

Article

Transient Propagation of Spherical Waves in Porous Material: Application of Fractional Calculus

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Abstract: A fractional-order wave equation is established and solved for a space of three dimensions using spherical coordinates. An equivalent fluid model is used in which the acoustic wave propagates only in the fluid saturating the porous medium; this model is a special case of Biot's theory obtained by the symmetry of the Lagrangian (invariance by translation and rotation). The basic solution of the wave equation is obtained in the time domain by analytically calculating Green's function of the porous medium and using the properties of the Laplace transforms. Fractional derivatives are used to describe, in the time domain, the fluid–structure interactions, which are of the inertial, viscous, and thermal kind. The solution to the fractional-order wave equation represents the radiation field in the porous medium emitted by a point source. An important result obtained in this study is that the solution of the fractional equation is expressed by recurrence relations that are the consequence of the modified Bessel function of the third kind, which represents a physical solution of the wave equation. This theoretical work with analytical results opens up prospects for the resolution of forward and inverse problems allowing the characterization of a porous medium using spherical waves.

Keywords: porous materials; fractional calculus; spherical wave; viscous and thermal surfaces; dynamic tortuosity and compressibility



Citation: Fellah, Z.E.A.; Fellah, M.; Roncen, R.; Ongwen, N.O.; Ogam, E.; Depollier, C. Transient Propagation of Spherical Waves in Porous Material: Application of Fractional Calculus. *Symmetry* **2022**, *14*, 233. <https://doi.org/10.3390/sym14020233>

Academic Editor: Serkan Araci

Received: 18 December 2021

Accepted: 19 January 2022

Published: 25 January 2022

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1. Introduction

Acoustic absorbers, such as plastic foams or fibrous materials [1–4], are widely used for sound insulation, passive sound control [1], and noise reduction in the aeronautical and building industries [5]. These materials have the particularity of being porous [6,7] and consisting of a solid phase and a fluid phase, the fluid part generally being saturated with air. When an acoustic (sound) wave propagates in this type of medium, the solid generally remains motionless (does not vibrate), and the wave essentially propagates in the fluid that saturates the porous material [1,6,7]. The equivalent fluid model is used, which is a special case of the Biot theory obtained by the symmetry of the Lagrangian (by translation and rotation) [1].

Acoustic propagation in porous media involves geometrical parameters that are different depending on the frequency domain. For example, in high frequencies, there is the tortuosity and the viscous and thermal characteristic lengths [8–12], but also the viscous

and thermal surfaces for the most absorbent media [13,14]. At low frequencies, the most relevant parameters are the viscous and thermal permeabilities [15–17], the low-frequency inertial parameter, and a viscous parameter that was recently experimentally measured [18]. It is therefore important to model acoustic propagation in porous media in these two frequency regimes.

The modeling of acoustic propagation in porous materials allows us to predict sound absorption and acoustic attenuation [1]. Several methods have been developed for the resolution of the direct and inverse problems allowing the characterization of porous materials [2,4,9,10,14,16–24]. The direct problem consists in solving the propagation equation and deducing the reflected and transmitted acoustic fields [11,12,15,19,25–28]. Knowing the experimental incident signal and the solution of the propagation equation (Green function of the medium) makes it possible to simulate the acoustic field propagating in the porous material, as well as to predict the waves reflected and transmitted by a slab of material [19]. The solution of the propagation equation is therefore a key step in directly solving the problem [11,12]. To characterize the porous material using sound waves, the solution of the inverse problem becomes necessary. The inverse problem is posed as follows: Knowing the experimental incident signal, the reflected and/or transmitted signal, we go back by inversion to the geometric parameters describing the porous medium [2,4,14,16,19–21,24].

In the study of wave propagation phenomena, the search for the elementary solution to the equation of propagation is a fundamental problem [1,11,12]. In spherical coordinates [29], this solution represents the field radiating from a point source. The solution is also essential in perturbation theory [30] when one is interested in propagation in a weakly inhomogeneous (with punctual defects) medium. In this work, we establish and resolve a fractional propagation equation in the time domain for a three-dimensional space by using the properties of the Laplace transform. The problem of the propagation of spherical waves in a homogeneous and isotropic porous medium in the presence of losses is studied. The visco-thermal losses are modeled by fractional derivatives in the time domain. This work generalizes our previous approach, which was to solve the wave equation in one space dimension [11,12]. We show in this paper, by using analytical computational developments, that Green's function of the porous medium (in spherical coordinates) and the solution of its fractional wave equation can be expressed by recurrence relations and depend on Green's function of a unidirectional propagation problem and its fractional wave equation solution. These results open up new perspectives for the prediction of sound absorption in porous media and their ultrasonic characterization, since the resolution of the propagation equation is the basis for solving the direct and inverse problems.

2. Fractional Model for Porous Media

When porous media are saturated with air, solid-frame vibrations can be neglected in the absence of direct contact with the sound source [1]. It is then considered that the waves propagate only in the fluid saturating the porous medium [6,7]. We will use the equivalent fluid model, which is a special case of the Biot model in which the exchanges between fluid and structure are described by two relaxation functions: the dynamic tortuosity $\alpha(\omega)$ given by Johnson et al. [6] and the dynamic compressibility $\beta(\omega)$ given by Allard [1]. Biot's equations [31,32] are Euler's equations deduced from a Lagrangian taking into account the symmetry due to invariances by translation and rotation.

The dynamic tortuosity describes the inertial and viscous effects while the dynamic compressibility describes the thermal effects. In the frequency domain, these functions correct for the density of the fluid, as well as its compressibility, and they represent the deviation from free-space fluid behavior as the frequency changes. In this context, the equations for acoustics in a porous material are given by:

$$\rho\alpha(\omega)i\omega\mathbf{V}(\mathbf{r},\omega) = \nabla P(\mathbf{r},\omega), \quad \frac{\beta(\omega)}{K}i\omega P(\mathbf{r},\omega) = \nabla \cdot \mathbf{V}(\mathbf{r},\omega). \quad (1)$$

In Equation (1): $i^2 = -1$, ρ is the saturating fluid density and K is the compressibility modulus of the fluid. $P(\mathbf{r}, \omega)$ and $\mathbf{V}(\mathbf{r}, \omega)$ are the Fourier transforms of $\mathbf{v}(\mathbf{r}, t)$ and $p(\mathbf{r}, t)$. The expression of the dynamic compressibility is given by:

$$\beta(\omega) = \gamma - \frac{(\gamma - 1)}{\alpha'(\omega)}, \quad (2)$$

where $\alpha'(\omega)$ is the dynamic thermal tortuosity and γ is the heat capacity ratio (equal to 1.4 for air). In the high-frequency domain, which corresponds to viscous and thermal boundary layers ($\delta(\omega) = \sqrt{2\eta\rho\omega}$ and $\delta'(\omega) = \sqrt{2\eta\rho\omega P_r}$, respectively) that are small compared with the characteristic pore radius, we have [13,14]:

$$\alpha(\omega) = \alpha_\infty + \frac{2\alpha_\infty}{\Lambda} \left(\frac{\eta}{-i\omega\rho} \right)^{1/2} + \frac{3\alpha_\infty}{\Sigma} \left(\frac{\eta}{-i\omega\rho} \right) + \mathcal{O} \left(\frac{1}{-i\omega} \right)^{3/2}, \quad (3)$$

$$\alpha'(\omega) = 1 + \frac{2}{\Lambda'} \left(\frac{\eta}{-i\omega\rho P_r} \right)^{1/2} + \frac{3}{\Sigma'} \left(\frac{\eta}{-i\omega\rho P_r} \right) + \mathcal{O} \left(\frac{1}{-i\omega\rho P_r} \right)^{3/2}, \quad (4)$$

where: η is the fluid viscosity, ω is the frequency pulsation, α_∞ is the tortuosity, Λ is the viscous characteristic length, Λ' is the thermal characteristic length, Σ is the characteristic viscous surface, Σ' is the characteristic thermal surface, and P_r is the Prandtl number. From Equations (2) and (4), we can deduce the expression of $\beta(\omega)$:

$$\beta(\omega) = 1 + (\gamma - 1) \left(\frac{2}{\Lambda'} \left(\frac{\eta}{-i\omega\rho P_r} \right)^{1/2} + \frac{3}{\Sigma'} \left(\frac{\eta}{-i\omega\rho P_r} \right) \right) + \mathcal{O} \left(\frac{1}{-i\omega\rho P_r} \right)^{3/2}, \quad (5)$$

In the time domain, α and β become time operators whose expressions depend on fractional derivatives; Equations (3) and (5) are then in the time domain:

$$\tilde{\alpha}(t) = \alpha_\infty \delta(t) + \frac{2\alpha_\infty}{\Lambda} \left(\frac{\eta}{\rho} \right)^{1/2} \frac{\partial^{-1/2}}{\partial t^{-1/2}} + \frac{3\alpha_\infty}{\Sigma} \left(\frac{\eta}{\rho} \right) \frac{\partial^{-1}}{\partial t^{-1}} + \mathcal{O} \left(\frac{\partial^{-3/2}}{\partial t^{-3/2}} \right), \quad (6)$$

$$\tilde{\beta}(t) = \delta(t) + \frac{2(\gamma - 1)}{\Lambda'} \left(\frac{\eta}{P_r \rho} \right)^{1/2} \frac{\partial^{-1/2}}{\partial t^{-1/2}} + \frac{3(\gamma - 1)}{\Sigma'} \left(\frac{\eta}{\rho P_r} \right) \frac{\partial^{-1}}{\partial t^{-1}} + \mathcal{O} \left(\frac{\partial^{-3/2}}{\partial t^{-3/2}} \right), \quad (7)$$

In Equations (6) and (7), $\delta(t)$ is the Dirac function. In this model, $\frac{\partial^{-1}}{\partial t^{-1}}$ represents a temporal integral, and the term $\frac{\partial^{-1/2}}{\partial t^{-1/2}}$ is interpreted as a semi-derivative operator following the definition of the fractional derivative of order ν given in Samko et al. [33]:

$$\frac{\partial^\nu}{\partial t^\nu} [x(t)] = \frac{1}{\Gamma(-\nu)} \int_0^t (t-u)^{-\nu-1} x(u) du, \quad (8)$$

where $\Gamma(x)$ is the gamma function. In the time domain, the constitutive relations (1) in a porous medium are given by

$$\int_0^t \rho \tilde{\alpha}(t-t') \frac{\partial \mathbf{v}(\mathbf{r}, t')}{\partial t'} dt' = -\nabla p(\mathbf{r}, t), \quad (9)$$

$$\int_0^t \frac{1}{K} \tilde{\beta}(t-t') \frac{\partial p(\mathbf{r}, t')}{\partial t'} dt' = -K \nabla \cdot \mathbf{v}(\mathbf{r}, t). \quad (10)$$

Using the expressions (6) and (7) of the tortuosity and compressibility operators and the constitutive relations (9) and (10), we obtain the following relations:

$$\rho \alpha_\infty \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + 2 \left(\frac{\alpha_\infty \sqrt{\rho \eta}}{\Lambda} \right) \frac{\partial^{1/2} \mathbf{v}(\mathbf{r}, t)}{\partial t^{1/2}} + \frac{3\alpha_\infty}{\Sigma} \left(\frac{\eta}{\rho} \right) \mathbf{v}(\mathbf{r}, t) = -\nabla p(\mathbf{r}, t), \quad (11)$$

$$\frac{1}{K} \frac{\partial p(\mathbf{r}, t)}{\partial t} + 2 \frac{\gamma - 1}{K \Lambda'} \left(\frac{\eta}{Pr \rho} \right)^{1/2} \frac{\partial^{1/2} p(\mathbf{r}, t)}{\partial t^{1/2}} + \frac{3(\gamma - 1)}{\Sigma'} \left(\frac{\eta}{\rho Pr} \right) p(\mathbf{r}, t) = \nabla \cdot \mathbf{v}(\mathbf{r}, t). \quad (12)$$

Equations (11) and (12) generalize those obtained in previous work for a unidirectional propagation problem [11,12,19]. Moreover, here, the characteristic viscous and thermal surfaces are taken into account, which is not the case in the previous developments. The use of the fractional derivative in the time domain makes it possible to describe the viscous and thermal exchanges between fluid and structure in the constitutive equations. These losses are at the origin of the attenuation and the acoustic absorption in a porous medium. Using the general constitutive relations (11) and (12), the propagation equation in the time domain is given by:

$$\Delta p(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} - b \frac{\partial^{3/2} p(\mathbf{r}, t)}{\partial t^{3/2}} - a \frac{\partial p(\mathbf{r}, t)}{\partial t} = 0, \quad (13)$$

where $c = \sqrt{K/\rho\alpha_\infty}$ is the front wave velocity, $a = -\frac{1}{c^2} \left(\frac{\eta}{\rho} \right) \left(\frac{3(\gamma-1)}{\Sigma' Pr} + \frac{4(\gamma-1)}{\Lambda \Lambda' \sqrt{Pr}} + \frac{3}{\Sigma} \right)$ and $b = \frac{2\alpha_\infty \sqrt{\rho\eta}}{K} \left(\frac{1}{\Lambda} + \frac{(\gamma-1)}{\sqrt{Pr} \Lambda'} \right)$ are terms of attenuation and dispersion, which depend mainly on the tortuosity α_∞ and viscous and thermal characteristic lengths Λ and Λ' . In Equation (13), we have stopped at the term $\partial/\partial t$, which corresponds to the term $(1/-i\omega)$ in the expansion of Equations (4) and (5). This fractional wave equation describes the acoustic propagation in a porous material with a rigid frame. The viscous and thermal losses are modeled using the fractional derivatives in the time domain of order 3/2. Using the definition of the fractional derivative (8), the fractional propagation equation can be written as

$$\Delta p(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} - a \frac{\partial p(\mathbf{r}, t)}{\partial t} - b \int_0^t \frac{\partial^2 p(\mathbf{r}, t')}{\partial t'^2} \frac{dt'}{\sqrt{t-t'}} = 0. \quad (14)$$

The fractional differential Equation (14) describes the propagation of a spherical wave generated by a point-like source, with the spherical wavefronts (connected parts of the wave that all share the same displacement, such as the crests of water waves) spreading radially outward from the point source. The porous medium is isotropic and homogeneous. This propagation equation was solved for the case of unidirectional propagation [11,12]. We will try in this paper to solve it as a three-dimensional problem using spherical coordinates.

3. Solution of the Fractional Spherical-Wave Equation

We propose in this section the solution the fractional integro-differential equation describing the propagation of transient ultrasonic waves in porous materials with a rigid frame given by Equation (13) or by:

$$\mathbf{r} = \mathbf{r}(x, y, z); \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

We will solve this fractional wave equation with the method of the Laplace transform. We note $P(\mathbf{r}, s)$, the Laplace transform of $p(\mathbf{r}, t)$, defined by [34]

$$P(\mathbf{r}, s) = \mathcal{L}[p(\mathbf{r}, t)] = \int_0^\infty \exp(-st) p(\mathbf{r}, t) dt. \quad (15)$$

The Laplace transform of Equation (14) yields

$$\begin{aligned} \Delta P(\mathbf{r}, s) - s^2 \left(\frac{1}{c^2} + \frac{a}{s} + b \sqrt{\frac{\pi}{s}} \right) P(\mathbf{r}, s) \\ = - \left(\frac{1}{c^2} + b \sqrt{\frac{\pi}{s}} \right) \times \left(s p(\mathbf{r}, 0) + \frac{\partial p}{\partial t}(\mathbf{r}, 0) \right) - a p(\mathbf{r}, 0). \end{aligned} \quad (16)$$

The initial conditions are assumed as:

$$p(\mathbf{r}, 0) = \frac{\partial p(\mathbf{r}, 0)}{\partial t} = 0. \quad (17)$$

Equation (16) becomes:

$$\Delta P(\mathbf{r}, s) - s^2 \left(\frac{1}{c^2} + \frac{a}{s} + b \sqrt{\frac{\pi}{s}} \right) P(\mathbf{r}, s) = 0. \quad (18)$$

By putting:

$$k^2(s) = \frac{1}{c^2} (s^2 + b's\sqrt{s} + c's), \quad b' = bc^2\sqrt{\pi}, \quad c' = ac^2, \quad (19)$$

we obtain the following Helmholtz equation:

$$(\Delta - k^2(s))P(\mathbf{r}, s) = 0. \quad (20)$$

Using the spherical coordinates $\mathbf{r} = \mathbf{r}(r, \theta, \varphi)$, the Laplacian is given by:

$$\Delta = P_r^2 + \frac{\ell^2}{r^2}, \quad (21)$$

where

$$P_r = \frac{1}{r} \frac{\partial}{\partial r} r = \frac{\partial}{\partial r} + \frac{1}{r},$$

is an operator depending only on the radial variable r , and ℓ is an operator depending only on the angular variables θ and φ and is given by:

$$\begin{aligned} \ell^2 &= \frac{1}{\sin^2 \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial \varphi^2} \right) \\ &= \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \end{aligned} \quad (22)$$

It is also in Cartesian coordinates:

$$\ell = \mathbf{r} \wedge \nabla, \quad \text{where} \quad \nabla^t = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \text{and by noting} \quad \ell^t = (l_x, l_y, l_z),$$

we have:

$$\begin{aligned} \ell_x &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} = - \left(\sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right), \\ \ell_y &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} = \cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi}, \\ \ell_z &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} = \frac{\partial}{\partial \varphi}. \end{aligned}$$

Equation (20) is, in fact, an eigenvalue equation, which can be written as

$$H P(\mathbf{r}, s) = 0, \quad (23)$$

where H is an operator given by

$$H = \Delta - k^2(s) = P_r^2 + \frac{\ell^2}{r^2} - k^2(s). \quad (24)$$

It can be seen from Equations (22) and (24) that H and ℓ^2 commute because H commutes with l_x , l_y , and l_z , and thus commutes with any function of these operators and, in particular, ℓ^2 . The operators H , ℓ^2 , and l_z have at least one complete set of eigenfunctions in common. We can therefore find the eigenfunctions of H from those of ℓ^2 and l_z . However, these are none other than the spherical harmonics $Y_\ell^m(\theta, \varphi)$ and are such that

$$\begin{aligned}\ell^2 Y_\ell^m(\theta, \varphi) &= -\ell(\ell+1) Y_\ell^m(\theta, \varphi), \\ l_z Y_\ell^m(\theta, \varphi) &= im Y_\ell^m(\theta, \varphi), \quad i^2 = -1,\end{aligned}\quad (25)$$

where l is all positive or zero integers, $\ell = 0, 1, 2, \dots$, and m is all integer values in the range $(-\ell, +\ell) - \ell \leq m \leq +\ell$.

We look for a solution in the form

$$P(\mathbf{r}, s) = P(r, \theta, \varphi, s) = R_\ell(\mathbf{r}, s) Y_\ell^m(\theta, \varphi); \quad l = 0, 1, 2, \dots, \text{ and } -\ell \leq m \leq +\ell. \quad (26)$$

By using the two Equations (23) and (26), we find that taking (25) into account satisfies the differential equation:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} - k^2(s) \right) R_\ell(\mathbf{r}, s) = 0, \quad (27)$$

which is a Bessel differential equation. Its special solutions are modified spherical Bessel functions of the first kind, $\sqrt{\frac{\pi}{2kr}} I_{\ell+1/2}(kr)$, of the second kind, $\sqrt{\frac{\pi}{2kr}} I_{-\ell-1/2}(kr)$, and of the third kind, $\sqrt{\frac{\pi}{2kr}} K_{\ell+1/2}(kr)$ [35].

Two linearly independent solutions are formed by the two functions $\sqrt{\frac{\pi}{2kr}} I_{\ell+1/2}(kr)$ and $\sqrt{\frac{\pi}{2kr}} I_{-\ell-1/2}(kr)$, or by both functions: $\sqrt{\frac{\pi}{2kr}} I_{\ell+1/2}(kr)$ and $\sqrt{\frac{\pi}{2kr}} K_{\ell+1/2}(kr)$. Keeping only the solutions that are finite at infinity, the general solution of (27) is given by:

$$U_\ell(r, s) = \varphi(s) R_\ell(r, s) = \varphi(s) K_{\ell+1/2}(r, s); \quad l = 0, 1, 2, \dots \quad (28)$$

where $\varphi(s)$ is a constant of integration determined by the initial conditions. It is known that the Bessel function can be calculated, for example, from the recurrence relation [35]:

$$R_\ell(r, s) = R_{\ell-2}(r, s) + \frac{(2\ell-1)}{kr} R_{\ell-1}(r, s). \quad (29)$$

where $R_0(\mathbf{r}, s)$ and $R_1(\mathbf{r}, s)$ are given by:

$$R_0(r, s) = -\sqrt{\frac{\pi}{2rk(s)}} K_{1/2}(rk(s)) = -\frac{\pi}{2rk(s)} \exp(-rk(s)), \quad (30)$$

$$R_1(r, s) = -\sqrt{\frac{\pi}{2rk(s)}} K_{3/2}(rk(s)) = \frac{\pi}{2rk(s)} \left(1 + \frac{1}{rk(s)} \right) \exp(-rk(s)). \quad (31)$$

4. Green Function of the Porous Medium in the Time Domain

Determining Green's function is an important step [11,12,19] in finding the solution to the fractional wave equation. Indeed, Green's function, sometimes called a "propagator", describes the propagation of a wave in the porous medium (velocity, attenuation, and dispersion) and depends on the wave number $k(s)$ (Equation (19)). The solution of the wave equation is thus obtained by carrying out a time convolution product of the incident signal with Green's function of the medium.

The Green function of the porous medium in the time domain is given by taking the inverse Laplace transform to solution (27):

$$G_\ell(r, t) = \mathcal{L}^{-1}(R_\ell(r, s)). \quad (32)$$

The inverse Laplace transforms of the relations (29)–(31) are used to calculate $G_\ell(r, t)$ from $G_0(r, t)$ and $G_1(r, t)$. Indeed, the Green function of the porous medium $G_\ell(r, t)$ satisfies the recurrence relation:

$$G_\ell(r, t) = G_{\ell-2}(r, t) + \frac{(2\ell-1)}{r} \mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) * G_{\ell-1}(r, t), \quad (33)$$

with $G_0(r, t)$ and $G_1(r, t)$ such that:

$$G_0(r, t) = -\frac{\pi}{2r} \mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) * \mathcal{L}^{-1} \exp(rk(s)), \quad (34)$$

and

$$G_1(r, t) = \frac{\pi}{2r} \left[\mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) + \frac{1}{r} \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right) \right] * \mathcal{L}^{-1}(\exp(-rk(s))). \quad (35)$$

In Equations (33)–(35), $*$ denotes the time convolution product. The term $\mathcal{L}^{-1}[\exp(-rk(s))] = F(r, t)$ corresponds to the Green function of the porous medium for a one-dimensional problem, which has already been calculated analytically in our previous work [11,12] and is given by:

$$F(r, \tau) = \begin{cases} 0, & \text{if } 0 \leq \tau \leq r/c, \\ \frac{r}{c} \left(\frac{b'}{4\sqrt{\pi}} (\tau - r/c)^{-3/2} \exp\left(-\frac{b'^2 r^2}{16c^2(\tau - r/c)}\right) + \Delta \int_0^{\tau - r/c} h(\xi) d\xi \right), & \text{if } t \geq r/c, \end{cases} \quad (36)$$

where $h(\xi)$ has the following form :

$$h(\xi) = -\frac{1}{\pi^{3/2}} \frac{1}{\sqrt{(\tau - \xi)^2 - r^2/c^2}} \frac{1}{\sqrt{\xi}} \int_0^1 \exp\left(-\frac{\chi(\mu, \tau, \xi)}{2}\right) (\chi(\mu, \tau, \xi) - 1) \frac{\mu d\mu}{\sqrt{1 - \mu^2}}, \quad (37)$$

and where $\chi(\mu, \tau, \xi) = \left(\Delta \mu \sqrt{(\tau - \xi)^2 - r^2/c^2} + b'(\tau - \xi) \right)^2 / 8\xi$, $\Delta = \sqrt{b'^2 - 4c'}$.

5. Analytical Solution in the Time Domain

The solution to Equations (26) and (27) in the time domain is obtained by the time convolution product of the Green functions given by relations (33)–(35) with the incident signal $p(0, t) = \mathcal{L}^{-1}(\varphi(s))$,

$$U_\ell(r, t) = \mathcal{L}^{-1}[U_\ell(r, s)] = p(0, t) * G_\ell(r, t) = \int_0^t G_\ell(r, \tau) p(0, t - \tau) d\tau. \quad (38)$$

We can therefore express the solution $U_\ell(r, t)$ with the recurrence relations

$$U_\ell(r, t) = U_{\ell-2}(r, t) + \frac{(2\ell-1)}{r} \mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) * U_{\ell-1}(r, t), \quad (39)$$

and

$$U_0(r, t) = -\frac{\pi}{2r} \mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) * \int_0^t F(r, \tau) p(0, t - \tau) d\tau, \quad (40)$$

$$U_1(r, t) = \frac{\pi}{2r} \left(\mathcal{L}^{-1}\left(\frac{1}{k(s)}\right) + \frac{1}{r} \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right) \right) * \int_0^t F(r, \tau) p(0, t - \tau) d\tau. \quad (41)$$

The convolution product used in Equations (40) and (41) of the Green function $F(r, t)$ with the incident signal $p(0, t)$ gives the solution $u(r, t)$ to the propagation equation in a 1D porous medium [11,12]:

$$u(r, t) = \int_0^t F(r, \tau) p(0, t - \tau) d\tau. \quad (42)$$

It is interesting to see here how the solution of the propagation equation for a spherical wave depends on the solution of the unidirectional propagation problem.

To express $U_0(r, t)$ and $U_1(r, t)$ given by Equations (40) and (41), we need to calculate the inverse Laplace transforms of $\frac{1}{k(s)}$ and $\frac{1}{k^2(s)}$, where the wavenumber $k(s)$ is given by Equation (19).

Let $D(t)$ be the inverse Laplace transform of $\frac{1}{k(s)}$; we have (Appendix A):

$$D(t) = \mathcal{L}^{-1}\left[\frac{1}{k(s)}\right] = \begin{cases} c \exp\left(-\frac{b'}{2}t\right) J_0\left(\frac{t}{2}\sqrt{4c' - b'^2}\right), & \text{if } 4c' - b'^2 > 0, \\ \frac{c}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{u^2}{4t}\right) \exp\left(-\frac{b'u}{2}\right) J_0\left(\frac{u}{2}\sqrt{4c' - b'^2}\right) du, & 4c' - b'^2 < 0, \end{cases} \quad (43)$$

when $4c' - b'^2 < 0$, we use the relation

$$J_0\left(\frac{U}{2}\sqrt{4c' - b'^2}\right) = J_0\left(i\frac{U}{2}\sqrt{b'^2 - 4c'}\right) = I_0\left(\frac{U}{2}\sqrt{b'^2 - 4c'}\right),$$

where J_0 is the Bessel function and I_0 is the modified Bessel function [34].

To calculate $U_1(r, t)$ given by Equation (48), we need to calculate the inverse Laplace transform of $\frac{1}{k^2(s)}$. Let us put $S(t) = \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right)$; according to the calculations made in Appendix B, one has

$$S(t) = \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right) = \begin{cases} \frac{2c}{\sqrt{b'^2 - 4c'}} \int_0^\infty \text{Erfc}\left(\frac{u}{2\sqrt{t}}\right) \exp\left(-\frac{b'u}{2}\right) \sinh\left(\frac{u}{2}\sqrt{b'^2 - 4c'}\right) du & \text{if } b'^2 - 4c' > 0, \\ \frac{-2c}{\sqrt{4c' - b'^2}} \int_0^\infty \text{Erfc}\left(\frac{y}{2\sqrt{t}}\right) \exp\left(-\frac{b'y}{2}\right) \sin\left(\frac{y}{2}\sqrt{4c' - b'^2}\right) dy & \text{if } 4c' - b'^2 > 0. \end{cases} \quad (44)$$

The solution to Equation (13) is obtained by taking the inverse Laplace transform of Equation (26):

$$p_{\ell,m}(\mathbf{r}, t) = \mathcal{L}^{-1}[P(r, \theta, \varphi, s)] = U_\ell(\mathbf{r}, t) Y_\ell^m(\theta, \varphi), \quad \ell = 0, 1, 2, \dots, -\ell \leq m \leq \ell, \quad (45)$$

where the radial solution $U_\ell(\mathbf{r}, t)$ is given by Equation (28) and satisfies the recurrence relation:

$$U_\ell(r, t) = U_{\ell-2}(r, t) + (2\ell - 1) \int_0^t D(\tau) U_{\ell-1}(r, t - \tau) d\tau, \quad \ell = 2, 3, \dots \quad (46)$$

The expression of $D(t)$ is given by (43). The functions $U_0(r, t)$ and $U_1(r, t)$ are given by:

$$U_0(r, t) = -\frac{\pi c}{2r} \int_0^t D(\tau) u(r, t - \tau) d\tau, \quad (47)$$

$$U_1(r, t) = \frac{\pi c}{2r} \int_0^t \left(D(\tau) + \frac{c}{r} S(\tau)\right) u(r, t - \tau) d\tau, \quad (48)$$

where $S(t)$ is given by Equation (44) and $u(r, t)$ by Equation (42).

Using solution (45), it is easy to calculate the first terms for $\ell = 0$, which induces $m = 0$. Knowing that $Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$, one has

$$p_{0,0}(r, t) = U_0(r, t)Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}U_0(r, t).$$

When $\ell = 1$, $m = -1, 0, 1$, and knowing that

$$Y_1^0(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos \theta \quad \text{and} \quad Y_1^{\pm 1}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin \theta \exp(\pm i\varphi),$$

we have the following solutions:

$$p_{1,0}(r, t) = \sqrt{\frac{3}{4\pi}} \cos \theta \cdot U_1(r, t), \quad (49)$$

$$p_{1,\pm 1}(r, t) = \sqrt{\frac{3}{8\pi}} \sin \theta \cdot \exp(\pm i\varphi) U_1(r, t), \quad (50)$$

where $U_1(r, t)$ is given by Equation (48).

The higher-order terms of the solution $p_{\ell,m}(r, t)$ (Equation (45)) can be calculated from the first two solutions $p_{0,0}(r, t)$ and $p_{1,m}(r, t)$ given by relations (50) and (49) and the recurrence relation (46), which originates from the properties of the Bessel function (28). The analytical calculation of the inverse Laplace transforms of expressions (43) and (44) makes it possible to obtain an analytical form in the time domain of the solution to the fractional spherical-wave equation. This solution allows us to predict the ultrasonic wave in spherical coordinates propagating in a porous medium by knowing the wave $p(0, t)$ incident in the medium given by the initial conditions and the properties of the material represented by the wave number $k(s)$, which essentially depends on the porosity, tortuosity, viscous and thermal characteristic lengths, and viscous and thermal surfaces.

6. Discussion and Conclusions

In this paper, we have solved the fractional propagation equation in spherical coordinates. This equation describes ultrasonic propagation in a porous material obeying the equivalent fluid model, in which the acoustic wave propagates only in the fluid saturating the porous medium, with the structure remaining motionless. This model is obtained through symmetry considerations (invariance by translation and rotation) of the Lagrangian of the Biot theory. We have shown that Green's function of the porous medium (in spherical coordinates) and the solution of its fractional wave equation are, respectively, expressed by recurrence relations and depend on Green's function of a unidirectional propagation problem and its fractional wave equation solution. Calculations are performed analytically in the Laplace domain, thus obtaining analytical expressions in the time domain of the response of the medium to an acoustic excitation. The time-domain analysis has the advantage of being the most suitable for transient signals, such as pulses or bursts, which are often used in experimental reality for the characterization and resolution of direct and inverse problems.

This essentially theoretical work with analytical results constitutes an important first step for the resolution of the direct problem allowing us to predict the ultrasonic propagation of an acoustic wave in a porous medium in spherical coordinates while knowing the incident wave and the physical properties of the porous medium (porosity ϕ , tortuosity α_∞ , viscous Λ and thermal lengths Λ' , and viscous Σ and thermal surfaces Σ').

In the future, we plan to perform numerical simulations to validate and compare the proposed approaches and consider the problem of reflection and transmission of an acoustic wave by a porous medium by associating the appropriate boundary conditions. Our objective is to propose a model in spherical coordinates that allows us to solve the

inverse problem based on experimental data of reflected and/or transmitted waves in order to recover the mechanical and acoustic properties as what currently exists for a unidirectional problem.

Author Contributions: Writing—original draft preparation, Z.E.A.F., M.F., E.O., N.O.O. and C.D.; analysis, analytical calculations, and processing of results, Z.E.A.F., R.R., M.F., E.O. and C.D.; writing—review and editing, Z.E.A.F., M.F., E.O. and C.D.; resources, Z.E.A.F., M.F., E.O. and C.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Calculus of $D(t) = \mathcal{L}^{-1}\left[\frac{1}{k(s)}\right]$

Let

$$d(s) = \mathcal{L}[D(t)] = \frac{c}{\sqrt{s^2 + b's + c'}},$$

we have:

$$D(t) = \mathcal{L}^{-1}[d(s)] = c \exp\left(-\frac{b'}{2}t\right) J_0\left(\frac{t}{2}\sqrt{4c' - b'^2}\right) \quad \text{if} \quad 4c' - b'^2 > 0.$$

Using the property

$$\mathcal{L}^{-1}\left[\frac{d(\sqrt{s})}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{u^2}{4t}\right) du,$$

we obtain:

$$\begin{aligned} D(t) &= \mathcal{L}^{-1}\left[\frac{1}{k(s)}\right] = \mathcal{L}^{-1}\left[\frac{c}{\sqrt{s^2 + b's + c'}}\right] \\ &= \frac{c}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{U^2}{4t}\right) \exp\left(-\frac{b'U}{2}\right) J_0\left(\frac{U}{2}\sqrt{4c' - b'^2}\right) dU, \end{aligned} \quad (\text{A1})$$

where $4c' - b'^2 < 0$, and we take into account the fact that

$$J_0\left(\frac{U}{2}\sqrt{4c' - b'^2}\right) = J_0\left(i\frac{U}{2}\sqrt{b'^2 - 4c'}\right) = I_0\left(\frac{U}{2}\sqrt{b'^2 - 4c'}\right).$$

Appendix B. Calculus of $S(t) = \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right)$

We have:

$$\mathcal{L}^{-1}\left(\frac{c}{s^2 + b's + c'}\right) = \begin{cases} \frac{2c}{\sqrt{b'^2 - 4c'}} \exp\left(-\frac{b'}{2}t\right) \sinh\left(\frac{t}{2}\sqrt{b'^2 - 4c'}\right), & \text{if } b'^2 - 4c' > 0, \\ \frac{-2c}{\sqrt{4c' - b'^2}} \exp\left(-\frac{b'}{2}t\right) \sin\left(\sqrt{4c' - b'^2}t\right), & \text{if } 4c' - b'^2 > 0. \end{cases}$$

Let be $E(s)$ the inverse Laplace transform of a function $e(t)$; we have the property:

$$\mathcal{L}^{-1}\left[\frac{E(\sqrt{s})}{s}\right] = \int_0^\infty \text{Erfc}\left(\frac{U}{2\sqrt{t}}\right) e(u) du,$$

where $\text{Erfc}(x)$ is the complementary error function given by:

$$\text{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-t^2) dt.$$

Tacking

$$E(s) = \frac{c}{s^2 + b's + c'},$$

and we have:

$$\frac{E(\sqrt{s})}{s} = \frac{1}{k^2(s)} = \frac{c}{z^2 + b'z\sqrt{z} + c'z'},$$

so that

$$S(t) = \mathcal{L}^{-1}\left(\frac{1}{k^2(s)}\right) = \begin{cases} \frac{2c}{\sqrt{b'^2 - 4c'}} \int_0^\infty \text{Erfc}\left(\frac{U}{2\sqrt{t}}\right) \exp\left(-\frac{b'}{2}U\right) \sinh\left(\frac{U}{2}\sqrt{b'^2 - 4c'}\right) dU & \text{if } b'^2 - 4c' > 0, \\ \frac{-2c}{\sqrt{4c' - b'^2}} \int_0^\infty \text{Erfc}\left(\frac{U}{2\sqrt{t}}\right) \exp\left(-\frac{b'}{2}U\right) \sin\left(\frac{U}{2}\sqrt{4c' - b'^2}\right) dU & \text{if } 4c' - b'^2 > 0. \end{cases}$$

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