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A Qualitative Study on Second-Order Nonlinear Fractional Differential Evolution Equations with Generalized ABC Operator

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Abstract: This research paper is dedicated to an investigation of an evolution problem under a new operator ($\mathfrak{g}$-Atangana–Baleanu–Caputo type fractional derivative)(for short, $\mathfrak{g}$-ABC). For the proposed problem, we construct sufficient conditions for some properties of the solution like existence, uniqueness and stability analysis. Existence and uniqueness results are proved based on some fixed point theorems such that Banach and Krasnoselskii. Furthermore, through mathematical analysis techniques, we analyze different types of stability results. The symmetric properties aid in identifying the best strategy for getting the correct solution of fractional differential equations. An illustrative example is discussed for the control problem.

Keywords: $\mathfrak{g}$-Atangana–Baleanu–Caputo type FD; integral boundary conditions; existence; stability results; fixed point theorem

1. Introduction

Arbitrary integration and differentiation are some of the most interesting research fields because they are suitable tools for modeling complex phenomena in a wide range of science and engineering fields, such as chemical engineering, electrodynamics, power systems, biological sciences, etc. (for more information see [1–9]). To meet the needs of modeling many practical problems in different fields of science and engineering, some researchers have realized the necessity development of the concept of fractional calculus by searching for new fractional derivatives with different singular or non-singular kernels. From this perspective, new fractional operators have turned into the best effective tool of numerous specialists and researchers with their contribution to physical phenomena and their performance in applying to real-world problems. Some applications of the operator used can be found in [10–13]. The symmetries can be found by solving a related set of partial fractional differential equations. Until 2015, all fractional derivatives had only singular kernels. Therefore, it is difficult to use these singularities to simulate physical phenomena. Caputo and Fabrizio in [14] introduced a new type of FD in the exponential kernel denoted by CF. The CF-fractional derivative has some problems regarding the locality of its kernel. Atangana and Baleanu (AB) in [15] investigsted new type and interesting FD with Mittag–Leffler kernels. The AB technique makes an outstanding memory description and inhold qualities for mean-square displacement using this generalised "Mittag–Leffler" function as a kernel [16,17]. Abdeljawad in [18] extended AB-fractional derivative type to higher arbitrary order and formulated their associated integral operators. There are some researchers who studied the properties of solution for some fractional differential equations via generalized fractional derivatives with respect to another function g, for example, [19,20]. The advantage of the operator ABC fractional derivative with respect to another function g (g-ABC) used in this work is the freedom of choice the suitable classical differentiation operator and the suitable function g to modeling some real-world problems such as various infectious diseases like Ebola virus, Leptospirosis, dynamics of smoking, etc in a more comprehensive way, see [21,22]. There are some researchers in the various area investigated some properties for evolution FDEs, for example, Zhao in [23] developed the adequate conditions of exact controllability for a new class of impulsive fractional functional evolution equations (IFFEEs) using resolvent operator theory. In [24], by using the resolvent operator theory, and the Picard type iterative methodology, the authors examined some properties of mild solutions for a class of R-L fractional stochastic evolution equations of Sobolev type in abstract spaces. Shokri in [25] developed a new class of two-step multiderivative methods for solving second-order initial value problems numerically. Recently, Almalahi et al. in [26] study some qualitative properties of solution for the following problem

$$\begin{cases} \left({}^{H}D^{\mathfrak{q},\beta;g}_{a^+}+\lambda\right)u(\varrho)=f(\varrho,\mu(\varrho)),\varrho\in J:=(a,T], \\ u(a)=0,u(T)=\sum_{i=1}^{m}\kappa_i I^{\zeta,g}_{a^+}\mu(\eta_i),\eta_i\in(a,T), \end{cases}$$

where ${}^{H}D^{\mathfrak{q},\beta,\phi}_{a^+}$denotes the ϕ-Hilfer FD of order $\mathfrak{q}\in(1,2)$, $\beta\in[0,1]$, $\gamma=\mathfrak{q}+2\beta-\mathfrak{q}\beta$, $\lambda<0$ and the integer $m\geq 1$. $f:J\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Raja et al. [27] proved some properties of the following fractional differential evolution equations

$$\begin{cases} \left({}^{C}\mathbf{D}^{\mathfrak{q}}_{0^+}-A(\varrho)\right)u(\varrho)=\mathcal{R}(\varrho)+f(\varrho,u(\varrho)),\varrho\in[0,T], \\ u(0)=u_0,u'(0)=u_1, \end{cases}$$

where ${}^{c}\mathbf{D}^{\mathfrak{q}}$ is the Mittag–Leffler-fractional derivative of order $\mathfrak{q}\in(1,2]$. Kamal Shah et al. [28] studied some properties of solutions for controllability problem of the following evolution equation

$$\begin{cases} \left({}^{ML}\mathbf{D}^{\mathfrak{q}}_{0^+}-\Phi(\varrho)\right)u(\varrho)=f(\varrho,u(\varrho),Zu(\varrho)),\varrho\in[0,T], \\ u(0)=u_0+\int_0^T\frac{(T-\eta)^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\hbar(u(\eta))d\eta, \end{cases}$$

where ${}^{ML}\mathbf{D}^{\mathfrak{q}}$ is the Mittag–Leffler-FD of order $\mathfrak{q}\in(0,1]$.

Motivated by the above argumentations, we investigate the sufficient conditions for the existence and uniqueness as well as different types of stability results for an important class of differential equations, called evolution equations which used to explain the law of differentiation to describe the development of dynamic systems described as follows

$$\begin{cases} \left({}^{ABC}\mathbf{D}^{\mathfrak{q};g}_{0^+}-\Phi(\varrho)\right)u(\varrho)=f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho)),\ \varrho\in J:=[0,T], \\ u(0)=0,u(T)=\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta, \end{cases} \tag{1}$$

where

(i) ${}^{ABC}\mathbf{D}^{\mathfrak{q};g}$ is the g-ABC-fractional derivative of order $\mathfrak{q}\in(1,2]$.
(ii) ${}^{RL}I^{\delta,g}$ is g-R-L fractional integral of order $\delta\in(0,1]$.
(iii) g is an increasing function, having a continuous derivative g' on $(0,T)$ such that $g'(\varrho)\neq 0$, for all $\varrho\in J$.

(iv) $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function fulfilled some conditions described later.
(v) $\Phi, \hbar : J \to \mathbb{R}$ are continuous functions.

To the best of our knowledge, this is the first work considering fractional-order equation with AB fractional derivative with respect to another function g. Various approaches to the definition of the fractional derivative with different kernels have been proposed in the literature. As a result, the researchers are motivated to use the operator that is best suited to the model they are studying. Fractional derivative with a nonsingular kernel has attracted a lot of attention in the recent past due to some physical phenomena that are difficult to model as a result of the singularities. The proposed problem (1) for different values of a function g includes the study of problems involving the results in [28] and many other results which do not study yet.

Observe that our approach used in this work is new because we prove the existence, uniqueness, and different types of stability results without using semigroup property and relies on a minimum number of hypotheses. More than that, the proposed problem (1) for different values of a function g includes the study of problems involving the results in [28] and many other results which do not study yet.

This paper has the following structure: In Section 2, we will introduce some symbols, auxiliary lemmas, and some basic definitions used throughout this paper. In addition, we deduced the equivalent solution formula of the g-ABC problem (1). The existence and uniqueness results for the g-ABC problem (1) have been discussed in Section 3. In Section 4, we analyse the stability results in the sense of Ulam-Hyers. Some examples provide to illustrate our results in section 5. In the last section, we present a summary comment on the results.

2. Preliminaries

This part is dedicated to reviewing some concepts, definitions, and auxiliary propositions that will be used later. Let $J = [0, T] \subset \mathbb{R}$. Let $\mathcal{X}$ be a Banach space with the norm $\|\cdot\|$. Let $C(J, \mathcal{X})$ be a Banach space of all continuous functions $u : J \to \mathcal{X}$ equipped with the norm

$$\|u\| = \sup\{\|u(\varrho)\| : \varrho \in J\},$$

where $u \in C(J, \mathcal{X})$.

Definition 1 ([2,5])**.** *Let $\mathfrak{q} > 0$ and $u \in L_1(J)$. Then, the following expression*

$$^{RL}I_{0^+}^{\mathfrak{q},g} u(\varrho) = \int_0^{\varrho} g'(s) \frac{(g(\varrho) - g(s))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} u(s) ds,$$

is called the left g-RL-FI of u of order $\mathfrak{q}$. Furthermore, the g-RL-FD of u of order $\mathfrak{q} \in (n, n+1], n \in \mathbb{N}$ is given by

$$^{RL}\mathbf{D}_{0^+}^{\mathfrak{q},g} u(\varrho) = \left(\frac{1}{g'(\varrho)} \frac{d}{d\varrho}\right)^n \left(^{RL}I_{0^+}^{n-\mathfrak{q},g} u(\varrho)\right).$$

Definition 2 ([29] Theorem 4)**.** *The left-sided g-ABC-FD of a function u of order $\mathfrak{q} \in (0, 1]$ are defined respectively by*

$$^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g} u(\varrho) = \frac{\mathfrak{B}(\mathfrak{q})}{1 - \mathfrak{q}} \sum_{n=0}^{\infty} \left(\frac{-\mathfrak{q}}{1 - \mathfrak{q}}\right)^n \mathbf{D}_{0^+}^{-n\mathfrak{q}-1} \frac{u'(\varrho)}{g'(\varrho)},$$

where the normalization function $\mathfrak{B}(\mathfrak{q})$ satisfied $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

Definition 3 ([30] Lemma 3.1)**.** *The correspondent g-AB fractional integral of order $\mathfrak{q} \in (0, 1]$ of the left-sided g-ABC-FD of a function u is defined by*

$$^{AB}I_{0^+}^{\mathfrak{q},g} u(\varrho) = \frac{1 - \mathfrak{q}}{\mathfrak{B}(\mathfrak{q})} u(\varrho) + \frac{\mathfrak{q}}{\mathfrak{B}(\mathfrak{q})} {}^{RL}I_{0^+}^{\mathfrak{q},g} u(\varrho).$$

Lemma 1 ([29])**.** *Let $0 < \mathfrak{q} \le 1$. If g-ABC fractional derivative exists, then we have*

$$^{AB}I_{0^+}^{\mathfrak{q},g}\ {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}u(\varrho) = u(\varrho) - u(a).$$

Lemma 2 ([15,18])**.** *Let $u(\varrho)$ be a function defined on J and $n < \mathfrak{q} \le n+1$, for some $n \in \mathbb{N}_0$, we have*

$$\left({}^{AB}I_{0^+}^{\mathfrak{q},ABC}\mathbf{D}_{0^+}^{\mathfrak{q}}u\right)(\varrho) = u(\varrho) - \sum_{i=0}^{n}\frac{u^{(i)}(0)}{i!}[\varrho]^i.$$

Theorem 1 ([31])**.** *Let $\mathbf{K}$ be a closed subspace of a Banach space $\mathbf{X}$. If there is a contraction mapping $\mathbf{G} : \mathbf{K} \to \mathbf{K}$, then $\mathbf{G}$ has a fixed point in $\mathbf{K}$.*

Theorem 2 ([32])**.** *Assume that K is a closed, convex, bounded and nonempty subset of space $\mathbf{X}$. If there is two operators Ξ^1, Ξ^2 such that $\Xi^1 u + \Xi^2 v \in \mathbf{X}$, $u, v \in \mathbf{X}$, Ξ^1 is completely continuous and Ξ^2 is contraction mapping, then there exists a solution $z \in K$ such that $z = \Xi^1 z + \Xi^2 z$.*

Lemma 3 ([18] Example 3.3)**.** *Let $\mathfrak{q} \in (1,2]$ and $w \in C(J, \mathcal{X})$, $w(0) = 0$. Then the solution of the following linear problem*

$$\begin{cases} {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q}}u(\varrho) = w(\varrho), \\ u(0) = c_1, u'(0) = c_2, \end{cases}$$

is given by

$$u(\varrho) = c_1 + c_2\varrho + \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho} w(\eta)d\eta + \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}\frac{(\varrho-\eta)^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}w(\eta)d\eta.$$

3. Equivalent Integral Equation

In this section, we obtain the equivalent integral equation of the problem (1).

Theorem 3. *Let $\mathfrak{q} \in (1,2]$, $\delta \in (0,1]$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\Phi, \hbar : J \to \mathbb{R}$ be continuous functions. Then $u \in C(J, \mathcal{X})$ is a solution of the following g-ABC-problem*

$$\begin{cases} \left({}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g} - \Phi(\varrho)\right)u(\varrho) = f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)), \varrho \in [0,T], \\ \quad u(0) = 0, u(T) = \int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta, \end{cases} \tag{2}$$

if and only if, u satisfies the following fractional integral equation

$$\begin{aligned} u(\varrho) \quad = \quad & \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta \\ & -\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\right]d\eta \\ & -\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\right]d\eta\Bigg] \\ & +\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}\left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\right]d\eta \\ & +\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho} g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ & \left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\right]d\eta. \end{aligned} \tag{3}$$

Proof. First, we assume that u is a function satisfies (2). Inserting the operator ${}^{AB}I_{0^+}^{\mathfrak{q},g}$ on both sides of the following equation

$${}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}u(\varrho) = \Phi(\varrho)u(\varrho) + f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)).$$

With the help of Lemma 3, we get

$$\begin{aligned} u(\varrho) &= c_1 + c_2(g(\varrho) - g(0)) \\ &+ \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho \Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &+ \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta. \end{aligned} \tag{4}$$

By the first condition $(u(0) = 0)$, we get $c_1 = 0$ and hence Equation (4) reduce to the following equation

$$\begin{aligned} u(\varrho) &= c_2(g(\varrho) - g(0)) \\ &+ \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho \Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &+ \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta. \end{aligned}$$

By the second condition $\left(u(T) = \int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta\right)$, we obtain

$$\begin{aligned} c_2 &= \frac{1}{(g(T)-g(0))}\Bigg[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta \\ &- \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^T \Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &- \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\Bigg]. \end{aligned}$$

Substitute c_1, c_2 in (4), we get (3).

Conversely, assume that u satisfies integral Equation (3). Then, by applying the operator ${}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}$ on both sides of (3), we get

$$\begin{aligned} {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}u(\varrho) &= {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}\frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta \\ &- \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^T \Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &- \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\Bigg] \\ &+ {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}\Bigg(\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho \Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &+ \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\Bigg). \end{aligned}$$

It follows from the fact ${}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}(g(\varrho)-g(0))=0$, that

$$\begin{aligned}{}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}u(\varrho) &= {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}\bigg(\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &\quad +\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\quad \Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\bigg) \\ &= {}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q},g}\,{}^{AB}I_{0^+}^{\mathfrak{q},g}\Big[\Phi(\varrho)u(\varrho)+f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho))\Big].\end{aligned}$$

Thus, we get

$$\Big({}^{ABC}\mathbf{D}_{0^+}^{\mathfrak{q};g}-\Phi(\varrho)\Big)u(\varrho)=f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho)).$$

This means that u satisfies the fractional boundary value problem (2).

Next, to prove Atangana–Baleanu fractional integral conditions, by take $\varrho=0$ in (3), we get $u(0)=0$. On the other hand, taking again $\varrho=T$ in (3), we get

$$u(T)=\int_0^{T}g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta.$$

Thus, the Atangana–Baleanu fractional integral conditions are satisfied. The proof is completed. □

4. Existence and Uniqueness Results

We devote our intention in this part to prove the existence and uniqueness of solutions for the g-ABC-fractional differential Equation (1). To our analysis, we present the following necessary assumptions;

Hypothesis 1 (H1). *Let $\hbar$ be bounded linear operator and there exists constant $\mathfrak{Z}_{\hbar}>0$ such that for any $u,\widehat{u}\in C(J,\mathcal{X})$, we have*

$$\|\hbar(u)-\hbar(\widehat{u})\|\leq\mathfrak{Z}_{\hbar}\|u-\widehat{u}\|.$$

Hypothesis 2 (H2). *For a bounded linear operator $\hbar$, there exist constants $\mathcal{W}_{\hbar}>0$, $\mathcal{A}_{\hbar}\geq 0$, such that for any $u\in C(J,\mathcal{X})$, we have*

$$\|\hbar(u)\|\leq\mathcal{W}_{\hbar}\|u\|+\mathcal{A}_{\hbar}.$$

Hypothesis 3 (H3). *Let f be a continuous function and there exists constants $\mathfrak{L}_1,\mathfrak{L}_2>0$ such that for any $u,\widehat{u},v,\widehat{v}\in C(J,\mathcal{X})$, we have*

$$\|f(\varrho,u,v)-f(\varrho,\widehat{u},\widehat{v})\|\leq\mathfrak{L}_1\|u-\widehat{u}\|+\mathfrak{L}_2\|v-\widehat{v}\|.$$

Hypothesis 4 (H4). *Let f be a continuous function and there exists $\mathcal{W}_f(\varrho)\in C(J,\mathcal{X})$ such that*

$$\|f(\varrho,u,\widehat{u})\|\leq\mathcal{W}_f(\varrho),$$

with $\sup_{\varrho\in J}\Big|\mathcal{W}_f(\varrho)\Big|=\mathcal{W}_f^*$.

To simplify our analysis, we used the following notations

$$\mathcal{G}=\left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathfrak{Z}_{\hbar}+\Pi_{g,\mathfrak{q}}\left(\mathcal{K}_{\Phi}+\mathfrak{L}_1+\mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\right)\right],$$

and

$$\Pi_{g,\mathfrak{q}} = 2\left[\frac{(\mathfrak{q}-1)}{\mathfrak{B}(\mathfrak{q}-1)}\frac{(g(T)-g(\eta))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)} + \frac{(2-\mathfrak{q})}{\mathfrak{B}(\mathfrak{q}-1)}\right]. \tag{5}$$

Theorem 4. *Assume that (H1)–(H3) hold. Then, the g-ABC-fractional differential Equation (1) has a unique solution provided that $\mathcal{G} < 1$.*

Proof. We first convert the equivalent g-ABC-fractional differential Equation (1) to a fixed-point problem. In view of Theorem 3, we define the operator $\Xi : C(J,\mathcal{X}) \to C(J,\mathcal{X})$ by

$$\begin{aligned}
\Xi u(\varrho) \quad = \quad & \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\left[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta\right. \\
& -\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\left[\Phi(\eta)u(\eta) + f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta \\
& -\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\
& \left.\left[\Phi(\eta)u(\eta) + f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta\right] \\
& +\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho\left[\Phi(\eta)u(\eta) + f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta \\
& +\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\
& \left[\Phi(\eta)u(\eta) + f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta.
\end{aligned} \tag{6}$$

Define a closed ball Π_φ as

$$\Pi_\varphi = \{u \in C(J,\mathcal{X}) : \|u\| \le \varphi\},$$

with radius $\varphi \ge \frac{\mathcal{G}_1}{1-\mathcal{G}}$, where

$$\mathcal{G}_1 = \frac{(g(T)-g(0))^\delta}{\Gamma(\delta+1)}\mathcal{A}_\hbar + \Pi_{g,\mathfrak{q}}\omega_f,$$

and $\omega_f = \max_{\varrho\in J}\|f(\varrho,0,0)\|$. Clearly, Π_φ is nonempty, bounded, convex and closed. Now, in order to apply Theorem 1, we divided the proof into two steps as follows.

Step (1): We will show that $\Xi\Pi_\varphi \subset \Pi_\varphi$. Since $\Phi(\varrho)$ is bounded function, then, there exists constant number $\mathcal{K}_\Phi > 0$, such that $|\Phi(\varrho)| \le \mathcal{K}_\Phi$ for each $\varrho \in J$. By (H3), we obtain

$$\begin{aligned}
\|f(\varrho,u(\varrho),Zu(\varrho))\| \quad & \le \quad \left\|f(\varrho,u(\varrho),^{RL}I^{\delta,g}u(\varrho)) - f(\varrho,0,0)\right\| + \|f(\varrho,0,0)\| \\
& \le \quad \mathfrak{L}_1\|u\| + \mathfrak{L}_2\left\|^{RL}I^{\delta,g}u\right\| + \omega_f.
\end{aligned}$$

Now, for all $u \in \Pi_{\varphi}$, $\varrho \in J$, we have

$$
\begin{aligned}
\|\Xi u\| \leq\ & \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\|\hbar(u(\eta))\|d\eta \\
&+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta \\
&+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\
&\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta\Bigg] \\
&+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta \\
&+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho} g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\
&\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta \\
\leq\ & \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}(\mathcal{W}_{\hbar}\varphi+\mathcal{A}_{\hbar}) \\
&+\frac{(2-\mathfrak{q})}{\mathfrak{B}(\mathfrak{q}-1)}\left(\mathcal{K}_{\Phi}\varphi+\mathfrak{L}_1\varphi+\mathfrak{L}_2\frac{[g(T)-g(0)]^{\delta}}{\Gamma(\delta+1)}\varphi+\omega_f\right) \\
&+\frac{(\mathfrak{q}-1)}{\mathfrak{B}(\mathfrak{q}-1)}\frac{(g(T)-g(\eta))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)}\left(\mathcal{K}_{\Phi}\varphi+\mathfrak{L}_1\varphi+\mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\varphi+\omega_f\right)\Bigg] \\
&+\frac{(2-\mathfrak{q})}{\mathfrak{B}(\mathfrak{q}-1)}\left(\mathcal{K}_{\Phi}\varphi+\mathfrak{L}_1\varphi+\mathfrak{L}_2\frac{[g(T)-g(0)]^{\delta}}{\Gamma(\delta+1)}\varphi+\omega_f\right) \\
&+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\frac{(g(T)-g(\eta))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)}\left(\mathcal{K}_{\Phi}\varphi+\mathfrak{L}_1\varphi+\mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\varphi+\omega_f\right).
\end{aligned}
$$

We used fact that g is an increasing function, it follows that $\frac{(g(\varrho)-g(0))}{(g(T)-g(0))} < 1$, for $\varrho < T$. Consequently,

$$
\begin{aligned}
\|\Xi u\| \leq\ & \left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{W}_{\hbar}+\Pi_{g,\mathfrak{q}}\left(\mathcal{K}_{\Phi}+\mathfrak{L}_1+\mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\right)\right]\varphi \\
&+\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{A}_{\hbar}+\Pi_{g,\mathfrak{q}}\omega_f \\
\leq\ & \mathcal{G}\varphi+\mathcal{G}_1<\varphi.
\end{aligned} \tag{7}
$$

Thus $\Xi\Pi_{\varphi} \subset \Pi_{\varphi}$.

Step (2): We will show that Ξ is a contraction mapping. Let $u, \widehat{u} \in \Pi_{\varphi}$ and $\varrho \in J$. Then, we estimate

$$
\begin{aligned}
& \|\Xi u - \Xi\widehat{u}\| \\
\leq\ & \int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\|\hbar(u(\eta)) - \hbar(\widehat{u}(\eta))\|d\eta \\
& + \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T [|\Phi(\eta)|\|u(\eta) - \widehat{u}(\eta)\| \\
& + \left\|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)) - f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho))\right\|\Big]d\eta \\
& + \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}[|\Phi(\eta)|\|u(\eta) - \widehat{u}(\eta)\| \\
& + \left\|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)) - f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho))\right\|\Big]d\eta \\
& + \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho [|\Phi(\eta)|\|u(\eta) - \widehat{u}(\eta)\| \\
& + \left\|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)) - f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho))\right\|\Big]d\eta \\
& + \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}[|\Phi(\eta)|\|u(\eta) - \widehat{u}(\eta)\| \\
& + \left\|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)) - f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho))\right\|\Big]d\eta.
\end{aligned}
$$

From (H3), we obtain

$$
\begin{aligned}
& \left\|f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta)) - f(\eta, \widehat{u}(\eta), {}^{RL}I^{\delta,g}\widehat{u}(\eta))\right\| \\
\leq\ & \mathfrak{L}_1\|u(\eta) - \widehat{u}(\eta)\| + \mathfrak{L}_2\left\|{}^{RL}I^{\delta,g}u(\eta) - {}^{RL}I^{\delta,g}\widehat{u}(\eta)\right\| \\
\leq\ & \mathfrak{L}_1\|u - \widehat{u}\| + \mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\|u - \widehat{u}\| \\
\leq\ & \left(\mathfrak{L}_1 + \mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\right)\|u - \widehat{u}\|. \qquad (8)
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \|\Xi u - \Xi\widehat{u}\| = \sup_{\varrho\in J}|\Xi u(\varrho) - \Xi\widehat{u}(\varrho)| \\
\leq\ & \left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathfrak{Z}_{\hbar} + \Pi_{g,\mathfrak{q}}\left(\mathcal{K}_{\Phi} + \mathfrak{L}_1 + \mathfrak{L}_2\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\right)\right]\|u - \widehat{u}\| \\
\leq\ & \mathcal{G}\|u - \widehat{u}\|.
\end{aligned}
$$

Due to $\mathcal{G} < 1$, we conclude that Ξ is a contraction operator. Hence, Theorem 1, implies that Ξ has a unique fixed point. □

Theorem 5. *Assume that (H1)–(H4) hold. If $\mathcal{Q} = \mathcal{T} + \mathcal{Y} < 1$, where*

$$
\begin{aligned}
\mathcal{T} &= \frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathfrak{Z}_{\hbar}, \\
\mathcal{Y} &= \frac{\Pi_{g,\mathfrak{q}}}{2}\left(\mathcal{K}_{\Phi} + \mathfrak{L}_1 + \mathfrak{L}_2\frac{[g(T)-g(0)]^{\delta}}{\Gamma(\delta+1)}\right),
\end{aligned}
$$

and

$$\left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{W}_{\hbar}+\Pi_{g,\mathfrak{q}}\mathcal{K}_{\Phi}\right]<1,$$

($\Pi_{g,\mathfrak{q}}$ is given by (5)), then the g-ABC-fractional differential Equation (1) has at least one solution.

Proof. We consider the operator Ξ defined in Theorem 4 and will divide it into two operators Ξ_1 and Ξ_2 such that

$$(\Xi u)(\varrho)=(\Xi_1 u)(\varrho)+(\Xi_2 u)(\varrho),$$

where

$$\begin{aligned}(\Xi_1 u)(\varrho) &= \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\left[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta\right.\\ &\quad-\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\left[\Phi(\eta)u(\eta)+f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta\\ &\quad-\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\\ &\quad\left.\left[\Phi(\eta)u(\eta)+f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta\right],\end{aligned}$$

and

$$\begin{aligned}(\Xi_2 u)(\varrho) &= \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho}\left[\Phi(\eta)u(\eta)+f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta\\ &\quad+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho} g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\\ &\quad\left[\Phi(\eta)u(\eta)+f(\eta,u(\eta),^{RL}I^{\delta,g}u(\eta))\right]d\eta.\end{aligned}$$

Define a closed ball Π_r as

$$\Pi_r=\{u\in C(J,\mathcal{X}):\|u\|\le r\},$$

with

$$r\ge\frac{\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{A}_{\hbar}+\Pi_{g,\mathfrak{q}}\mathcal{W}_f^*}{1-\left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{W}_{\hbar}+\Pi_{g,\mathfrak{q}}\mathcal{K}_{\Phi}\right]}.$$

Clearly, Π_r is nonempty, bounded, convex and closed. In order to apply Theorem 2, we will divide the proof into three steps as follows

Step 1: We shall show that $\Xi_1 u+\Xi_2\widehat{u}\in\Pi_r$. For all $u,\widehat{u}\in\Pi_r$, we have

$$\begin{aligned}\|(\Xi_1 u)\| &\le \frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}(\mathcal{W}_{\hbar}r+\mathcal{A}_{\hbar})\\ &\quad+\frac{(\mathfrak{q}-1)}{\mathfrak{B}(\mathfrak{q}-1)}\frac{(g(T)-g(\eta))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)}\left(\mathcal{K}_{\Phi}r+\mathcal{W}_f^*\right)\\ &\quad+\frac{(2-\mathfrak{q})}{\mathfrak{B}(\mathfrak{q}-1)}\left(\mathcal{K}_{\Phi}r+\mathcal{W}_f^*\right),\end{aligned} \tag{9}$$

and

$$\begin{aligned}\|(\Xi_2 u)\| &\leq \frac{(2-\mathfrak{q})}{\mathfrak{B}(\mathfrak{q}-1)}\Big(\mathcal{K}_{\Phi} r+\mathcal{W}_f^*\Big) \\ &\quad+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\frac{(g(T)-g(\eta))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)}\Big(\mathcal{K}_{\Phi} r+\mathcal{W}_f^*\Big). \end{aligned} \tag{10}$$

Thus, by (9) and (10), we obtain

$$\begin{aligned}\|\Xi_1 u+\Xi_2 \widehat{u}\| &\leq \|(\Xi_1 u)\|+\|(\Xi_2 u)\| \\ &\leq \left[\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{W}_{\hbar}+\Pi_{g,\mathfrak{q}}\mathcal{K}_{\Phi}\right] r \\ &\quad+\frac{(g(T)-g(0))^{\delta}}{\Gamma(\delta+1)}\mathcal{A}_{\hbar}+\Pi_{g,\mathfrak{q}}\mathcal{W}_f^* \\ &\leq r. \end{aligned}$$

This implies that $\Xi_1 u+\Xi_2 \widehat{u}\in \Pi_r$.

Step 2: Ξ_1 is a contraction map. Let $u,\widehat{u}\in\Pi_r$ and $\varrho\in J$. Then, we estimate

$$\begin{aligned}&\|\Xi_1 u-\Xi_1\widehat{u}\| \\ \leq\ & \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\left[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\|\hbar(u(\eta))-\hbar(\widehat{u}(\eta))\|d\eta\right. \\ &+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T[|\Phi(\eta)|\|u(\eta)-\widehat{u}(\eta)\| \\ &+\Big\|f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho))-f(\varrho,\widehat{u}(\varrho),{}^{RL}I^{\delta,g}\widehat{u}(\varrho))\Big\|\Big]d\eta \\ &+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}[|\Phi(\eta)|\|u(\eta)-\widehat{u}(\eta)\| \\ &\left.+\Big\|f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho))-f(\varrho,\widehat{u}(\varrho),{}^{RL}I^{\delta,g}\widehat{u}(\varrho))\Big\|\Big]d\eta\right].\end{aligned}$$

We used fact that g is an increasing function, it follows that $\frac{(g(\varrho)-g(0))}{(g(T)-g(0))}<1$, for $\varrho<T$. Consequently,

$$\|(\Xi_1 u)(\varrho)-(\Xi_1\widehat{u})\|\leq\mathcal{Q}\|u-\widehat{u}\|.$$

Due to $\mathcal{Q}<1$, we conclude that the operator Ξ_1 is a contraction map.

Step 3: Ξ_2 is completely continuous. From the continuity of f and by the Lebesgue dominated convergence theorem, we conclude that Ξ_2 is continuous too. In addition, by (10), Ξ_2 is uniformly bounded on Π_r. Next, we will show that $\Xi_2(\Pi_r)$ is equicontinuous. For this purpose, let $u\in\Pi_r, 0\leq\varrho_1<\varrho_2\leq T$. Then, we obtain

$$\begin{aligned}&\|(\Xi_2 u)(\varrho_2)-(\Xi_2 u)(\varrho_1)\| \\ \leq\ & \left|\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho_2}\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\right. \\ &\left.-\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho_1}\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\right| \\ &+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^{\varrho_1}g'(\eta)\frac{(g(\varrho_2)-g(\eta))^{\mathfrak{q}-1}-(g(\varrho_1)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})} \\ &\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta \\ &+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_{\varrho_1}^{\varrho_2}g'(\eta)\frac{(g(\varrho_2)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\Big[|\Phi(\eta)|\|u(\eta)\|+\Big\|f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big\|\Big]d\eta\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^{\varrho_2} \left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta)) \right] d\eta \right. \\
&\quad \left. - \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)} \int_0^{\varrho_1} \left[\Phi(\eta)u(\eta) + f(\eta, u(\eta), {}^{RL}I^{\delta,g}u(\eta)) \right] d\eta \right| \\
&\quad + \frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)} \frac{(g(\varrho_2)-g(0))^{\mathfrak{q}} - (g(\varrho_1)-g(0))^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+1)} \left(\mathcal{K}_\Phi r + \mathcal{W}_f^* \right) \\
&\to 0 \text{ as } \varrho_2 \to \varrho_1.
\end{aligned}
$$

According to the above steps with Arzela–Ascoli theorem, we understand that $(\Xi_2\Pi_r)$ is relatively compact. Consequently, Ξ_2 is completely continuous. Thus, by Theorem 2, we infer that g-ABC-fractional differential Equation (1) has at least one solution on J. □

5. Stability Results

Ulam's question about the stability of group homomorphisms in 1940 [33] inspired the problem of functional equation stability. Hyers [34] presented a positive interpretation of the Ulam question within Banach spaces the next year, which was the first important advance and step toward more solutions in this topic. Many studies on various generalisations of the Ulam problem and Hyers theory have been published since then. Rassias [35] was the first to extend Hyers idea of mappings across Banach spaces in 1978. Rassias result drew the attention of many mathematicians all over the world, who began looking into the difficulties of functional equation stability. For more information about the stability of solutions, we refer the readers to the papers [36,37].

This part is dedicated to studying different kinds of stability for (1), namely Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability. Before that, for $\varepsilon > 0$, we define a continuous function $\alpha_\phi : J \to \mathbb{R}^+$ such that satisfy the following inequalities

$$\left| \left({}^{ABC}\mathbf{D}^{\mathfrak{q};g} - \Phi(\varrho) \right) \widehat{u}(\varrho) = f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho)) \right| \leq \varepsilon, \quad \varrho \in J, \tag{11}$$

and

$$\left| \left({}^{ABC}\mathbf{D}^{\mathfrak{q};g} - \Phi(\varrho) \right) \widehat{u}(\varrho) = f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho)) \right| \leq \varepsilon \alpha_\phi(\varrho). \tag{12}$$

Definition 4. *The g-ABC-fractional differential Equation (1) is Ulam-Hyers stable if there exists a positive number $C_f > 0$ such that, for each $\varepsilon > 0$ and for each $\widehat{u} \in C(J, \mathcal{X})$ satisfies the inequality (11), there exist a unique solution $u \in C(J, \mathcal{X})$ of g-ABC-fractional differential Equation (1) such that*

$$\|\widehat{u} - u\| \leq C_f \varepsilon.$$

Also, the g-ABC-fractional differential Equation (1) is generalized Ulam-Hyers stable if there exists $\varphi_f : (0, \infty] \to (0, \infty]$ with $\varphi_f(0) = 0$ such that

$$\|\widehat{u} - u\| \leq \varphi_f \varepsilon.$$

Remark 1. *If there exists a function $\mathcal{P} \in C(J, \mathcal{X})$, then $\widehat{u} \in C(J, \mathcal{X})$ is a solution of (11) if and only if*

(i) $|\mathcal{P}(\varrho)| \leq \varepsilon$ *for all* $\varrho \in J$,
(ii) ${}^{ABC}\mathbf{D}^{\mathfrak{q};g}\widehat{u}(\varrho) = \Phi(\varrho)\widehat{u}(\varrho) + f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta,g}\widehat{u}(\varrho)) + \mathcal{P}(\varrho)$, $\varrho \in J$.

Proof. See [5]. □

Lemma 4. *If $u \in C(J, \mathcal{X})$ is a function that satisfies the inequality (11), then u satisfies the following inequality*

$$\|u - \Psi_u\| \leq \varepsilon \Pi_{g,\mathfrak{q}},$$

where

$$\begin{aligned}\Psi_u &= \frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\delta-1}}{\Gamma(\delta)}\hbar(u(\eta))d\eta \\ &-\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &-\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta\Bigg] \\ &+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta \\ &+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\Big[\Phi(\eta)u(\eta)+f(\eta,u(\eta),{}^{RL}I^{\delta,g}u(\eta))\Big]d\eta.\end{aligned}$$

Proof. In view of Remark 1, we have

$$\begin{cases} ({}^{ABC}\mathbf{D}^{\mathfrak{q};g}-\Phi(\varrho))u(\varrho)=f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho))+\mathcal{P}(\varrho), \\ \widehat{u}(0)=u(0)=0, \\ u(T)=\widehat{u}(T)=\int_0^T\frac{[g(T)-g(\eta)]^{\delta-1}}{\Gamma(\delta)}g'(\eta)\hbar(u(\eta))d\eta. \end{cases} \tag{13}$$

Then, by Theorem 3, the solution of problem (13) is given by

$$\begin{aligned}u(\varrho) &= \Psi_u-\frac{(g(\varrho)-g(0))}{(g(T)-g(0))}\Bigg[\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T\mathcal{P}(\eta)d\eta \\ &-\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\mathcal{P}(\eta)d\eta\Bigg] \\ &+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho\mathcal{P}(\eta)d\eta+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\mathcal{P}(\eta)d\eta.\end{aligned}$$

Due to the fact $\left(\frac{(g(\varrho)-g(0))}{(g(T)-g(0))}<1,\varrho<T\right)$, we obtain

$$\begin{aligned}\|u-\Psi_u\| &\leq \frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T|\mathcal{P}(\eta)|d\eta+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^T g'(\eta)\frac{(g(T)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}\mathcal{P}(\eta)d\eta \\ &+\frac{2-\mathfrak{q}}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho|\mathcal{P}(\eta)|d\eta+\frac{\mathfrak{q}-1}{\mathfrak{B}(\mathfrak{q}-1)}\int_0^\varrho g'(\eta)\frac{(g(\varrho)-g(\eta))^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}|\mathcal{P}(\eta)|d\eta \\ &\leq \varepsilon\Pi_{g,\mathfrak{q}}.\end{aligned}$$

□

Theorem 6. *Assume that (H1)–(H4) hold. Under the Lemma 4, the following equation*

$${}^{ABC}\mathbf{D}^{\mathfrak{q};g}u(\varrho)=\Phi(\varrho)u(\varrho)+f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho)),\varrho\in[0,T], \tag{14}$$

is Ulam-Hyers stable as well as generalized Ulam-Hyers stable provided that $\mathcal{G}<1$.

Proof. Let $\widehat{u}$ be satisfied (11), let $u\in C(J,\mathcal{X})$ be a unique solution to the following problem

$$\begin{cases} ({}^{ABC}\mathbf{D}^{\mathfrak{q};g}-\Phi(\varrho))u(\varrho)=f(\varrho,u(\varrho),{}^{RL}I^{\delta,g}u(\varrho)),\varrho\in[0,T] \\ u(0)=\widehat{u}(0)=0 \\ u(T)=\widehat{u}(T)=\int_0^T\frac{[g(T)-g(\eta)]^{\delta-1}}{\Gamma(\delta)}g'(\eta)\hbar(u(\eta))d\eta. \end{cases}$$

Then, by Theorem 3, we get

$$u(\varrho) = \Psi_u.$$

It follows from Theorem 4 and Lemma 4, that

$$\begin{aligned}\|u-\widehat{u}\| &= \sup_{\varrho\in J}|u(\varrho)-\Psi_{\widehat{u}}| \leq \sup_{\varrho\in J}|u(\varrho)-\Psi_u| + \sup_{\varrho\in J}|\Psi_u-\Psi_{\widehat{u}}| \\ &\leq \varepsilon\Pi_{g,\mathfrak{q}} + \mathcal{G}\|u-\widehat{u}\|.\end{aligned}$$

Thus

$$\|u-\widehat{u}\| \leq C_f\varepsilon,$$

where

$$C_f = \frac{\Pi_{g,\mathfrak{q}}}{1-\mathcal{G}} > 0.$$

Now, by choosing $\varphi_f(\varepsilon) = C_f\varepsilon$ such that $\varphi_f(0) = 0$, then the g-ABC problem (14) has generalized Ulam-Hyers stability. □

To prove the Ulam-Hyers-Rassias stability, we need the following hypothesis;

Hypothesis 5 (H5). *Let $\alpha_\phi \in C(J,\mathcal{X})$ be an increasing function. Then, there exists $\mathcal{R} > 0$ such that for any $\varrho \in J$, we have*

$$^{AB}I_{0^+}^{\mathfrak{q},g}\alpha_\phi(\varrho) \leq \mathcal{R}\alpha_\phi(\varrho). \tag{15}$$

Definition 5. *Let $\widehat{u} \in C(J,\mathcal{X})$ be a function satisfies (12) and $u \in C(J,\mathcal{X})$ be a solution of g-ABC-fractional differential Equation (1). If there exists $0 < \mathcal{N} \in \mathbb{R}$ and non-decreasing function $\alpha_\phi(\varrho)$ such that*

$$\|u-\widehat{u}\| \leq \mathcal{N}\varepsilon\alpha_\phi(\varrho), \quad \varrho \in J,\ \varepsilon > 0,$$

then, the g-ABC-fractional differential Equation (14) is Ulam-Hyers-Rassias stable with respect to $\alpha_\phi(\varrho)$.

Remark 2. *A function $\widehat{u} \in C(J,\mathcal{X})$ satisfies (12) if and only if there exists a function $z \in C(J,\mathcal{X})$ such that*

(i) $|z(\varrho)| \leq \varepsilon\alpha_\phi(\varrho),\ \varkappa \in J,$
(ii) $^{ABC}\mathbf{D}^{\mathfrak{q};g}\widehat{u}(\varrho) = \Phi(\varrho)\widehat{u}(\varrho) + f(\varrho,\widehat{u}(\varrho),{}^{RL}I^{\delta,g}\widehat{u}(\varrho)) + z(\varrho),\ \varrho \in J.$

Lemma 5. *If $u \in C(J,\mathcal{X})$ is a solution to inequality (11), then u satisfies the following inequality*

$$\|u-\Psi_u\| \leq \varepsilon\mathcal{R}\alpha_\phi(\varrho).$$

Proof. Indeed, by Remark 2 and Theorem 3, one can easily prove that

$$\|u-\Psi_u\| \leq \varepsilon\mathcal{R}\alpha_\phi(\varrho).$$

□

Theorem 7. *Suppose that (H1)–(H5) hold. If $\mathcal{G} < 1$, then g-ABC-fractional differential Equation (14) is Ulam-Hyers-Rassias as well as generalized Ulam-Hyers-Rassias stable.*

Proof. Let $\widehat{u}$ be satisfied (12), let $u \in C(J,\mathcal{X})$ be a unique solution to the following problem

$$\begin{cases} \left(^{ABC}\mathbf{D}^{\mathfrak{q};g} - \Phi(\varrho)\right)u(\varrho) = f(\varrho,u(\varrho),Zu(\varrho)), \varrho \in [0,T], \\ u(0) = \widehat{u}(0) = 0, \\ u(T) = \widehat{u}(T) = \int_0^T \frac{[g(T)-g(\eta)]^{\delta-1}}{\Gamma(\delta)}g'(\eta)\hbar(u(\eta))d\eta. \end{cases}$$

Then, by Theorem 3, we get
$$u(\varrho) = \Psi_u.$$
It follows from Theorem 4 and Lemma 5 that
$$\begin{aligned} \|u - \widehat{u}\| &= \sup_{\varrho\in J}|u(\varrho) - \Psi_{\widehat{u}}| \leq \sup_{\varrho\in J}|u(\varrho) - \Psi_u| + \sup_{\varrho\in J}|\Psi_u - \Psi_{\widehat{u}}| \\ &\leq \varepsilon\mathcal{R}\alpha_\phi(\varrho) + \mathcal{G}\|u - \widehat{u}\|. \end{aligned}$$
Thus
$$\|u - \widehat{u}\| \leq \mathcal{N}\varepsilon\alpha_\phi(\varrho),$$
where
$$\mathcal{N} = \frac{\mathcal{R}}{1-\mathcal{G}} > 0.$$

Hence, the g-ABC-fractional differential Equation (14) is Ulam-Hyers-Rassias stable as well as generalized Ulam-Hyers-Rassias stable. □

6. An Application

Taking $\mathfrak{q} = \frac{9}{7}$, $g(\varrho) = e^{\varrho}$, With the choice of operator $\Phi(\varrho) = \frac{1}{100e^{\varrho-1}}$. Then, we have the following problem
$$\begin{cases} \left({}^{ABC}\mathbf{D}_{0^+}^{\frac{9}{7},e^{\varrho}} - \frac{1}{100e^{\varrho-1}}\right)u(\varrho) = \frac{1}{50e^{\varrho-1}}\left(\frac{|u(\varrho)|}{1+|u(\varrho)|} + \frac{{}^{RL}I^{\delta,g}|u(\varrho)|}{1+|Zu(\varrho)|}\right) + \varrho + 1,\ \varrho\in[0,1], \\ u(0) = \int_0^1 e^{\eta}\frac{(e-e^{\eta})^{\frac{9}{7}-1}}{\Gamma(\frac{9}{10})}\frac{u(\eta)}{50}d\eta. \end{cases} \tag{16}$$
Here, we have
$$f(\varrho, u(\varrho), Zu(\varrho)) = \frac{1}{50e^{\varrho-1}}\left(\frac{|u(\varrho)|}{1+|u(\varrho)|} + \frac{{}^{RL}I^{\delta,g}|u(\varrho)|}{1+|Zu(\varrho)|}\right) + \varrho + 1,$$
and
$${}^{RL}I^{\delta,g}u(\varrho) = \int_0^{\varrho} e^{\eta}\frac{(e-e^{\eta})^{\frac{9}{7}-1}}{\Gamma(\frac{9}{10})}\frac{u(\eta)}{50}d\eta.$$
In addition, we have
$$\hbar(u(\varrho)) = \frac{u(\varrho)}{5}.$$
Let $u, \overline{u} \in C(J, \mathcal{X})$. Then, for all $\varrho \in J$, we obtain
$$\begin{aligned} &\left|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho)) - f(\varrho, \overline{u}(\varrho), {}^{RL}I^{\delta,g}\overline{u}(\varrho))\right| \\ &\leq \frac{e}{50}\left[|u(\varrho) - \overline{u}(\varrho)| + \left|{}^{RL}I^{\delta,g}u(\varrho) - {}^{RL}I^{\delta,g}\overline{u}(\varrho)\right|\right], \end{aligned}$$
and
$$\left|f(\varrho, u(\varrho), {}^{RL}I^{\delta,g}u(\varrho))\right| \leq \frac{e}{50}\left[|u(\varrho)| + \left|{}^{RL}I^{\delta,g}u(\varrho)\right|\right].$$
In addition, it is easy to see that $|\Phi(\varrho)| \leq \frac{e}{100}$, $|\hbar(u(\varrho))| \leq \frac{|u(\varrho)|}{50}$ and
$$|\hbar(u(\varrho)) - \hbar(\overline{u}(\varrho))| \leq \frac{|u(\varrho) - \overline{u}(\varrho)|}{50}.$$
Therefore, the hypotheses (H1)–(H4) hold with $\mathcal{K}_{\Phi} = \frac{e}{100}$, $\mathfrak{Z}_{\hbar} = \mathcal{W}_{\hbar} = \frac{1}{50}$, $\mathcal{A}_{\hbar} = 0$, $\mathfrak{L}_1 = \mathfrak{L}_2 = \frac{e}{50}$. According to the given data with some simple calculations, we find that $\Pi_{g,\mathfrak{q}} = 1.36$. It follows that $\mathcal{G} = 0.19$. Thus, all conditions in Theorem 4 are satisfied. Hence,

the g-ABC-fractional differential Equation (16) has a unique solution. Moreover, for every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{u} \in C(J, \mathcal{X})$ satisfies

$$\left| {}^{ABC}\mathbf{D}^{\mathfrak{q};g}\widehat{u}(\varrho) - \Phi(\varrho)\widehat{u}(\varrho) - f(\varrho, \widehat{u}(\varrho), {}^{RL}I^{\delta;g}\widehat{u}(\varrho)) \right| \leq \varepsilon, \ \ \varrho \in (0,1),$$

there exists a solution $u \in C(J, \mathcal{X})$ of the g-ABC-fractional differential Equation (16) with

$$|\widehat{u}(\varrho) - u(\varrho)| \leq C_f \varepsilon,$$

where $C_f = \frac{\Pi_{g,\mathfrak{q}}}{1-\mathcal{G}} = 1.68 > 0$.

7. Conclusions Remarks

The theory of fractional operators in the context of the Atangana–Baleanu operator has recently piqued researchers' interest, motivating them to look into and improve several qualitative properties of solutions to FDEs using such operators. To do this, we investigated adequate conditions for the existence and uniqueness of solutions for the evolution equation in the context of a novel nonsingular FD in ABC type fractional derivative with respect to another function.

Our approach was based on the reduction of the proposed problem into the fractional integral equation and using some standard fixed point theorems due to Banach-type and Krasnoselskii-type. Furthermore, through mathematical analysis techniques, we analyzed the stability results in Ulam-Hyers sense. An example was provided to justify the main results. In fact, our outcomes generalize those in [26,28]. The supposed problem with given integro-derivative boundary conditions can describe some mathematical models of real and physical processes in which some parameters are frequently acclimated to appropriate circumstances. So, the value of these parameters can change the impacts of fractional integrals and derivatives. The main results are illustrated with a numerical example. Further, our current problem was a general extension of the previous standard cases of FDEs by assigning different values for all existing orders and parameters or defining the $\mathfrak{g}$ function in the aforesaid problem. In future works, we will extend this work with a delay function.

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