_x - qv_x) - \alpha\beta(\beta up - \beta vq - 2)D_x \end{bmatrix}. \quad (23)$$

By the symmetries Equation (12), there are five symmetry components of Equation (5), and they can be expressed by:

$$\begin{aligned} \hat{\eta}^1 &= (\hat{\eta}_1^1, \hat{\eta}_2^1, \hat{\eta}_3^1, \hat{\eta}_4^1) = (-u_x, -v_x, -p_x, -q_x), \\ \hat{\eta}^2 &= (-u_t, -v_t, -p_t, -q_t), \quad \hat{\eta}^3 = (-v, u, q, -p), \quad \hat{\eta}^4 = (-u, -v, p, q), \\ \hat{\eta}^5 &= (-u - 2\alpha\beta tv + (x + 2\alpha t)u_x - tu_t, -v + 2\alpha\beta tu + (x + 2\alpha t)v_x - tv_t, \\ &\quad 2\alpha\beta tq + (x + 2\alpha t)up_x - tp_t, -2\alpha\beta tp + (x + 2\alpha t)q_x - tq_t). \end{aligned} \quad (24)$$

From Equation (23), one can obtain the adjoint linearizing system of Equation (5):

$$L^* \omega_\sigma = L^* \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (25)$$

with ω^i ($i = 1, 2, 3, 4$) being functions dependent on x, t, u, v, p and q . After complicated calculations, we find that there are two solutions to Equation (25):

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_1(p, q, -u, -v), \quad (26)$$

and:

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_2(q, -p, -v, u), \quad (27)$$

where C_1 and C_2 are constants. They are adjoint symmetries of Equation (5). We take $C_1 = 1$ and $C_2 = 1$ in the following for simplicity.

Based on the symmetry component $\hat{\eta}^1 = (\hat{\eta}_1^1, \hat{\eta}_2^1, \hat{\eta}_3^1, \hat{\eta}_4^1)$ in Equation (24) and the adjoint symmetry $(\omega^1, \omega^2, \omega^3, \omega^4)$ in Equation (26), one can obtain conservation laws of Equation (5) with respect to V_1 through the conservation laws identity of Theorem 1 in [6] as follows:

$$\begin{aligned} X_1 &= 2\alpha\beta^2 pqvu_x + 2\alpha\beta pu_x - \alpha\beta^2 up^2u_x + \alpha\beta^2 uq^2u_x + 2\alpha\beta^2 pquv_x + \alpha\beta^2 vp^2v_x - 2\alpha\beta qv_x \\ &\quad - \alpha\beta^2 vq^2v_x + 2\alpha\beta up_x - \alpha\beta^2 u^2pp_x + 2\alpha\beta^2 uvqp_x + \alpha\beta^2 pv^2p_x + \alpha\beta^2 qu^2q_x + 2\alpha\beta^2 uvpq_x \\ &\quad - \alpha\beta^2 qv^2q_x - 2\alpha\beta vq_x + \alpha qu_{xx} + \alpha pv_{xx} - \alpha vp_{xx} - \alpha uq_{xx} + u_x q_t + v_x p_t - p_x v_t - q_x u_t, \\ T_1 &= -u_{xx}q - v_{xx}p + p_{xx}v + q_{xx}u. \end{aligned}$$

Similarly, using the other symmetry components $\hat{\eta}^i$ ($i = 2, 3, 4, 5$) in Equation (24) and the adjoint symmetry $(\omega^1, \omega^2, \omega^3, \omega^4)$ in Equation (26), one can obtain other conservation laws of Equation (5) with respect to V_i ($i = 2, 3, 4, 5$):

$$\begin{aligned} X_2 &= 2\alpha\beta^2 pqvu_t + 2\alpha\beta pu_t - \alpha\beta^2 up^2u_t + \alpha\beta^2 uq^2u_t + 2\alpha\beta^2 pquv_t + \alpha\beta^2 vp^2v_t - 2\alpha\beta qv_t \\ &\quad - \alpha\beta^2 vq^2v_t + 2\alpha\beta up_t - \alpha\beta^2 u^2pp_t + 2\alpha\beta^2 uvqp_t + \alpha\beta^2 pv^2p_t + \alpha\beta^2 qu^2q_t + 2\alpha\beta^2 uvpq_t \\ &\quad - \alpha\beta^2 qv^2q_t - 2\alpha\beta vq_t + \alpha qu_{xt} + \alpha pv_{xt} - \alpha vp_{xt} - \alpha uq_{xt} - \alpha u_t q_x - \alpha v_t p_x + \alpha p_t v_x + \alpha q_t u_x, \\ T_2 &= -u_{xt}q - v_{xt}p + p_{xt}v + q_{xt}u. \\ X_3 &= vq_t - up_t + qv_t - pu_t, \quad T_3 = -qv_x + pu_x - vq_x + up_x. \\ X_4 &= uq_t + vp_t + pv_t + qu_t, \quad T_4 = -qu_x - pv_x - vp_x - uq_x. \\ X_5 &= (-\alpha xq - 2\alpha^2 tq)u_{xx} + (-\alpha xp - 2\alpha^2 tp)v_{xx} + (\alpha xv + 2\alpha^2 tv)p_{xx} + (\alpha xu + 2\alpha^2 tu)u_{xx} \\ &\quad + \alpha tq u_{xt} + \alpha tp v_{xt} - \alpha tv p_{xt} - \alpha tu q_{xt} + (-2\alpha\beta^2 xvpq - 4\alpha^2\beta^2 tvpq + \alpha\beta^2 xup^2 \\ &\quad - 2\alpha\beta xp - 4\alpha^2\beta tp - 2\alpha^2\beta^2 tuq^2 + 2\alpha^2\beta^2 t up^2 - \alpha\beta^2 xuq^2)u_x - (\alpha t + x)q_t u_x \\ &\quad + (2\alpha\beta^2 tvpq + \alpha\beta^2 tuq^2 - \alpha\beta^2 t up^2)u_t + (\alpha t + x)q_x u_t - (\alpha t + x)p_t v_x \\ &\quad + (2\alpha\beta xq + 4\alpha^2\beta tq - 4\alpha^2\beta^2 tvpq - 2\alpha\beta^2 xupq - 2\alpha^2\beta^2 tvp^2 + \alpha\beta^2 xvq^2 - \alpha\beta^2 xvp^2 \\ &\quad + 2\alpha^2\beta^2 tvq^2)v_x + (2\alpha\beta^2 tvpq - \alpha\beta^2 tvq^2 + \alpha\beta^2 tvp^2)v_t + (\alpha t + x)p_x v_t + (-2\alpha\beta xu \\ &\quad - 4\alpha^2\beta tu - 2\alpha\beta^2 xuvq - 4\alpha^2\beta^2 tvvq + \alpha\beta^2 xu^2p - \alpha\beta^2 xv^2p + 2\alpha^2\beta^2 tu^2p \\ &\quad - 2\alpha^2\beta^2 tv^2p)p_x + (2\alpha\beta^2 tvvq - \alpha\beta^2 tu^2p + \alpha\beta^2 tv^2p + v)p_t + (-4\alpha^2\beta^2 tvvp \\ &\quad - 2\alpha\beta^2 xuvp - \alpha\beta^2 xu^2q + \alpha\beta^2 xv^2q - 2\alpha^2\beta^2 tu^2q + 2\alpha^2\beta^2 tv^2q + 2\alpha\beta xv + 4\alpha^2\beta tv)q_x \\ &\quad + (2\alpha\beta^2 tvvp + \alpha\beta^2 tu^2q - \alpha\beta^2 tv^2q + u)q_t + 4\alpha\beta^2 uvvpq + 2\alpha\beta up - \alpha\beta^2 u^2p^2 + \alpha\beta^2 u^2q^2 \\ &\quad + \alpha\beta^2 v^2p^2 - 2\alpha\beta vq - \alpha\beta^2 v^2q^2, \\ T_5 &= (2\alpha t + x)qu_{xx} + (2\alpha t + x)pv_{xx} - (2\alpha t + x)vp_{xx} - (2\alpha t + x)uq_{xx} - tq u_{xt} - tp v_{xt} + tv p_{xt} \\ &\quad + tu q_{xt} - 2\alpha\beta tq v_x + 2\alpha\beta tp u_x + (-2\alpha\beta tv - u)q_x + (2\alpha\beta tu - v)p_x. \end{aligned}$$

It is pointed out that the conservation laws (X_3, T_3) and (X_4, T_4) are trivial, and the others are nontrivial. Since there are two adjoint symmetries for Equation (5), using the other pair Equation (24) and Equation (27), we can also obtain five other explicit conservation laws for Equation (5). For simplicity, we omit them here.

3.3. Conservation Laws of Equation (6)

From the definition of the Fréchet derivative, one obtains the linearizing operator of Equation (6):

$$L = \begin{bmatrix} (-2uq_x - qu_x - pv_x) & (-qv_x - 2vq_x + pu_x) & vu_x + 2 - 4pq_x & -vv_x - uu_x - 4qq_x \\ + (pv - qu)D_x & - (qv + pu)D_x & -uv_x - 2D_{xx} + D_{xt} & + (H - p^2 - q^2)D_x \\ (-2up_x - pu_x + qv_x) & (-2vp_x - pv_x - qu_x) & -uu_x - vv_x - 4pp_x & uv_x - vu_x - 4qp_x \\ - (qv + pu)D_x & + (qu - pv)D_x & + (H - p^2 - q^2)D_x & -2 + 2D_{xx} - D_{xt} \\ qp_x - pq_x - 4uv_x & -qq_x - pp_x - 4vv_x & (-uq_x - vp_x - 2pv_x) & (-vq_x + up_x - 2qv_x) \\ + 2 - 2D_{xx} + D_{xt} & + (H - u^2 - v^2)D_x & + (qu - vp)D_x & - (qv + pu)D_x \\ -pp_x - qq_x - 4uu_x & pq_x - qp_x - 4vu_x & (vq_x - up_x - 2pu_x) & (-vp_x - uq_x - 2qu_x) \\ + (H - u^2 - v^2)D_x & -2 + 2D_{xx} - D_{xt} & - (qv + pu)D_x & + (vp - uq)D_x \end{bmatrix}, \quad (28)$$

where $H = 4 - u^2 - v^2 - p^2 - q^2$.

From the definition of the adjoint Fréchet derivative, the adjoint linearizing operator of Equation (6) is:

$$L^* = \begin{bmatrix} -uq_x - 2pv_x - p_xv & q_xv - up_x + 2qv_x & -4uv_x - pq_x + 2 & pp_x + qq_x + 4vv_x \\ +(-pv + qu)D_x & +(pu + qv)D_x & +qp_x - 2D_{xx} + D_{xt} & +(G + u^2 + v^2)D_x \\ up_x - q_xv + 2pu_x & -2qu_x - p_xv - uq_x & pp_x + 4uu_x + qq_x & pq_x - qp_x - 4vu_x \\ +(pu + qv)D_x & +(-qu + pv)D_x & +(G + u^2 + v^2)D_x & -2 + 2D_{xx} - D_{xt} \\ -uv_x + vu_x + 2 & 4qq_x + uu_x + vv_x & (-2uq_x - pv_x - u_xq) & (-pu_x + v_xq + 2vq_x) \\ -4pq_x - 2D_{xx} + D_{xt} & +(G + p^2 + q^2)D_x & +(vp - uq)D_x & +(pu + qv)D_x \\ vv_x + uu_x + 4pp_x & uv_x - vu_x - 4qp_x & 2up_x - v_xq + pu_x & -2vp_x - u_xq - pv_x \\ +(G + p^2 + q^2)D_{xx} & -2 + 2D_{xx} - D_{xt} & +(qv + pu)D_x & +(-pv + uq)D_x \end{bmatrix}, \quad (29)$$

where $G = -4 + u^2 + v^2 + p^2 + q^2$.

Using the symmetries Equation (13), there are seven symmetry components of Equation (6), and they are given by:

$$\begin{aligned} \hat{\eta}^1 &= (\hat{\eta}_1^1, \hat{\eta}_2^1, \hat{\eta}_3^1, \hat{\eta}_4^1) = (-u_x, -v_x, -p_x, -q_x), \\ \hat{\eta}^2 &= (-u_t, -v_t, -p_t, -q_t), \\ \hat{\eta}^3 &= (0, 0, -q, p), \quad \hat{\eta}^4 = (-v, u, 0, 0), \\ \hat{\eta}^5 &= (-p, -q, u, v), \quad \hat{\eta}^6 = (-q, p, -v, u), \\ \hat{\eta}^7 &= (-4tv - \frac{u}{2} + (x + 4t)u_x - tu_t, 4tu - \frac{v}{2} + (x + 4t)v_x - tv_t, \\ &\quad -4tq - \frac{p}{2} + (x + 4t)p_x - tp_t, 4tp - \frac{q}{2} + (x + 4t)q_x - tq_t). \end{aligned} \quad (30)$$

From Equation (29), we know that the adjoint linearizing system is:

$$L^* \omega_\sigma = L^* \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (31)$$

with ω^i ($i = 1, 2, 3, 4$) being functions dependent on x, t, u, v, p and q . After complicated calculations, we find four different solutions to Equation (31), and they are listed as follows:

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_1(-u, v, p, -q), \quad (32)$$

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_2(q, p, 0, 0), \quad (33)$$

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_3(v, u, q, p), \quad (34)$$

$$(\omega^1, \omega^2, \omega^3, \omega^4) = C_4(0, 0, v, u), \quad (35)$$

where C_i ($i = 1, 2, 3, 4$) are real numbers. They are all adjoint symmetries of Equation (6). Substituting $\hat{\eta}^i = (\hat{\eta}_1^i, \hat{\eta}_2^i, \hat{\eta}_3^i, \hat{\eta}_4^i)$ in Equation (30) and $(\omega^1, \omega^2, \omega^3, \omega^4)$ in Equations (32)–(35) into the conservation laws identity of Theorem 1 in [6], one can obtain explicit conservation laws of Equation (6) for V_i . For example, using Equations (30) and (33) ($C_2 = 1$), we can obtain the following conservation laws:

$$\begin{aligned} X_1 &= uu_x(p^2 + q^2) + vv_x(p^2 + q^2) + p_x(-4p + 2p^3 + 2pq^2 + pu^2 + pv^2) \\ &\quad + q_x(-4q + 2q^3 + 2qp^2 + qu^2 + qv^2) + 2qp_{xx} - 2pq_{xx} + p_xq_t - q_xp_t, \\ T_1 &= -qp_{xx} + pq_{xx}, \\ X_2 &= uu_t(p^2 + q^2) + vv_t(p^2 + q^2) + p_t(-4p + 2p^3 + 2pq^2 + pu^2 + pv^2) \\ &\quad + q_t(-4q + 2q^3 + 2qp^2 + qu^2 + qv^2) + 2qp_{xt} - 2pq_{xt} + 2p_xq_t - 2q_xp_t, \\ T_2 &= -qp_{xt} + pq_{xt}. \end{aligned}$$

$$\begin{aligned} X_3 &= qq_t + pp_t, \\ T_3 &= -qq_x - pp_x. \end{aligned}$$

$$X_4 = 0,$$

$$T_4 = 0.$$

$$X_5 = -2qu_x + 2pv_x + 2uq_x - 2vp_x - uq_t + vp_t - upq^2 - up^3 - vq^3 - vqp^2 + 4pu - pu^3 - pu^2v + 4qv - vqu^2 - qv^3,$$

$$T_5 = qu_x - pv_x.$$

$$X_6 = 2qv_x + 2pu_x - 2up_x - 2vq_x + up_t + vq_t - q^3u - qp^2u + pq^2v + p^3v - 4pv + pu^2v + pv^3 + 4qu - qu^3 - quv^2,$$

$$T_6 = -qv_x - pu_x.$$

$$X_7 = (8t + 2x)(pq_{xx} - qp_{xx}) + 2qt p_{xt} - 2pt q_{xt} - (4tq^2 + 4tp^2 + xq^2 + xp^2)(uu_x + vv_x) + (-8tp^2 + 16t - 2xp^2 + 4x - 8tq^2 - 4tu^2 - 4tv^2 - 2xq^2 - xu^2 - xv^2)(pp_x + qq_x) + (x + 2t)(p_t q_x - p_x q_t) + t(p^2 + q^2)(uu_t + vv_t) + (2tpq^2 + tu^2p + tv^2p + 2tp^3 - \frac{q}{2})p_t + (2tp^2q + tu^2q + tv^2q + 2tq^3 + \frac{p}{2})q_t + (u^2 + v^2)(p^2 + q^2) + (p^2 + q^2)^2 - 2p^2 - 2q^2, \\ T_7 = -(4t + x)pq_{xx} + (4t + x)qq_{xx} - tq p_{xt} + tp q_{xt} - (4tq + \frac{p}{2})q_x + (-4tp + \frac{q}{2})p_x.$$

Here, (X_3, T_3) and (X_4, T_4) are trivial conservation laws, others are nontrivial.

Remark 3. There are seven symmetry components of Equation (6), and they are expressed by Equation (30). Using Equation (30) and other adjoint symmetries in Equations (32)–(35), we can obtain twenty-one other conservation laws for Equation (6). The effectiveness of the SA method in seeking the conservation laws of multi-component coupled systems has been illustrated.

4. Conclusions and Discussions

Recently, the FL equation and coupled FL equations have attracted the interest of many researchers. They can be regarded as integrable analogs of the nonlinear Schrödinger (NLS) equation and its coupled forms in the ultra-short regime. In this paper, Lie symmetry analyses for the FL equation and two coupled FL equations are performed, and we have shown that they all form a closed Lie algebra. Explicit conservation laws for the three FL equations have been obtained by the SA method. The correctness of the derived conservation laws has been tested by a mathematic software. The obtained conservation laws for the FL equation and coupled FL systems may be used to explain some practical physical problems.

The SA method and the multiplier method can both derive explicit conservation laws for PDEs, whether the PDEs have a Lagrangian or not. Taking the FL equation as an example, the relationships between the SA method and the multiplier method are further investigated. We conclude that the adjoint symmetry in the SA method is also the multiplier in the multiplier method. Using the multiplier method, every multiplier can only derive one conservation law. Using the SA method, any pair of symmetry and adjoint symmetry can derive one conservation law, so it can derive more conservation laws than the multiplier method. As illustrated in Sections 3.2 and 3.3, it is very effective to search for the conservation laws of multi-component coupled systems by the SA method. However, when we seek conservation laws by the SA method, the adjoint symmetries cannot be zero. If the adjoint symmetry is zero, the obtained conservation law is trivial, and then we will have to use other methods.

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