Article

Existence and Uniqueness Theorem for Uncertain Wave Equation

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Abstract: In the real world, the indeterminate phenomenon and determinate phenomenon are symmetric; however, the indeterminate phenomenon absolutely exists. Hence, the indeterminate dynamic phenomenon is studied in this paper by using uncertainty theory, where the indeterminate dynamic phenomenon is associated with the belief degree and called the uncertain dynamic phenomenon. Based on uncertainty theory, the uncertain wave equation derived by the Liu process is constructed to model the propagation of various types of wave with uncertain disturbance in nature, where the Liu process is Lipschitz-continuous and has stationary and independent increments. First important of all, only the equation has solution which can be used to clearly depict the wave propagation influenced by uncertain disturbance. Therefore, the aims of this paper is to propose and prove a theorem of existence and uniqueness with Lipschitz and linear growth conditions.

Keywords: existence and uniqueness; Liu process; uncertain wave equation; partial differential equation

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1. Introduction

Uncertainty theory was founded by Liu [1] to model the indeterminacy associated with belief degrees. A belief degree is a chance that a possible event may happen and the chance is estimated by a human. So, uncertainty theory could be regarded as an instrument to describe uncertain phenomena in human systems. For example, Liu [2] uses uncertain variable to model indeterminate factors and established uncertain programming, which has been applied to green supply chain management [3]. After uncertainty theory was constructed, some other scholars had also done much meaningful work in many fields, such as reliability analysis [4], portfolio optimization [5,6], vehicle routing problems [7] and uncertain risk analysis [8].

An uncertain process is a collection of uncertain variables varying with space or time to model dynamic uncertain phenomena. Its investigation was pioneered by Liu [9]. Following that, Liu [10] proposed the Liu process, a special type of uncertain process, which is Lipschitz-continuous and has independent and stationary increments which are actually uncertain normal variables. On the base of the Liu process, Liu [10] initialized uncertain calculus to handle integration and differentiation of some uncertain processes.

The uncertain differential equation was established by Liu [9] which is one kind of differential equation motivated by the Liu process. After that, when satisfying linear growth and Lipschitz continuity, Chen and Liu [11] presented and verified the existence and uniqueness theorem of an uncertain differential equation. Following that, the property of stability was studied, including stability in uncertain measures [10], stability in the p th moment [12] and stability in mean [13]. Additionally, it is significant for us to obtain the solution. Hence, the analytic solution was studied in [11] for a linear uncertain differential equation and two special classes of nonlinear differential equations, respectively. To obtain the numerical solution, the Yao-Chen formula was derived by Yao and Chen [14] by which we can derive a class of equivalent ordinary differential equations. Then, Gao [15] applied the Yao-Chen formula to design an algorithm and obtain the numerical solution of an

uncertain differential equation. Recently, it has been used in high numbers of research fields, such as finance [16] and optimal control [17].

The partial differential equation has been applied widely in many fields, while disturbances are always around us in real world. So, deterministic partial differential equations are unable to model some phenomena with indeterminate disturbance. With the development of the Wiener Process, the partial differential equation found it was extended to the stochastic partial differential equation. For example, Walsh [18] investigated the stochastic heat equation for dealing with heat conduction. In addition, Cabaña [19] studied the stochastic wave equation driven by a white noise for modeling wave propagation phenomena. Then stochastic wave equation was also studied by Orsingher [20], Marcus [21] and Czapor [22].

Probability theory is employed to describe randomness in relation to frequencies, while uncertainty theory is applied to model uncertainty associated with belief degrees. So, to describe phenomena with uncertain disturbance, Yang and Yao [23] introduced uncertainty theory into the partial differential equation and the presented uncertain partial differential equation. They also studied the uncertain heat equation, a type of uncertain differential equation. In addition, Gao and Ralescu [24] established the uncertain wave equation involving the Liu process to describe wave propagation influenced by uncertain disturbance. In this paper, the existence and uniqueness theorem will be presented and proved regarding a nonlinear uncertain wave equation. Moreover, the paper will be arranged as follows. Section 2 will introduce some necessarily fundamental knowledge used in other sections. Then we show the process of proof of the existence and uniqueness theorem with some conditions and give some examples in Section 3. Ultimately, a concise conclusion is drawn in Section 4.

2. Preliminaries

The uncertain wave equation involving the Liu process plays a significant role in modeling wave propagation with uncertain disturbances. Consequently, some basic knowledge will be introduced firstly.

Definition 1 (Liu [9]). Suppose that T is a totally ordered set and suppose that $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space. If function $X_\tau : T \times (\Gamma, \mathcal{L}, \mathcal{M}) \rightarrow \mathbb{R}$ is measurable, that is, $\{X_\tau \in B\}$ is an event for each Borel set B for every t , then X_τ is called an uncertain process.

If $X_{\tau+s} - X_s$ have a common distribution for each $\tau > 0$ and $s > 0$, then we say that X_τ is a stationary uncertain process. If for any $\tau_1, \tau_2, \dots, \tau_k$ with $\tau_0 < \tau_1 < \dots < \tau_k$, $X_{\tau_0}, X_{\tau_1} - X_{\tau_0}, \dots, X_{\tau_k} - X_{\tau_{k-1}}$ are independent, then we say X_τ has independent increments, where τ_0 is the initial time.

Definition 2 (Liu [10]). If the uncertain process C_τ satisfies the following conditions

- (i) Almost all of its sample paths are Lipschitz continuous and $C_0 = 0$;
- (ii) C_τ has independent increments and is also stationary;
- (iii) The expected value and variance of every increment $C_{s+\tau} - C_s$ are 0 and τ^2 , and uncertainty distribution of $C_{s+\tau} - C_s$ is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}\tau}\right)\right)^{-1}, \quad x \in \mathbb{R}, \quad (1)$$

then C_τ is called the Liu process.

Definition 3 (Liu [10]). Assume that C_τ is a Liu process and assume that X_τ is an uncertain process. For any partition of $[a, b]$ satisfying $a = \tau_1 < \tau_2 < \dots < \tau_{n+1} = b$, we write the net as follows

$$\Delta = \max_{1 \leq i \leq n} |\tau_{i+1} - \tau_i|. \quad (2)$$

Then the Liu integral of X_τ in relation to C_τ is

$$\int_a^b X_\tau dC_\tau = \lim_{\Delta \rightarrow 0} \sum_{i=1}^n X_{\tau_i} \cdot (C_{\tau_{i+1}} - C_{\tau_i}) \quad (3)$$

providing that the limit almost surely exists and it is finite. Furthermore, we say X_τ integrable.

Definition 4 (Liu [9]). Let C_τ be a Liu process. Assume that functions g and h are continual. Then

$$dX_\tau = g(\tau, X_\tau) d\tau + h(\tau, X_\tau) dC_\tau \quad (4)$$

is called an uncertain differential equation.

Definition 5 (Liu [25]). Suppose there is a partially ordered set T and an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If the function $X_\tau(\gamma) : T \times (\Gamma, \mathcal{L}, \mathcal{M}) \rightarrow \mathfrak{R}$ is measurable, that is, for any Borel set B $\{X_\tau \in B\}$ is an event for each τ , then X_τ is said to be an uncertain field.

3. Existence and Uniqueness

This section is employed to show a theorem regarding existence and uniqueness for the uncertain nonlinear wave equation [24] as follows.

$$\begin{cases} \frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + g(\tau, y, X(\tau, y)) + h(\tau, y, X(\tau, y)) \dot{C}_\tau, & (\tau, y) \in \mathfrak{R}^+ \times \mathfrak{R} \\ X(0, y) = \psi(y), & y \in \mathfrak{R} \\ \frac{\partial X}{\partial \tau}(0, y) = \varphi(y), & y \in \mathfrak{R}. \end{cases} \quad (5)$$

where $\dot{C}_\tau = dC_\tau/d\tau$. Then, Equation (5) is equivalent to

$$\begin{aligned} X(\tau, y) = & \frac{1}{2} [\psi(y + a\tau) + \psi(y - a\tau)] + \frac{1}{2a} \int_{y-a\tau}^{y+a\tau} \varphi(z) dz + \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} \\ & g(\tau, z, X(t, z)) dz + \frac{1}{2a} \int_0^\tau dC_t \int_{y-a(\tau-t)}^{y+a(\tau-t)} h(t, z, X(t, z)) dz. \end{aligned} \quad (6)$$

Firstly, a lemma is introduced before proving the theorem of existence and uniqueness.

Lemma 1 (Chen and Liu [11]). Let the uncertain process X_τ be integrable on $[a, b]$ in relation to τ and assume that C_τ is a Liu process. Then, we have

$$\left| \int_a^b X_\tau(\gamma) dC_\tau(\gamma) \right| \leq K_\gamma \int_a^b |X_\tau(\gamma)| d\tau. \quad (7)$$

Here, K_γ is the Lipschitz constant of $X_\tau(\gamma)$.

Theorem 1. The solution of Equation (5) is unique if $\psi(y)$ and $\varphi(y)$ satisfy

$$|\psi(y)| \leq P, \quad |\varphi(y)| \leq Q, \quad \forall y \in \mathfrak{R}$$

where $P, Q > 0$, $g(\tau, y, x)$ and $h(\tau, y, x)$ satisfy

$$|g(\tau, y, x) - g(\tau, y, v)| + |h(\tau, y, x) - h(\tau, y, v)| \leq L|x - v|, \quad \forall x, v \in \mathfrak{R}, \tau \geq 0 \quad (8)$$

and

$$|g(\tau, y, x)| + |h(\tau, y, x)| \leq L(1 + |x|), \quad \forall x \in \mathfrak{R}, \tau \geq 0 \quad (9)$$

for some crisp number L . Furthermore, the solution of Equation (5) is sample-continuous.

Proof. Firstly, we testify the existence through the method of successive approximation. Conveniently, $X(\tau, y)$ is simplified as $X_{\tau, y}$.

Let

$$X_{\tau, y}^{(0)} = X_{0, y} = \psi(y), \quad (10)$$

and

$$\begin{aligned} X_{\tau, y}^{(n)} = & \frac{1}{2} [\psi(y + a\tau) + \psi(y - a\tau)] + \frac{1}{2a} \int_{y-a\tau}^{y+a\tau} \varphi(z) dz + \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} g(t, z, X_{\tau, z}^{(n-1)}) dz \\ & + \frac{1}{2a} \int_0^\tau dC_t \int_{y-a(\tau-t)}^{y+a(\tau-t)} h(t, z, X_{\tau, z}^{(n-1)}) dz, \quad n = 1, 2, \dots \end{aligned} \quad (11)$$

Define

$$D_\tau^{(n)}(\gamma) = \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} |X_{s, y}^{(n+1)}(\gamma) - X_{s, y}^{(n)}(\gamma)|, \quad \forall \gamma \in \Gamma. \quad (12)$$

By Equation (9) and Lemma 1, we have

$$\begin{aligned} D_\tau^{(0)}(\gamma) = & \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2} [\psi(y + as) + \psi(y - as)] + \frac{1}{2a} \int_{y-as}^{y+as} \varphi(z) dz \right. \\ & + \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} g(t, z, X_{t, z}^{(0)}(\gamma)) dz \\ & \left. + \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} h(t, z, X_{t, z}^{(0)}(\gamma)) dz - \psi(y) \right| \\ \leq & 2P + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_{y-as}^{y+as} \varphi(z) dz \right| + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \right. \\ & \left. g(t, z, \psi(z)) dz \right| + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} h(t, z, \psi(z)) dz \right| \\ \leq & 2P + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_{y-as}^{y+as} |\varphi(z)| dz + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \\ & |g(t, z, \psi(z))| dz + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} K_\gamma |h(t, z, \psi(z))| dz \\ \leq & 2P + Q\tau + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} L(1 + |\psi(z)|)(1 + K_\gamma) dz \\ \leq & 2P + Q\tau + \max_{y \in \mathbb{R}} \int_0^\tau L(1 + P)(1 + K_\gamma)(\tau - t) dt \\ = & 2P + Q\tau + \frac{L(1 + P)(1 + K_\gamma)}{2} \tau^2. \end{aligned} \quad (13)$$

For $n = 1, 2, \dots$, we claim that

$$D_\tau^{(n)}(\gamma) \leq \frac{2L^n(1 + K_\gamma)^n P}{(2n)!} \tau^{2n} + \frac{L^n(1 + K_\gamma)^n Q}{(2n + 1)!} \tau^{2n+1} + \frac{L^{n+1}(1 + K_\gamma)^{n+1}(1 + P)}{(2n + 2)!} \tau^{2n+2}, \quad (14)$$

where $0 \leq \tau \leq T$ and T is a crisp number. If $k = 1$, it follows from Equations (11) and (12) and Lemma 1 that

$$\begin{aligned}
 D_{\tau}^{(1)}(\gamma) &= \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[g(t, z, X_{t,z}^{(1)}(\gamma)) - g(t, z, X_{t,z}^{(0)}(\gamma)) \right] dz \right. \\
 &\quad \left. + \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left[h(t, z, X_{t,z}^{(1)}(\gamma)) - h(t, z, X_{t,z}^{(0)}(\gamma)) \right] dz \right| \\
 &\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[g(t, z, X_{t,z}^{(1)}(\gamma)) - g(t, z, X_{t,z}^{(0)}(\gamma)) \right] dz \right| \\
 &\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left[h(t, z, X_{t,z}^{(1)}(\gamma)) - h(t, z, X_{t,z}^{(0)}(\gamma)) \right] dz \right| \\
 &\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| g(t, z, X_{t,z}^{(1)}(\gamma)) - g(t, z, X_{t,z}^{(0)}(\gamma)) \right| dz \\
 &\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left| h(t, z, X_{t,z}^{(1)}(\gamma)) - h(t, z, X_{t,z}^{(0)}(\gamma)) \right| dz \\
 &\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| g(t, z, X_{t,z}^{(1)}(\gamma)) - g(t, z, X_{t,z}^{(0)}(\gamma)) \right| dz \\
 &\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt K_{\gamma} \int_{y-a(s-t)}^{y+a(s-t)} \left| h(t, z, X_{t,z}^{(1)}(\gamma)) - h(t, z, X_{t,z}^{(0)}(\gamma)) \right| dz. \quad (15)
 \end{aligned}$$

Then, by using Equation (8) and Equation (13), we have

$$\begin{aligned}
 D_{\tau}^{(1)}(\gamma) &\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} L \left| X_{t,z}^{(1)}(\gamma) - X_{t,z}^{(0)}(\gamma) \right| dz \\
 &\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{K_{\gamma}}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} L \left| X_{t,z}^{(1)}(\gamma) - X_{t,z}^{(0)}(\gamma) \right| dz \\
 &= \frac{1+K_{\gamma}}{2a} L \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| X_{t,z}^{(1)}(\gamma) - X_{t,z}^{(0)}(\gamma) \right| dz \\
 &\leq \frac{1+K_{\gamma}}{2a} L \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[2P + Qt + \frac{(1+P)(1+K_{\gamma})}{2} Lt^2 \right] dz \\
 &\leq \frac{1+K_{\gamma}}{2a} L \int_0^{\tau} 2a(\tau-t) \left[2P + Qt + \frac{\tau^2}{2} L(1+P)(1+K_{\gamma}) \right] dt \\
 &= L(1+K_{\gamma}) \left[P\tau^2 + \frac{Q}{6}\tau^3 + \frac{L(1+K_{\gamma})(1+P)}{24}\tau^4 \right]. \quad (16)
 \end{aligned}$$

Equation (14) holds for $k = 1$ via Equation (16). If we think that Equation (14) holds for $k = n - 1$, then we will prove that it also holds for $k = n$ according to the induction method. Through Equations (11) and (12) and Lemma 1, we derive

$$\begin{aligned}
D_{\tau}^{(n)}(\gamma) &= \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[g(t, z, X_{t,z}^{(n)}(\gamma)) - g(t, z, X_{t,z}^{(n-1)}(\gamma)) \right] dz \right. \\
&\quad \left. + \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left[h(t, z, X_{t,z}^{(n)}(\gamma)) - h(t, z, X_{t,z}^{(n-1)}(\gamma)) \right] dz \right| \\
&\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[g(t, z, X_{t,z}^{(n)}(\gamma)) - g(t, z, X_{t,z}^{(n-1)}(\gamma)) \right] dz \right| \\
&\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \left| \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left[h(t, z, X_{t,z}^{(n)}(\gamma)) - h(t, z, X_{t,z}^{(n-1)}(\gamma)) \right] dz \right| \\
&\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| g(t, z, X_{t,z}^{(n)}(\gamma)) - g(t, z, X_{t,z}^{(n-1)}(\gamma)) \right| dz \\
&\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{y-a(s-t)}^{y+a(s-t)} \left| h(t, z, X_{t,z}^{(n)}(\gamma)) - h(t, z, X_{t,z}^{(n-1)}(\gamma)) \right| dz \\
&\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| g(t, z, X_{t,z}^{(n)}(\gamma)) - g(t, z, X_{t,z}^{(n-1)}(\gamma)) \right| dz \\
&\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt K_{\gamma} \int_{y-a(s-t)}^{y+a(s-t)} \left| h(t, z, X_{t,z}^{(n)}(\gamma)) - h(t, z, X_{t,z}^{(n-1)}(\gamma)) \right| dz. \tag{17}
\end{aligned}$$

According to Equation (14) for $k = n - 1$ and Equation (8), we obtain

$$\begin{aligned}
D_{\tau}^{(n)}(\gamma) &\leq \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{1}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} L \left| X_{t,z}^{(n)}(\gamma) - X_{t,z}^{(n-1)}(\gamma) \right| dz \\
&\quad + \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \frac{K_{\gamma}}{2a} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} L \left| X_{t,z}^{(n)}(\gamma) - X_{t,z}^{(n-1)}(\gamma) \right| dz \\
&= \frac{1 + K_{\gamma}}{2a} L \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left| X_{t,z}^{(n)}(\gamma) - X_{t,z}^{(n-1)}(\gamma) \right| dz \\
&\leq \frac{1 + K_{\gamma}}{2a} L \max_{y \in \mathbb{R}} \max_{0 \leq s \leq \tau} \int_0^s dt \int_{y-a(s-t)}^{y+a(s-t)} \left[\frac{2L^{n-1}(1 + K_{\gamma})^{n-1}N}{(2n-2)!} \tau^{2n-2} \right. \\
&\quad \left. + \frac{L^{n-1}(1 + K_{\gamma})^{n-1}Q}{(2n-1)!} \tau^{2n-1} + \frac{L^n(1 + K_{\gamma})^n(1 + P)}{(2n)!} \tau^{2n} \right] dz \\
&\leq \frac{1 + K_{\gamma}}{2a} L \int_0^{\tau} 2a(\tau - t) \left[\frac{2L^{n-1}(1 + K_{\gamma})^{n-1}P}{(2n-2)!} \tau^{2n-2} + \frac{L^{n-1}(1 + K_{\gamma})^{n-1}Q}{(2n-1)!} \tau^{2n-1} \right. \\
&\quad \left. + \frac{L^n(1 + K_{\gamma})^n(1 + P)}{(2n)!} \tau^{2n} \right] dt \\
&= \frac{2L^n(1 + K_{\gamma})^nP}{(2n)!} \tau^{2n} + \frac{L^n(1 + K_{\gamma})^nQ}{(2n+1)!} \tau^{2n+1} + \frac{L^{n+1}(1 + K_{\gamma})^{n+1}(1 + P)}{(2n+2)!} \tau^{2n+2}. \tag{18}
\end{aligned}$$

Thus, we show that Equation (14) is true for all n . For any sample γ , noting that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{\tau}^{(n)}(\gamma) &\leq \sum_{n=0}^{\infty} \frac{2L^n(1 + K_{\gamma})^nP}{(2n)!} \tau^{2n} + \frac{L^n(1 + K_{\gamma})^nQ}{(2n+1)!} \tau^{2n+1} \\
&\quad + \frac{L^{n+1}(1 + K_{\gamma})^{n+1}(1 + P)}{(2n+2)!} \tau^{2n+2}. \tag{19}
\end{aligned}$$

Clearly, according to Weierstrass' criterion, we know that $X_{\tau,y}^{(n)}(\gamma)$ uniformly converges on $[0, T]$ for every given γ .

Denote

$$X_{\tau,y}(\gamma) = \lim_{n \rightarrow \infty} X_{\tau,y}^{(n)}(\gamma), \quad \gamma \in \Gamma, \quad \tau \in [0, T]. \quad (20)$$

Then, we have

$$\begin{aligned} X_{\tau,y}(\gamma) = & \frac{1}{2}[\psi(y + a\tau) + \psi(y - a\tau)] + \frac{1}{2a} \int_{y-a\tau}^{y+a\tau} \varphi(z) dz + \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} g(t, z, X_{t,z}) dz \\ & + \frac{1}{2a} \int_0^\tau dC_t(\gamma) \int_{y-a(\tau-t)}^{y+a(\tau-t)} h(t, z, X_{t,z}) dz. \end{aligned} \quad (21)$$

Hence, for all $\tau > 0$, Equation (5) has the solution $X_{\tau,y}$ because of T being arbitrary.

Next, uniqueness is going to be proved. Assume that the uncertain equation has two solutions $X_{\tau,y}$ and $X_{\tau,y}^*$. According to Equations (9) and (8), we have

$$\begin{aligned} & |X_{\tau,y}(\gamma) - X_{\tau,y}^*(\gamma)| \\ &= \left| \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} [g(t, z, X_{t,z}) - g(t, z, X_{t,z}^*)] dz \right. \\ &\quad \left. + \frac{1}{2a} \int_0^\tau dC_\tau(\gamma) \int_{y-a(\tau-t)}^{y+a(\tau-t)} [h(t, z, X_{t,z}) - h(t, z, X_{t,z}^*)] dz \right| \\ &\leq \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} |g(t, z, X_{t,z}) - g(t, z, X_{t,z}^*)| dz \\ &\quad + \frac{1}{2a} \int_0^\tau dC_\tau(\gamma) \int_{y-a(\tau-t)}^{y+a(\tau-t)} |h(t, z, X_{t,z}) - h(t, z, X_{t,z}^*)| dz \\ &\leq L(1 + K_\gamma) \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} |X_{t,z}(\gamma) - X_{t,z}^*(\gamma)| dz. \end{aligned} \quad (22)$$

It follows from the Grönwall inequality that

$$|X_{\tau,y}(\gamma) - X_{\tau,y}^*(\gamma)| \leq 0. \quad (23)$$

Therefore, $X_{\tau,y} = X_{\tau,y}^*$. Thus, we prove the uniqueness.

Finally, we will verify that $X_{\tau,y}$ is sample-continuous. According to the above proof,

$$\begin{aligned} & X_{\tau,y}(\gamma) \\ &\leq \sum_{n=0}^{\infty} \left[\frac{2L^n(1 + K_\gamma)^nP}{(2n)!} \tau^{2n} + \frac{L^n(1 + K_\gamma)^nQ}{(2n+1)!} \tau^{2n+1} + \frac{L^{n+1}(1 + K_\gamma)^{n+1}(1 + P)}{(2n+2)!} \tau^{2n+2} \right] \\ &\leq (1 + 2P + Q) \exp\left(\sqrt{L(1 + K_\gamma)}\tau\right) \end{aligned} \quad (24)$$

holds. Suppose that $0 < s < \tau$ and $z < y$, we have

$$\begin{aligned}
& |X_{\tau,y}(\gamma) - X_{s,z}^*(\gamma)| \\
&= \left| \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} g(t,v, X_{t,v}) dv - \frac{1}{2a} \int_0^s dt \int_{z-a(s-t)}^{z+a(s-t)} g(t,v, X_{t,v}) dv \right. \\
&\quad \left. + \frac{1}{2a} \int_0^\tau dC_t(\gamma) \int_{y-a(\tau-t)}^{y+a(\tau-t)} h(t,v, X_{t,v}) dv - \frac{1}{2a} \int_0^s dC_t(\gamma) \int_{z-a(s-t)}^{z+a(s-t)} h(t,v, X_{t,v}) dv \right| \\
&= \frac{1}{2a} \left| \int_s^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} g(t,v, X_{t,v}) dv - \int_0^s dt \int_{z-a(s-t)}^{y-a(\tau-t)} g(t,v, X_{t,v}) dv \right. \\
&\quad \left. + \int_0^s dt \int_{z+a(s-t)}^{y+a(\tau-t)} g(t,v, X_{t,v}) dv + \int_s^t dC_t(\gamma) \int_{y-a(\tau-t)}^{y+a(\tau-t)} h(t,v, X_{t,v}) dv \right. \\
&\quad \left. - \int_0^s dC_t(\gamma) \int_{z-a(s-t)}^{y-a(\tau-t)} h(t,v, X_{t,v}) dv + \int_0^s dC_t(\gamma) \int_{z+a(s-t)}^{y+a(\tau-t)} h(t,v, X_{t,v}) dv \right| \quad (25)
\end{aligned}$$

according to the interval additivity of the integral. By using Lemma 1, Equations (9) and (24), we have

$$\begin{aligned}
& |X_{\tau,y}(\gamma) - X_{s,z}^*(\gamma)| \\
&\leq \frac{1}{2a} \left[\int_s^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} |g(t,v, X_{t,v})| dv + \int_0^s dt \int_{z-a(s-t)}^{y-a(\tau-t)} |g(t,v, X_{t,v})| dv \right. \\
&\quad \left. + \int_0^s dt \int_{z+a(s-t)}^{y+a(\tau-t)} |g(t,v, X_{t,v})| dv + \int_s^\tau dt K_\gamma \int_{y-a(\tau-t)}^{y+a(\tau-t)} |h(t,v, X_{t,v})| dv \right. \\
&\quad \left. + \int_0^s dt K_\gamma \int_{z-a(s-t)}^{y-a(\tau-t)} |h(t,v, X_{t,v})| dv + \int_0^s dt K_\gamma \int_{z+a(s-t)}^{y+a(\tau-t)} |h(t,v, X_{t,v})| dv \right] \\
&\leq \frac{1}{2a} (1 + K_\gamma) L \left[\int_s^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} (1 + |X_{t,v}|) dv + \int_0^s dt \int_{z-a(s-t)}^{y-a(\tau-t)} (1 + |X_{t,v}|) dv \right. \\
&\quad \left. + \int_0^s dt \int_{z+a(s-t)}^{y+a(\tau-t)} (1 + |X_{t,v}|) dv \right] \\
&\leq \frac{(1 + 2P + Q) \exp(\sqrt{L(1 + K_\gamma)}\tau)}{2} \left[(\tau - s)^2 + \frac{1}{a} s[(y - z) - a(\tau - s)] \right. \\
&\quad \left. + \frac{1}{a} s[(y - z) + a(\tau - s)] \right] \\
&= \frac{(1 + 2P + Q) \exp(\sqrt{L(1 + K_\gamma)}\tau)}{2} \left[(\tau - s)^2 + \frac{2}{a} s(y - z) \right]. \quad (26)
\end{aligned}$$

Thus $|X_{\tau,y} - X_{s,z}^*| \rightarrow 0$ as $(s, z) \rightarrow (\tau, y)$. That is, $X_{\tau,y}$ is sample-continuous. Thus, we proved the theorem. $\square$

Example 1. Considering

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + X(\tau, y) + \dot{C}_\tau, \quad X(0, y) = 0, \quad \frac{\partial X}{\partial \tau}(0, y) = 0, \quad (27)$$

Then, the unique solution of Equation (27) is

$$X(\tau, y) = \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} X(t, z) dz + \int_0^\tau (\tau - t) dC_t \quad (28)$$

according to Theorem 1.

Proof. Now we verify that Equation (28) is the solution of Equation (27). Obviously, we have $X(0, y) = 0$ from Equation (28). Noting that

$$\begin{aligned}\frac{\partial X}{\partial \tau}(\tau, y) &= \frac{1}{2a} \left[\frac{\partial}{\partial \tau} \int_0^\tau dC_t \int_{y-a(\tau-t)}^{y+a(\tau-t)} X(t, z) dz \right] \\ &= \frac{1}{2} \int_0^\tau [X(t, y + a(\tau - t)) + X(t, y - a(\tau - t))] dt + C_\tau,\end{aligned}\quad (29)$$

we have

$$\frac{\partial X}{\partial \tau}(0, y) = 0. \quad (30)$$

Since

$$\begin{aligned}\frac{\partial^2 X}{\partial \tau^2}(\tau, y) &= \frac{\partial}{\partial \tau} \left[\frac{1}{2} \int_0^\tau [X(t, y + a(\tau - t)) + X(t, y - a(\tau - t))] dt + C_\tau \right] \\ &= X(\tau, y) + \frac{a}{2} \int_0^\tau \left[\frac{\partial X}{\partial y}(t, y + a(\tau - t)) - \frac{\partial X}{\partial y}(t, y - a(\tau - t)) \right] dt + \frac{dC_\tau}{d\tau},\end{aligned}\quad (31)$$

$$\begin{aligned}\frac{\partial X}{\partial y}(\tau, y) &= \frac{\partial}{\partial x} \left[\frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} X(t, z) dz + \int_0^\tau (\tau - t) dC_t \right] \\ &= \frac{1}{2a} \int_0^\tau [X(t, y + a(\tau - t)) - X(t, y - a(\tau - t))] dt,\end{aligned}\quad (32)$$

and

$$\begin{aligned}\frac{\partial^2 X}{\partial y^2}(\tau, y) &= \frac{1}{2a} \left[\frac{\partial}{\partial y} \int_0^\tau [X(t, y + a(\tau - t)) - X(t, y - a(\tau - t))] dt \right] \\ &= \frac{1}{2a} \int_0^\tau \left[\frac{\partial X}{\partial y}(t, y + a(\tau - t)) - \frac{\partial X}{\partial y}(t, y - a(\tau - t)) \right] dt.\end{aligned}\quad (33)$$

Then, we have

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + X(\tau, y) + \dot{C}_\tau. \quad (34)$$

Thus, we prove that Equation (28) is the solution of Equation (27). Next we prove the uniqueness. Since $g(\tau, y, x) = x$ and $h(\tau, y, x) = 1$, we obtain

$$|g(\tau, y, x)| + |h(\tau, y, x)| = 1 + |x|, \quad \forall x \in \mathbb{R}, \quad (35)$$

and

$$|g(\tau, y, x) - g(\tau, y, v)| + |h(\tau, y, x) - h(\tau, y, v)| = |x - v|, \quad \forall x, v \in \mathbb{R}. \quad (36)$$

Due to Theorem 1, Equation (27) has a unique solution which is Equation (28). $\square$

Example 2. The solution of the following uncertain wave equation

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + \dot{C}_\tau, \quad X(0, y) = \sin y, \quad \frac{\partial X}{\partial \tau}(0, y) = 0. \quad (37)$$

is

$$X(\tau, y) = \sin y \cos a\tau + \int_0^\tau (\tau - t) dC_t \quad (38)$$

and it is unique from Theorem 1.

Proof. Obviously, we have $X(0, y) = \sin y$ from Equation (38). Noting that

$$\frac{\partial X}{\partial \tau}(\tau, y) = \frac{\partial}{\partial \tau} [\sin y \cos a\tau + \int_0^\tau (\tau - t) dC_t] = -a \sin y \sin a\tau + C_\tau, \quad (39)$$

we have

$$\frac{\partial X}{\partial \tau}(0, y) = 0. \quad (40)$$

Since

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = \frac{\partial}{\partial \tau}[-a \sin y \sin a\tau + C_\tau] = -a^2 \sin y \cos a\tau + \dot{C}_\tau, \quad (41)$$

$$\frac{\partial X}{\partial y}(\tau, y) = \frac{\partial}{\partial y}[\sin y \cos a\tau + \int_0^\tau (\tau - t) dC_t] = \cos y \cos a\tau + \int_0^\tau (\tau - t) dC_t, \quad (42)$$

and

$$\frac{\partial^2 X}{\partial y^2}(\tau, y) = \frac{\partial}{\partial y}[\cos y \cos a\tau + \int_0^\tau (\tau - t) dC_t] = -\sin y \cos a\tau + \int_0^\tau (\tau - t) dC_t, \quad (43)$$

we have

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + \dot{C}_\tau. \quad (44)$$

Thus, we prove that Equation (38) is the solution of Equation (37). Next, we prove the uniqueness. Since $g(\tau, y, x) = 0$ and $h(\tau, y, x) = 1$, we obtain

$$|g(\tau, y, x)| + |h(\tau, y, x)| = 1 \leq (1 + |x|), \quad \forall x \in \mathfrak{R}, \quad (45)$$

and

$$|g(\tau, y, x) - g(\tau, y, v)| + |h(\tau, y, x) - h(\tau, y, v)| = 0, \quad \forall x, v \in \mathfrak{R}. \quad (46)$$

According to Theorem 1, Equation (37) has a unique solution which is Equation (38). $\square$

Example 3. Considering

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + \sin^2 X(\tau, y) + \dot{C}_\tau, \quad X(0, y) = 0, \quad \frac{\partial X}{\partial \tau}(0, y) = 0, \quad (47)$$

Then, the unique solution of Equation (47) is

$$X(\tau, y) = \frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} \sin^2 X(t, z) dz + \int_0^\tau (\tau - t) dC_t \quad (48)$$

according to Theorem 1.

Proof. Now we verify that Equation (48) is the solution of Equation (47). Obviously, we have $X(0, y) = 0$ from Equation (48). Noting that

$$\begin{aligned} \frac{\partial X}{\partial \tau}(\tau, y) &= \frac{1}{2a} \left[\frac{\partial}{\partial \tau} \int_0^\tau dC_t \int_{y-a(\tau-t)}^{y+a(\tau-t)} \sin^2 X(t, z) dz \right] \\ &= \frac{1}{2} \int_0^\tau [\sin^2 X(t, y + a(\tau - t)) + X(t, y - a(\tau - t))] dt + C_\tau, \end{aligned} \quad (49)$$

we have

$$\frac{\partial X}{\partial \tau}(0, y) = 0. \quad (50)$$

Since

$$\begin{aligned}
& \frac{\partial^2 X}{\partial \tau^2}(\tau, y) \\
&= \frac{\partial}{\partial \tau} \left[\frac{1}{2} \int_0^\tau [\sin^2 X(t, y + a(\tau - t)) + \sin^2 X(t, y - a(\tau - t))] dt + C_\tau \right] \\
&= \sin^2 X(\tau, y) + a \int_0^\tau \left[\sin X(t, y + a(\tau - t)) \cos X(t, y + a(\tau - t)) \frac{\partial X}{\partial y}(t, y + a(\tau - t)) \right. \\
&\quad \left. - \sin X(t, y - a(\tau - t)) \cos X(t, y - a(\tau - t)) \frac{\partial X}{\partial y}(t, y - a(\tau - t)) \right] dt + \frac{dC_\tau}{d\tau}, \quad (51)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial X}{\partial y}(\tau, y) &= \frac{\partial}{\partial x} \left[\frac{1}{2a} \int_0^\tau dt \int_{y-a(\tau-t)}^{y+a(\tau-t)} \sin^2 X(t, z) dz + \int_0^\tau (\tau - t) dC_t \right] \\
&= \frac{1}{2a} \int_0^\tau [\sin^2 X(t, y + a(\tau - t)) - \sin^2 X(t, y - a(\tau - t))] dt, \quad (52)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2 X}{\partial y^2}(\tau, y) \\
&= \frac{1}{2a} \left[\frac{\partial}{\partial y} \int_0^\tau [\sin^2 X(t, y + a(\tau - t)) - \sin^2 X(t, y - a(\tau - t))] dt \right] \\
&= \frac{1}{a} \int_0^\tau \left[\sin X(t, y + a(\tau - t)) \cos X(t, y + a(\tau - t)) \frac{\partial X}{\partial y}(t, y + a(\tau - t)) \right. \\
&\quad \left. - \sin X(t, y - a(\tau - t)) \cos X(t, y - a(\tau - t)) \frac{\partial X}{\partial y}(t, y - a(\tau - t)) \right] dt. \quad (53)
\end{aligned}$$

Then, we have

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + \sin^2 X(\tau, y) + \dot{C}_\tau. \quad (54)$$

Thus, we prove that Equation (48) is the solution of Equation (47). Next we prove the uniqueness. Since $g(\tau, y, x) = \sin^2 x$ and $h(\tau, y, x) = 1$, we obtain

$$|g(\tau, y, x)| + |h(\tau, y, x)| = 1 + \sin^2 x \leq (1 + |x|), \quad \forall x \in \mathbb{R}, \quad (55)$$

and

$$|g(\tau, y, x) - g(\tau, y, v)| + |h(\tau, y, x) - h(\tau, y, v)| \leq 2|x - v|, \quad \forall x, v \in \mathbb{R}. \quad (56)$$

Due to Theorem 1, Equation (47) has a unique solution which is Equation (48). $\square$

Corollary 1. The solution of Equation (5) is unique if functions $\psi(x)$ and $\varphi(x)$ satisfy

$$|\psi(y)| \leq P, \quad |\varphi(y)| \leq Q, \quad \forall y \in \mathbb{R} \quad (57)$$

for $P, Q > 0$, and $g(\tau, y, x)$ and $h(\tau, y, x)$ are bounded and its partial differential is continuous in relation to x . Furthermore, the solution of Equation (5) is sample-continuous.

Proof. There is a crisp number K such that

$$|g(\tau, y, x)| \leq K, \text{ and } |h(\tau, y, x)| \leq K, \quad \forall x \in \mathbb{R}, \quad (58)$$

due to $g(\tau, y, x)$ and $h(\tau, y, x)$ being bounded in relation to x . Thus we have

$$|g(\tau, y, x)| + |h(\tau, y, x)| \leq 2K \leq 2K(1 + |x|). \quad (59)$$

Since the partial differential of $g(\tau, y, x)$ is continuous with in relation to x , there is a crisp number H such that

$$\left| \frac{\partial g}{\partial x}(\tau, y, x) \right| \leq H, \quad \forall x \in \mathbb{R}. \quad (60)$$

It follows from the Lagrange Mean Value Theorem that

$$|g(\tau, y, x) - g(\tau, y, v)| = \left| \frac{\partial g(\tau, y, x)}{\partial x}(x - v) \right| \leq H|x - v|, \quad \forall x, v \in \mathbb{R}. \quad (61)$$

Denoting $L = \max\{2K, H\}$, we derive the result via Theorem 1. In addition the proof of sample-continuity for the solution is as the same as one in Theorem 1. Therefore the proof is completed. $\square$

Corollary 2. *The solution of the following linear uncertain wave equation*

$$\frac{\partial^2 X}{\partial \tau^2}(\tau, y) = a^2 \frac{\partial^2 X}{\partial y^2}(\tau, y) + g(\tau, y) + h(\tau, y)\dot{C}_\tau, \quad X(0, y) = \psi(y), \quad \frac{\partial X}{\partial \tau}(0, y) = \varphi(y) \quad (62)$$

is unique if functions $\psi(y)$ and $\varphi(y)$ are bounded

$$|\psi(y)| \leq P, \quad |\varphi(y)| \leq Q, \quad \forall y \in \mathbb{R} \quad (63)$$

for $P, Q > 0$, and functions $g(\tau, y)$ and $h(\tau, y)$ are also bounded. Furthermore, the solution of Equation (62) is sample-continuous.

Proof. There is a crisp number K such that

$$|g(\tau, y, x)| \leq K, \quad \text{and} \quad |h(\tau, y, x)| \leq K, \quad \forall x \in \mathbb{R}, \quad (64)$$

because $g(\tau, y)$ and $h(\tau, y)$ are bounded. Following that, we have

$$|g(\tau, y, x)| + |h(\tau, y, x)| \leq 2K \leq 2K(1 + |x|) \stackrel{\text{Let } L=2K}{=} L(1 + |x|), \quad \forall x \in \mathbb{R}. \quad (65)$$

Denoting

$$g(\tau, y, x) = g(\tau, y), \quad h(\tau, y, x) = h(\tau, y), \quad \forall x \in \mathbb{R}, \quad (66)$$

we have

$$|g(\tau, y, x) - g(\tau, y, v)| + |h(\tau, y, x) - h(\tau, y, v)| = 0, \quad \forall x, v \in \mathbb{R}. \quad (67)$$

Hence we obtain the corollary by using Theorem 1. $\square$

Remark 1. *The proofs show that Corollaries 1 and 2 are special cases of Theorem 1. Theorem 1 is more general and it is convenient for the complex case. However, Corollaries 1 and 2 are more convenient for special cases.*

4. Discussion and Conclusions

The uncertain wave equation is established to model wave propagation with uncertain disturbances. For an equation, the property existence and uniqueness of the solution is important. Hence, the existence and uniqueness theorem was proposed and proved under the Lipschitz condition and linear growth condition. Then, some examples were given to illustrate how to obtain an uncertain wave equation's solution. Finally, two corollaries were derived, which is the special case of the general existence and uniqueness theorem and is convenient for judging whether the special wave equation has a unique solution or not.

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