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A Quadruple Integral Containing the Gegenbauer Polynomial $C_n^{(\lambda)}(x)$: Derivation and Evaluation

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Abstract: A four-dimensional integral containing $g(x,y,z,t)C_n^{(\lambda)}(x)$ is derived. $C_n^{(\lambda)}(x)$ is the Gegenbauer polynomial, $g(x,y,z,t)$ is a product of the generalized logarithm quotient functions and the integral is taken over the region $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1, 0 \leq t \leq 1$. The integral is difficult to compute in general. Special cases are given and invariant index forms are derived. The zero distribution of almost all Hurwitz–Lerch zeta functions is asymmetrical. All the results in this work are new.

Keywords: Gegenbauer polynomial; apéry's constant; cauchy integral; quadruple integral

MSC: Primary 30E20; 33-01; 33-03; 33-04; 33-33B; 33E20

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1. Significance Statement

The definite integral of the Gegenbauer polynomial is evaluated in the work by Askey et al. [1]. In the work by Srivastava [2] the author obtained an inversion formula for a singular integral transform involving Gegenbauer polynomials. In the work done by Bingham [3] the author performed the passage to the limit so as to obtain a complete and explicit description of measures which is of importance in probabilistic work on random walks on spheres. The Gegenbauer polynomial has many mathematical applications which are detailed in Andrews et al. [4] (1999, Chapter 9). See also section (14.30) in [5]. Physical applications of these polynomials are detailed in section (18.38) in [5]. Other applications are detailed in section (18.39) in [5]. We extend the previous important work by adding three more dimensions to the previously derived integrals in this paper. A quadruple integral will be derived and expressed in terms of a Hurwitz–Lerch zeta function. The Hurwitz–Lerch zeta function $zeta(s,v)$, the digamma function $psi(0)(s)$, the Riemann zeta function $zeta(k)$, and $log(2)$ are used to deduce special cases.

2. Introduction

In this paper, we derive the quadruple definite integral given by

$$\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-m}\left(\frac{1}{t}\right)C_n^{(\lambda)}(x)\log^{\frac{1}{2}(m-n-1)}\left(\frac{1}{z}\right)$$
$$\log^{\lambda+\frac{1}{2}(m+n-1)}\left(\frac{1}{y}\right)\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt \quad (1)$$

where the parameters k, a, λ, n are general complex numbers and $1/2 < Re(m) < 1$. The method used by us in [6] are followed in the derivations. This method employs a form of the generalized Cauchy's integral formula, which is given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \quad (2)$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour.

3. Definite Integral of the Contour Integral

We use the method in [6,7]. The variable of integration in the contour integral is $r = w + m$. Using a generalization of Cauchy's integral formula we form the quadruple integral by replacing y by

$$\log\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)$$

and multiplying by

$$x^{m-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-m}\left(\frac{1}{t}\right)C_n^{(\lambda)}(x)\log^{\frac{1}{2}(m-n-1)}\left(\frac{1}{z}\right)\log^{\lambda+\frac{1}{2}(m+n-1)}\left(\frac{1}{y}\right)$$

then taking the definite integral with respect to $x \in [0,1]$, $y \in [0,1]$, $z \in [0,1]$ and $t \in [0,1]$ to obtain

$$\frac{1}{\Gamma(k+1)}\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-m}\left(\frac{1}{t}\right)C_n^{(\lambda)}(x)\log^{\frac{1}{2}(m-n-1)}\left(\frac{1}{z}\right)$$
$$\log^{\lambda+\frac{1}{2}(m+n-1)}\left(\frac{1}{y}\right)\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt$$

$$=\frac{1}{2\pi i}\int_0^1\int_0^1\int_0^1\int_0^1\int_C a^w w^{-k-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}x^{m+w-1}C_n^{(\lambda)}(x)\log^{-m-w}\left(\frac{1}{t}\right)$$
$$\log^{\frac{1}{2}(m-n+w-1)}\left(\frac{1}{z}\right)\log^{\lambda+\frac{1}{2}(m+n+w-1)}\left(\frac{1}{y}\right)dwdxdydzdt$$

$$=\frac{1}{2\pi i}\int_C\int_0^1\int_0^1\int_0^1\int_0^1 a^w w^{-k-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}x^{m+w-1}C_n^{(\lambda)}(x)\log^{-m-w}\left(\frac{1}{t}\right)$$
$$\log^{\frac{1}{2}(m-n+w-1)}\left(\frac{1}{z}\right)\log^{\lambda+\frac{1}{2}(m+n+w-1)}\left(\frac{1}{y}\right)dxdydzdtdw$$

$$=\frac{1}{2\pi i}\int_C\frac{\pi^2 a^w w^{-k-1}2^{-2\lambda-m-w+1}\csc(\pi(m+w))\Gamma(n+2\lambda)}{n!\Gamma(\lambda)}dw \quad (3)$$

from Equation (18.17.37) in [5] and Equation (4.215.1) in [8] where $Re(m+w) > 0$, $Re(\lambda) > 0$, $Re(n) > 0$, $1/2 < Re(m) < 1$ and using the reflection Formula (8.334.3) in [8] for the Gamma function. The reversal of the order of integration over x, y, z and t is done by using Fubini's theorem for multiple integrals see (9.112) in [9], since the integrand is of bounded measure over the space $\mathbb{C} \times [0,1] \times [0,1] \times [0,1] \times [0,1]$.

4. The Hurwitz–Lerch Zeta Function and Infinite Sum of the Contour Integral

In this section, we use Equation (2) to derive the contour integral representations for the Hurwitz–Lerch Zeta function.

4.1. The Hurwitz–Lerch Zeta Function

The Hurwitz–Lerch zeta function see [5,10] has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \tag{4}$$

where $|z| < 1, vs. \neq 0, -1, \ldots$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(v-1)t}}{e^t - z} dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Infinite Sum of the Contour Integral

Using Equation (2) and replacing y by $\log(a) + i\pi(2y+1) - \log(2)$ then multiplying both sides by $-\frac{i\pi^2 2^{-2\lambda-m+2} e^{i\pi m(2y+1)} \Gamma(n+2\lambda)}{n!\Gamma(\lambda)}$ taking the infinite sum over $y \in [0,\infty)$ and simplifying in terms of the Hurwitz–Lerch zeta function we obtain

$$\begin{aligned}
&\frac{i^{k-1}\pi^{k+2}e^{i\pi m}2^{k-2\lambda-m+2}\Gamma(n+2\lambda)\Phi\left(e^{2im\pi},-k,\frac{-i\log(a)+i\log(2)+\pi}{2\pi}\right)}{k!n!\Gamma(\lambda)} \\
&= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C \frac{i\pi^2 a^w w^{-k-1}2^{-2\lambda-m-w+2}e^{i\pi(2y+1)(m+w)}\Gamma(n+2\lambda)}{n!\Gamma(\lambda)}dw \\
&= -\frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} \frac{i\pi^2 a^w w^{-k-1}2^{-2\lambda-m-w+2}e^{i\pi(2y+1)(m+w)}\Gamma(n+2\lambda)}{n!\Gamma(\lambda)} \\
&= \frac{1}{2\pi i}\int_C \frac{\pi^2 a^w w^{-k-1}2^{-2\lambda-m-w+1}\csc(\pi(m+w))\Gamma(n+2\lambda)}{n!\Gamma(\lambda)}dw
\end{aligned} \tag{6}$$

from Equation (1.232.2) in [8] where $Im(\pi(m+w)) > 0$ in order for the sum to converge.

5. Definite Integral in Terms of the Hurwitz–Lerch Zeta Function

Theorem 1. *For all* $k, a, \lambda, n \in \mathbb{C}, 1/2 < Re(m) < 1$ *then,*

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-m}\left(\frac{1}{t}\right)C_n^{(\lambda)}(x)\log^{\frac{1}{2}(m-n-1)}\left(\frac{1}{z}\right) \\
&\qquad \log^{\lambda+\frac{1}{2}(m+n-1)}\left(\frac{1}{y}\right)\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt \\
&= \frac{i^{k-1}\pi^{k+2}e^{i\pi m}2^{k-2\lambda-m+2}\Gamma(n+2\lambda)\Phi\left(e^{2im\pi},-k,\frac{-i\log(a)+i\log(2)+\pi}{2\pi}\right)}{n!\Gamma(\lambda)}
\end{aligned} \tag{7}$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion. □

Example 1. *The degenerate case*

$$\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-m}\left(\frac{1}{t}\right)C_n^{(\lambda)}(x)\log^{\frac{1}{2}(m-n-1)}\left(\frac{1}{z}\right)\log^{\lambda+\frac{1}{2}(m+n-1)}\left(\frac{1}{y}\right)dxdydzdt$$

$$=\frac{\pi^2 2^{-2\lambda-m+1}\csc(\pi m)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)} \tag{8}$$

Proof. Use Equation (7) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [11]. □

Example 2. *The Hurwitz zeta function* $\zeta(s,v)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}}dxdydzdt$$

$$=\frac{i^k\pi^{k+2}2^{2k-2\lambda+\frac{3}{2}}\Gamma(n+2\lambda)\zeta\left(-k,\frac{-i\log(a)+i\log(2)+\pi}{4\pi}\right)}{n!\Gamma(\lambda)}-\frac{i^k\pi^{k+2}2^{2k-2\lambda+\frac{3}{2}}\Gamma(n+2\lambda)\zeta\left(-k,\frac{1}{2}\left(\frac{-i\log(a)+i\log(2)+\pi}{2\pi}+1\right)\right)}{n!\Gamma(\lambda)} \tag{9}$$

Proof. Use Equation (7) and set $m=1/2$ and simplify using entry (4) in Table below (64:12:70) in [11]. □

Example 3. *The Digamma function* $\psi^{(0)}(s)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}\log\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}dxdydzdt$$

$$=\frac{i\pi 2^{-2\lambda-\frac{1}{2}}\psi^{(0)}\left(\frac{-i\log(a)+i\log(2)+\pi}{4\pi}\right)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)}-\frac{i\pi 2^{-2\lambda-\frac{1}{2}}\psi^{(0)}\left(\frac{-i\log(a)+i\log(2)+3\pi}{4\pi}\right)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)} \tag{10}$$

Proof. Use Equation (9) and apply l'Hopital's rule as $k\to -1$ and simplify using Equation (64:4:1) in [11]. □

Example 4. *The fundamental constant* $\log(2)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)}{\sqrt{x}\sqrt{\log(t)}\left(\log\left(\frac{2x\sqrt{\log(y)}\sqrt{\log(z)}}{\log(t)}\right)+i\pi\right)}dxdydzdt$$

$$=\frac{\pi 2^{\frac{1}{2}-2\lambda}\log(2)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)} \quad (11)$$

Proof. Use Equation (10) and set $a=-2$ and simplify. □

Example 5. *The Riemann zeta function* $\zeta(s)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)\log^k\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}}dxdydzdt$$

$$=-\frac{i^k\left(2^{k+1}-1\right)\pi^{k+2}2^{k-2\lambda+\frac{3}{2}}\zeta(-k)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)} \quad (12)$$

Proof. Use Equation (7) and set $m=1/2$ and simplify using entry (4) in Table below (64:12:7) in [11]. □

Example 6. *Apéry's constant* $\zeta(3)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}\log^3\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}dxdydzdt$$

$$=\frac{3i2^{-2\lambda-\frac{7}{2}}\zeta(3)\Gamma(n+2\lambda)}{\pi n!\Gamma(\lambda)} \quad (13)$$

Proof. Use Equation (12) and set $k=-3$ and simplify. □

Example 7. *The fundamental constant* $\zeta(5)$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}\log^{-\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{z}\right)C_n^{(\lambda)}(x)\log^{\lambda+\frac{n}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}\log^5\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}dxdydzdt$$

$$=-\frac{15i2^{-2\lambda-\frac{15}{2}}\zeta(5)\Gamma(n+2\lambda)}{\pi^3 n!\Gamma(\lambda)} \quad (14)$$

Proof. Use Equation (12) and set $k=-5$ and simplify. □

6. Discussion

This paper uses our contour integral method for deriving a new quadruple integral containing the Gegenbauer polynomial $C_n^{(\lambda)}(x)$, along with some interesting special cases with many more possible. The evaluations in this present work were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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