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# Dynamic Inequalities of Two-Dimensional Hardy Type via Alpha-Conformable Derivatives on Time Scales

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**Abstract:** We established some new $\alpha$-conformable dynamic inequalities of Hardy–Knopp type. Some new generalizations of dynamic inequalities of $\alpha$-conformable Hardy type in two variables on time scales are established. Furthermore, we investigated Hardy's inequality for several functions of $\alpha$-conformable calculus. Our results are proved by using two-dimensional dynamic Jensen's inequality and Fubini's theorem on time scales. When $\alpha = 1$, then we obtain some well-known time-scale inequalities due to Hardy. As special cases, we derived Hardy's inequality for $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

**Keywords:** Hardy's inequality; Jensen's inequality; dynamic inequality; time scale



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## 1. Introduction

The renowned discrete Hardy's inequality [1] states that:

**Theorem 1.** *If $\{b_n\}$ is a nonnegative real sequence and $p > 1$, then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} b_n^p, \quad p > 1. \tag{1}$$

Hardy discovered this inequality while attempting to sketch an easier proof of Hilbert's inequality for double series, which was known at that time.

Using the calculus of variations, Hardy himself in [2] gave the following integral analogous of inequality (1).

**Theorem 2.** *If $\phi$ is a nonnegative integrable function over a finite interval $(0, \chi)$ such that $\phi \in L^p(0, \infty)$ and $p > 1$, then*

$$\int_0^{\infty}\left(\frac{1}{\chi}\int_0^{\chi}\phi(\Im)d\Im\right)^p d\chi \le \left(\frac{p}{p-1}\right)^p \int_0^{\infty}\phi^p(\chi)d\chi. \tag{2}$$

It is worth mentioning that inequalities (1) and (2) are sharp in the sense that the constant $(p/p-1)^p$ in each of them cannot be replaced by a smaller one.

In [3], Hardy and Littlewood extended inequality (1) and obtained the following two discrete inequalities.

**Theorem 3.** *Let $\{b_n\}$ be a nonnegative real sequence.*

**(*i*)** *If $p > 1$ and $c > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^c} \left( \sum_{k=1}^{n} b_k \right)^p \leq K(p, \gamma) \sum_{n=1}^{\infty} \frac{1}{n^{c-p}} b_n^p. \tag{3}$$

**(*ii*)** *If $p > 1$ and $c < 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^c} \left( \sum_{k=n}^{\infty} b_k \right)^p \leq K(p, \gamma) \sum_{n=1}^{\infty} \frac{1}{n^{c-p}} b_n^p, \tag{4}$$

*where $K(p, c)$ in inequalities (3) and (4) is a nonnegative constant depending on $p$ and $c$.*

In [2], Hardy established the continuous versions of inequalities (3) and (4) as follows:

**Theorem 4.** *Let $\phi$ be a nonnegative integrable function $\phi$ on $(0, \infty)$.*

**(*i*)** *If $p > 1$ and $m > 1$, then*

$$\int_0^{\infty} \frac{1}{\chi^m} \left( \int_0^{\chi} \phi(\Im) d\Im \right)^p dx \leq \left( \frac{p}{m-1} \right)^p \int_0^{\infty} \frac{1}{\chi^{m-p}} \phi^p(\chi) d\chi; \tag{5}$$

**(*ii*)** *If $p > 1$ and $m < 1$, then*

$$\int_0^{\infty} \frac{1}{\chi^m} \left( \int_{\chi}^{\infty} \phi(\Im) d\Im \right)^p dx \leq \left( \frac{p}{1-m} \right)^p \int_0^{\infty} \frac{1}{\chi^{m-p}} \phi^p(\chi) d\chi. \tag{6}$$

The reverse of inequality (2) was proven by Hardy and Littlewood in [3]. Their result can be written as:

**Theorem 5.** *If $0 < p < 1$ and $\phi$ is a nonnegative integrable function on $(\chi, \infty)$ such that $\phi \in L^p(0, \infty)$, then*

$$\int_0^{\infty} \left( \frac{1}{\chi} \int_{\chi}^{\infty} \phi(\Im) d\Im \right)^p d\chi \geq \left( \frac{p}{1-p} \right)^p \int_0^{\infty} \phi^p(\chi) d\chi. \tag{7}$$

In the same paper [3], the authors proved the following sharp inequality.

**Theorem 6.** *If $p > 1$ and $\phi$ is a nonnegative integrable function on $(\chi, \infty)$ such that $\phi \in L^p(0, \infty)$, then*

$$\int_0^{\infty} \left( \frac{1}{\chi} \int_{\chi}^{\infty} \phi(\Im) d\Im \right)^p d\chi \leq p^p \int_0^{\infty} \phi^p(\chi) d\chi, \tag{8}$$

*which by a trivial transformation can be written as*

$$\int_0^{\infty} \left( \int_{\chi}^{\infty} \phi(\Im) d\Im \right)^p d\chi \leq p^p \int_0^{\infty} (x f(\chi))^p dx. \tag{9}$$

The discrete version of inequality (9) was given in [4] as follows:

**Theorem 7.** *If $\{b_n\}$ is a nonnegative real sequence and $p > 1$, then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} b_k\right)^p \le p^p \sum_{n=1}^{\infty}(na_n)^p. \tag{10}$$

Hardy [5] generalized (1) and proved the following result.

**Theorem 8.** *If $p > 1, b_n > 0, \lambda_n > 0$ for $n \ge 1$ and $\Lambda_n = \sum_{k=1}^{n} \lambda_n$, then*

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n^p}\left(\sum_{k=1}^{n}\lambda_k b_k\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\lambda_n b_n^p. \tag{11}$$

The study of Hardy type inequalities has attracted the attention of many researchers. Over several decades, many generalizations, extensions, and refinements have been made to the above inequalities; we refer the reader to the papers [1,2,5–10], the books [4,11], and the references cited therein.

Time-scale calculus with the objective to unify discrete and continuous analysis was introduced by S. Hilger [12]. For additional subtleties on time scales, we direct the peruser to the books by Bohner and Peterson [13,14].

In [15], Řehák has given the time scales version of Hardy's inequality as follows:

**Theorem 9.** *Let $\mathbb{T}$ be a time scale and $\psi \in C_{rd}([a,\infty)_{\mathbb{T}}, [0,\infty)), \Lambda(\Im) = \int_a^{\Im} \psi(\zeta)\Delta\zeta$ for $\Im \in [a,\infty)_{\mathbb{T}}$.*

$$\int_a^{\infty}\left(\frac{\Lambda^{\sigma}(\Im)}{\sigma(\Im)-a}\right)^{\beta}\Delta\Im < \left(\frac{\beta}{\beta-1}\right)^{\beta}\int_a^{\infty}\psi^{\beta}(\Im)\Delta\Im, \quad \beta > 1, \tag{12}$$

*unless $\psi \equiv 0$.*

*Furthermore, if $\mu(\Im)/\Im \to 0$ as $\Im \to \infty$, then inequality (12) is sharp.*

In [16], Saker and O'Regan established a generalization of Řehák's result in the following form.

**Theorem 10.** *Let $\mathbb{T}$ be time scale and $1 \leqslant c \leqslant k$. Let*

$$\chi(\Im) = \int_a^{\Im}\lambda(s)\Delta s \tag{13}$$

*for any $\Im \in [a,\infty)_{\mathbb{T}}$ and define*

$$\Theta(\Im) = \int_a^{\Im}\lambda(s)\xi(s)\Delta s \tag{14}$$

*for any $\Im \in [a,\infty)_{\mathbb{T}}$. Then*

$$\int_a^{\infty}\frac{\lambda(\Im)}{(\chi^{\sigma}(\Im))^c}(\Theta^{\sigma}(\Im))^k\Delta\Im \leqslant \frac{k}{c-1}\int_a^{\infty}\chi^{1-c}(\Im)\lambda(\Im)\xi(\Im)(\Theta(\Im))^{k-1}\Delta\Im$$

*and*

$$\int_a^{\infty}\frac{\lambda(\Im)}{(\chi^{\sigma}(\Im))^c}(\Theta^{\sigma}(\Im))^k\Delta\Im \leqslant \left(\frac{k}{c-1}\right)^k\int_a^{\infty}\frac{(\chi^{\sigma}(\Im))^{(k-1)c}}{(\chi(\Im))^{k(c-1)}}\lambda(\Im)\xi^k(\Im)\Delta\Im.$$

**Theorem 11.** *Let $\mathbb{T}$ be a time scale and $k > 1$ and $0 \leqslant c < 1$. Let $\chi$ be defined as in (13) and define*

$$\overline{\Theta}(\Im) = \int_{\Im}^{\infty} \lambda(s)\xi(s)\Delta s$$

*for any $\Im \in [a, \infty)_{\mathbb{T}}$. Then,*

$$\int_a^{\infty} \frac{\lambda(\Im)}{(\chi^{\sigma}(\Im))^c} \left(\overline{\Theta}(\Im)\right)^k \Delta\Im \leqslant \frac{k}{1-c} \int_a^{\infty} (\chi^{\sigma}(\Im))^{1-c} \lambda(\Im)\xi(\Im) \left(\overline{\Theta}(\Im)\right)^{k-1} \Delta\Im$$

*and*

$$\int_a^{\infty} \frac{\lambda(\Im)}{(\chi^{\sigma}(\Im))^c} \left(\overline{\Theta}(\Im)\right)^k \Delta\Im \leqslant \left(\frac{k}{1-c}\right)^k \int_a^{\infty} (\chi^{\sigma}(\Im))^{k-c} \lambda(\Im)\xi^k(\Im)\Delta\Im.$$

Recently, in 2017, Agarwal et al. [17] gave the inequality.

**Theorem 12.** *Suppose $\mathbb{T}$ is a time scale such that $0 \in \mathbb{T}$. Further, assume $\eta$ is a nonincreasing nonnegative function on $[0, \infty)_{\mathbb{T}}$. If $p > 1$, then*

$$\int_0^{\infty} \frac{1}{\Im^p} \left( \int_0^{\Im} \eta(s)\Delta s \right)^p \Delta\Im \geq \frac{p}{p-1} \int_0^{\infty} \eta^p(\Im)\Delta\Im. \tag{15}$$

In 2020, El-Deeb et al. [18] established a generalization of inequality (15).

**Theorem 13.** *Suppose that $\mathbb{T}$ is a time scale with $0 \leq a \in \mathbb{T}$. Moreover, assume that $\eta$ and $\lambda$ are nonnegative rd-continuous functions on $[a, \infty)_{\mathbb{T}}$ with $\eta$ nonincreasing. If $p \geq 1$ and $\gamma > 1$, then*

$$\int_a^{\infty} \frac{\lambda(\Im)\Psi^p(\Im)}{\Lambda^{\gamma}(\Im)} \Delta\Im \geq \frac{p}{\gamma - 1} \int_a^{\infty} \lambda(\Im)\Lambda^{p-\gamma}(\Im)\eta^p(\Im)\Delta\Im, \tag{16}$$

*where*

$$\Psi(\Im) = \int_a^{\Im} \lambda(s)\eta(s)\Delta s \qquad \text{and} \qquad \Lambda(\Im) = \int_a^{\Im} \lambda(s)\Delta s.$$

In 2020, Saker [19] proved the following theorem.

**Theorem 14.** *Assume that $\mathbb{T}$ is a time scale with $\omega \in (0, \infty)_{\mathbb{T}}$. If $m \leqslant 0 < h < 1$, $\chi(\Im) = \int_{\Im}^{\infty} \lambda(s)\Delta s$ and $\Theta(\Im) = \int_{\omega}^{\Im} \lambda(s)\xi(s)\Delta s$, then*

$$\int_{\omega}^{\infty} \frac{\lambda(\Im)}{\chi^m(\Im)} (\Theta^{\sigma}(\Im))^h \Delta\Im \geqslant \left(\frac{h}{1-m}\right)^h \int_{\omega}^{\infty} \lambda(\Im)\xi^h(\Im)\chi^{h-m}(\Im)\Delta\Im.$$

*If $0 < h < 1 < m$, $\chi(\Im) = \int_{\Im}^{\infty} \lambda(s)\Delta s$ and $\overline{\Theta}(\Im) = \int_{\Im}^{\infty} \lambda(s)\xi(s)\Delta s$, then*

$$\int_{\omega}^{\infty} \frac{\lambda(\Im)}{\chi^m(\Im)} (\overline{\Theta}(\Im))^h \Delta\Im \geqslant \left(\frac{hM^m}{m-1}\right)^h \int_{\omega}^{\infty} \lambda(\Im)\xi^h(\Im)\chi^{h-m}(\Im)\Delta\Im,$$

*where*

$$M := \inf_{\Im \in \mathbb{T}} \frac{\chi^{\sigma}(\Im)}{\chi(\Im)} > 0.$$

In [20], Ozkan and Yildirim gave the following result among many other results.

**Theorem 15.** *Let $a \in [0,\infty)_{\mathbb{T}}$ and $u,\varpi \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_+)$ such that the delta integral $\int_{\chi}^{b} \frac{u(\Im)}{(\Im-a)(\sigma(\Im)-a)}\Delta\Im$ converges. If $\psi \in C_{rd}([a,b]_{\mathbb{T}},(\theta,\beta))$ and $\Psi \in C((\theta,\beta),\mathbb{R})$ is convex, then*

$$\int_a^b \frac{u(\chi)}{\chi-a}\Psi\left(\frac{\int_a^{\sigma(\chi)}\psi(\Im)\Delta\Im}{\sigma(\chi)-a}\right)\Delta\chi \le \int_a^b \Psi(\psi(\chi))\left(\int_{\chi}^{b}\frac{u(\Im)}{(\Im-a)(\sigma(\Im)-a)}\Delta\Im\right)\Delta\chi.$$

Benkhettou et al. [21] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus.

We define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by

$$\sigma(t) := \inf\{s\in\mathbb{T} : s>t\}, \qquad t\in\mathbb{T} \tag{17}$$

and the backward jump operator $\rho:\mathbb{T}:\to\mathbb{T}$ is defined by

$$\rho(t) := \sup\{s\in\mathbb{T} : s<t\}, \qquad t\in\mathbb{T}. \tag{18}$$

**Definition 1.** *Let $\eta:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}^k$, and $\alpha\in(0,1]$. For $t>0$, we define $T_\alpha^\Delta(\eta)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon>0$, there is a $\delta$-neighborhood $U_t\subset\mathbb{T}$ of $t$, $\delta>0$, such that*

$$|[\eta(\sigma(t))-\eta(s)]t^{1-\alpha} - T_\alpha^\Delta(\eta)(t)[\sigma(t)-s]| \le \varepsilon|\sigma(t)-s|,$$

*for all $s\in U_t$. We call $T_\alpha^\Delta(\eta)(t)$ the conformable derivative of $\eta$ of order $\alpha$ at $t$, and we define a conformable derivative on $\mathbb{T}$ at 0, as $T_\alpha^\Delta(\eta)(0)=\lim_{t\to 0+}T_\alpha^\Delta(\eta)(t)$.*

**Remark 1.** *If $\alpha=1$, then we obtain from Definition 1 the delta derivative of time scales. The conformable derivative of order zero is defined by the identity operator $T_0^\Delta(\eta)=\eta$.*

**Theorem 16.** *Let $\alpha\in(0,1]$ and $\mathbb{T}$ be a time scale. Assume $\eta:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$. The following properties hold.*

*(i) If $\eta$ is conformable differentiable of order $\alpha$ at $t>0$, then $\eta$ is continuous at $t$;*
*(ii) If $\eta$ is continuous at $t$ and $t$ is right-scattered, then $\eta$ is conformable differentiable of order $\alpha$ at $t$ with*

$$T_\alpha^\Delta(\eta)(t) = \frac{\eta(\sigma(t))-\eta(t)}{\mu(t)}t^{1-\alpha};$$

*(iii) If $t$ is right-dense, then $\eta$ is conformable differentiable of order $\alpha$ at $t$ if and only if the limit $\lim_{s\longrightarrow t}\frac{\eta(t)-\eta(s)}{t-s}t^{1-\alpha}$ exists as a finite number. In this case,*

$$T_\alpha^\Delta(\eta)(t) = \lim_{s\longrightarrow t}\frac{\eta(t)-\eta(s)}{t-s}t^{1-\alpha};$$

*(iv) If $\eta$ is differentiable of order $\alpha$ at $t$, then*

$$\eta(\sigma(t)) = \eta(t)+\mu(t)t^{\alpha-1}T_\alpha^\Delta(\eta)(t).$$

The conformable derivative has the following properties.

**Theorem 17.** *Assume $\eta,\xi:\mathbb{T}\to\mathbb{R}$ are conformable differentiable of order $\alpha\in(0,1]$, then the following properties hold:*

(i) *The sum* $\eta + \xi : \mathbb{T} \to \mathbb{R}$ *is conformable differentiable with*

$$T_\alpha^\Delta(\eta + \xi) = T_\alpha^\Delta(\eta) + T_\alpha^\Delta(\xi);$$

(ii) *For any* $k \in \mathbb{R}, k\eta : \mathbb{T} \to \mathbb{R}$ *is conformable differentiable with*

$$T_\alpha^\Delta(k\eta) = kT_\alpha^\Delta(\eta);$$

(iii) *If* $\eta$ *and* $\xi$ *are continuous, then the product* $\eta\xi : \mathbb{T} \to \mathbb{R}$ *is conformable differentiable with*

$$T_\alpha^\Delta(\eta\xi) = T_\alpha^\Delta(\eta)\xi + \eta^\sigma T_\alpha^\Delta(\xi) = T_\alpha^\Delta(\eta)\xi^\sigma + \eta T_\alpha^\Delta(\xi);$$

(iv) *If* $\eta$ *is continuous, then* $1/\eta$ *is conformable differentiable with*

$$T_\alpha^\Delta\left(\frac{1}{\eta}\right) = \frac{-T_\alpha^\Delta(\eta)}{\eta(\eta \circ \sigma)}$$

*valid at all points* $t \in \mathbb{T}^k$ *for which* $\eta(\eta \circ \sigma) \neq 0$;

(v) *If* $\eta$ *and* $\xi$ *are continuous, then* $\eta/\xi$ *is conformable differentiable with*

$$T_\alpha^\Delta\left(\frac{\eta}{\xi}\right) = \frac{T_\alpha^\Delta(\eta)\xi - \eta T_\alpha^\Delta(\xi)}{\xi\xi^\sigma}$$

*valid for all* $t \in \mathbb{T}^k$ *for which* $\xi\xi^\sigma \neq 0$.

**Definition 2.** *Let* $\eta : \mathbb{T} \to \mathbb{R}$ *be regulated function. Then, the* $\alpha$*-conformable integral of* $\eta, 0 < \alpha \leqslant 1$, *is defined by*

$$\int \eta(t)\Delta_\alpha t = \int \eta(t)t^{\alpha-1}\Delta t.$$

**Definition 3.** *Suppose* $\eta : \mathbb{T} \to \mathbb{R}$ *is a regulated function. Denote the indefinite* $\alpha$*-conformable integral of* $\eta$ *of order* $\alpha$, $\alpha \in (0,1]$, *as follows:* $F_\alpha(t) = \int \eta(t)\Delta_\alpha t$. *Then, for all* $a, b \in \mathbb{T}$, *we define the Cauchy* $\alpha$*-conformable integral by*

$$\int_a^b \eta(t)\Delta_\alpha t = F_\alpha(b) - F_\alpha(a).$$

A few years ago, by using conformable calculus, a lot of papers were published on conformable inequalities and several authors investigated several new conformable inequalities of Hardy type. For example, in 2020, Saker et al. [22] gave an $\alpha$-conformable version of Theorems 10 and 11 on time scales as follows:

**Theorem 18.** *Let* $\mathbb{T}$ *be time scales and* $1 \leqslant c \leqslant k$, *define*

$$\chi(x) = \int_a^x \lambda(s)\Delta_\alpha s \text{ and } \Theta(x) = \int_a^x \lambda(s)\xi(s)\Delta_\alpha s.$$

*If* $\Theta(\infty) < \infty$ *and*

$$\int_a^\infty \frac{\lambda(s)}{(\chi^\sigma(s))^{c-\alpha+1}}\Delta_\alpha s < \infty,$$

*then*

$$\int_a^\infty \frac{\lambda(x)}{(\chi^\sigma(x))^{c-\alpha+1}}(\Theta(x))^k\Delta_\alpha x \leqslant \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty \frac{\lambda(x)(\chi(x))^{K(\alpha-c)}}{(\chi^\sigma(x))^{(1-k)(c-\alpha+1)}}\xi^k(x)\Delta_\alpha x.$$

**Theorem 19.** *Let $\mathbb{T}$ be time scales and $0 \leqslant c < 1$ and $k > 1$, define*

$$\chi(x) = \int_a^x \lambda(s)\Delta_\alpha s, \text{ and } \Theta(x) = \int_x^\infty \lambda(s)\xi(s)\Delta_\alpha s.$$

*If $\Theta(\infty) < \infty$ and*

$$\int_a^\infty \frac{\lambda(s)}{(\chi^\sigma(s))^{c-\alpha+1}}\Delta_\alpha s < \infty,$$

*then*

$$\int_a^\infty \frac{\lambda(x)}{(\chi^\sigma(x))^{c-\alpha+1}}(\Theta^\sigma(x))^k \Delta_\alpha x \leqslant \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty (\chi^\sigma(x))^{k-c+\alpha-1}\lambda(x)\xi^k(x)\Delta_\alpha x.$$

In 2021, Zakarya et al. [23] gave an $\alpha$-conformable version of Theorem 14 on time scales as follows:

**Theorem 20.** *Assume that $\mathbb{T}$ are time scales with $\omega \in (0,\infty)_\mathbb{T}, k \leqslant 0 < h < 1$ and $\alpha \in (0,1]$. Define*

$$\chi(\Im) = \int_\Im^\infty \lambda(s)\Delta_\alpha s \text{ and } \Theta(\Im) = \int_\omega^\Im \lambda(s)\xi(s)\Delta_\alpha s.$$

*Then,*

$$\int_\omega^\infty \frac{\lambda(\Im)}{\chi^{k-\alpha+1}(\Im)}(\Theta^\sigma(\Im))^h \Delta_\alpha \Im \geqslant \left(\frac{h}{\alpha-m}\right)^h \int_\omega^\infty \lambda(\Im)\xi^h(\Im)\chi^{h-m+\alpha-1}(\Im)\Delta_\alpha \Im.$$

**Theorem 21.** *Assume that $\mathbb{T}$ are time scales with $\omega \in (0,\infty)_\mathbb{T}, 0 < h < 1 < k$ and $\alpha \in (0,1]$. Define*

$$\chi(\Im) = \int_\Im^\infty \lambda(s)\Delta_\alpha s \text{ and } \overline{\Theta}(\Im) = \int_\Im^\infty \lambda(s)\xi(s)\Delta_\alpha s$$

*such that*

$$M := \inf_{\Im \in \mathbb{T}} \frac{\chi^\sigma(\Im)}{\chi(\Im)} > 0.$$

*Then,*

$$\int_\omega^\infty \frac{\lambda(\Im)}{\chi^{k-\alpha+1}(\Im)}(\overline{\Theta}(\Im))^h \Delta_\alpha \Im \geqslant \left(\frac{hM^{k-\alpha+1}}{k-\alpha}\right)^h \int_\omega^\infty \lambda(\Im)\xi^h(\Im)\chi^{h-k+\alpha-1}(\Im)\Delta_\alpha \Im.$$

As the same proof of Theorem 10, we can write the conformable version as follows:

**Theorem 22.** *Let $a \in [0,\infty)_\mathbb{T}$ and define, for $\Im \in [0,\infty)_\mathbb{T}$,*

$$\Phi(\Im) := \int_a^\Im \lambda(\zeta)g(\zeta)\Delta_\alpha \zeta \qquad \text{and} \qquad \Lambda(\Im) := \int_a^\Im \lambda(\zeta)\Delta_\alpha \zeta.$$

*If $p \geq c > 1$, then*

$$\int_a^\infty \lambda(\Im)\frac{(\Phi^\sigma(\Im))^p}{(\Lambda^\sigma(\Im))^q}\Delta_\alpha \Im \leq \left(\frac{p}{q-1}\right)^p \int_a^\infty \lambda(\Im)\frac{(\Lambda^\sigma(\Im))^{q(p-1)}}{(\Lambda(\Im))^{p(q-1)}} g^p(\Im)\Delta_\alpha \Im. \quad (19)$$

In this paper, we prove some generalizations of Hardy type dynamic inequalities that were given recently by Ozkan and Yildirim in [20]. The obtained results extend some known Hardy type integral inequalities and unify and extend some continuous inequalities and their corresponding discrete analogues. The paper is arranged as follows: In Section 2, we state and prove the main results. In Section 3, we state the conclusion.

**Lemma 1.** [24] (Fubini's Theorem on time scales) *Let $\psi$ be bounded and $\Delta$-integrable over $R = [a,b) \times [c,d)$ and suppose that the single integrals*

$$I(\Im) = \int_c^d \psi(\Im,\zeta)\Delta\zeta \qquad \text{and} \qquad K(\zeta) = \int_a^b \psi(\Im,\zeta)\Delta\Im$$

*exist for each $\Im \in [a,b)$ and for each $\zeta \in [c,d)$, respectively. Then, the iterated integrals*

$$\int_a^b \Delta\Im \int_c^d \psi(\Im,\zeta)\Delta\zeta \qquad \text{and} \qquad \int_c^d \Delta\zeta \int_a^b \psi(\Im,\zeta)\Delta\Im$$

*exist and the equality*

$$\int_a^b \Delta\Im \int_c^d \psi(\Im,\zeta)\Delta\zeta = \int_c^d \Delta\zeta \int_a^b \psi(\Im,\zeta)\Delta\Im$$

*holds.*

**Lemma 2.** [25] (Dynamic Jensen's Inequality) *Suppose that $a, b \in \mathbb{T}$ with $a < b$. Further, let $\psi \in C_{rd}\big([a,b]_{\mathbb{T}}, (\theta,\beta)\big)$ and $\varphi \in C_{rd}\big([a,b]_{\mathbb{T}}, \mathbb{R}_+\big)$. If $\Psi \in C\big((\theta,\beta), \mathbb{R}_+\big)$ is convex, then*

$$\Psi\left(\frac{\int_a^b \varphi(\zeta)\psi(\zeta)\Delta_\alpha\zeta}{\int_a^b \varphi(\zeta)\Delta_\alpha\zeta}\right) \le \int_a^b \frac{\varphi(\zeta)\Psi(\psi(\zeta))\Delta_\alpha\zeta}{\int_a^b \varphi(\zeta)\Delta_\alpha\zeta}. \tag{20}$$

We need the following lemma, which gives a two-dimensional dynamic Jensen's inequality, in the proof of our main results.

**Lemma 3** ([25]). *Suppose that $a, b, c, d \in \mathbb{T}$ with $a < b$ and $c < d$. Further, let $\psi \in C_{rd}([a,b)_{\mathbb{T}} \times [c,d]_{\mathbb{T}}, (\theta,\beta)), \phi \in C_{rd}([c,d]_{\mathbb{T}}, \mathbb{R}_+)$ and $\varphi \in C_{rd}([a,b)_{\mathbb{T}}, \mathbb{R}_+)$. If $\Psi \in C((\theta,\beta), \mathbb{R}_+)$ is convex, then*

$$\Psi\left(\frac{\int_a^b\int_c^d \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^b\int_c^d \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}\right) \le \int_a^b\int_c^d \frac{\phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta))\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^b\int_c^d \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}. \tag{21}$$

## 2. Main Results

**Theorem 23.** *Let $a \in [0,\infty)_{\mathbb{T}}$ and $u, \varpi \in C_{rd}([a,b)_{\mathbb{T}}, \mathbb{R}_+)$ such that the delta integral*

$$\int_\chi^b \frac{\varpi(\Im)u(\Im)}{\int_a^{\Im} \varpi(\chi)\Delta_\alpha\chi \int_a^{\sigma(\Im)} \varpi(\chi)\Delta_\alpha\chi}\Delta_\alpha\Im$$

*converges. If* $\psi \in C_{rd}([a,b)_{\mathbb{T}},(\theta,\vartheta))$ *and* $\Psi \in C((\theta,\vartheta),\mathbb{R})$ *is convex, then*

$$\int_a^b \frac{\varpi(\chi)u(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im}\Psi\left(\frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha\Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\right)\Delta_\alpha\chi$$
$$\leq \int_a^b \varpi(\chi)\Psi(\psi(\chi))\left(\int_\chi^b \frac{u(\Im)}{\int_a^\Im \varpi(\chi)\Delta_\alpha\chi \int_a^{\sigma(\Im)} \varpi(\chi)\Delta_\alpha\chi}\Delta_\alpha\Im\right)\Delta_\alpha\chi. \tag{22}$$

**Proof.** Employing the dynamic Jensen inequality (20) and Fubini's theorem on time scales, we obtain

$$\int_a^b \frac{\varpi(\chi)u(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im}\Psi\left(\frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha\Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\right)\Delta_\alpha\chi$$
$$\leq \int_a^b \frac{\varpi(\chi)u(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im \int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\left(\int_a^{\sigma(\chi)} \varpi(\Im)\Psi(\psi(\Im))\Delta_\alpha\Im\right)\Delta_\alpha\chi$$
$$= \int_a^b \varpi(\Im)\Psi(\psi(\Im))\left(\int_\Im^b \frac{u(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im \int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\Delta_\alpha\chi\right)\Delta_\alpha\Im,$$

which is our desired result. □

**Remark 2.** *If we put* $\varpi(\Im) = 1$, $\alpha = 1$ *in Theorem 23, then we recapture Theorem 15.*

Below, we present various applications of Theorem 23.

**(1)** In Theorem 23, if $u(\chi) = 1$ and $b$ is finite, then inequality (22) reads

$$\int_a^b \frac{\varpi(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im}\Psi\left(\frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha\Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\right)\Delta_\alpha\chi$$
$$\leq \int_a^b \varpi(\chi)\Psi(\psi(\chi))\left(\frac{1}{\int_a^\chi \varpi(\zeta)\Delta_\alpha\zeta} - \frac{1}{\int_a^b \varpi(\zeta)\Delta_\alpha\zeta}\right)\Delta_\alpha\chi, \tag{23}$$

while for $b \to \infty$, it becomes

$$\int_a^\infty \frac{\varpi(\chi)}{\int_a^\chi \varpi(\Im)\Delta_\alpha\Im}\Psi\left(\frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha\Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha\Im}\right)\Delta_\alpha\chi \leq \int_a^\infty \frac{\varpi(\chi)\Psi(\psi(\chi))}{\int_a^\chi \varpi(\zeta)\Delta_\alpha\zeta}\Delta_\alpha\chi. \tag{24}$$

**(2)** If we take $\Psi(u) = u^{\beta}$, where $\beta > 1$ is a constant, then inequalities (23) and (24) are, respectively,

$$\int_a^b \frac{\varpi(\chi)}{\int_a^{\chi} \varpi(\Im)\Delta_\alpha \Im} \left( \frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha \Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha \Im} \right)^{\beta} \Delta_\alpha \chi$$
$$\leq \int_a^b \varpi(\chi)\psi^{\beta}(\chi) \left( \frac{1}{\int_a^{\chi} \varpi(\zeta)\Delta_\alpha \zeta} - \frac{1}{\int_a^{b} \varpi(\zeta)\Delta_\alpha \zeta} \right) \Delta_\alpha \chi, \tag{25}$$

and

$$\int_a^{\infty} \frac{\varpi(\chi)}{\int_a^{\chi} \varpi(\Im)\Delta_\alpha \Im} \left( \frac{\int_a^{\sigma(\chi)} \varpi(\Im)\psi(\Im)\Delta_\alpha \Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha \Im} \right)^{\beta} \Delta_\alpha \chi \leq \int_a^{\infty} \frac{\varpi(\chi)\psi^{\beta}(\chi)}{\int_a^{\chi} \varpi(\zeta)\Delta_\alpha \zeta} \Delta_\alpha \chi. \tag{26}$$

**(3)** If we take $\Psi(u) = \exp(u)$ and replace $\psi$ by $\ln \psi$, then inequalities (23) and (24) are, respectively,

$$\int_a^b \frac{\varpi(\chi)}{\int_a^{\chi} \varpi(\Im)\Delta_\alpha \Im} \exp\left( \frac{\int_a^{\sigma(\chi)} \varpi(\Im)\ln \psi(\Im)\Delta_\alpha \Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha \Im} \right) \Delta_\alpha \chi$$
$$\leq \int_a^b \varpi(\chi)\psi(\chi) \left( \frac{1}{\int_a^{\chi} \varpi(\zeta)\Delta_\alpha \zeta} - \frac{1}{\int_a^{b} \varpi(\zeta)\Delta_\alpha \zeta} \right) \Delta_\alpha \chi$$

and

$$\int_a^{\infty} \frac{\varpi(\chi)}{\int_a^{\chi} \varpi(\Im)\Delta_\alpha \Im} \exp\left( \frac{\int_a^{\sigma(\chi)} \varpi(\Im)\ln \psi(\Im)\Delta_\alpha \Im}{\int_a^{\sigma(\chi)} \varpi(\Im)\Delta_\alpha \Im} \right) \Delta_\alpha \chi \leq \int_a^{\infty} \frac{\varpi(\chi)\psi(\chi)}{\int_a^{\chi} \varpi(\zeta)\Delta_\alpha \zeta} \Delta_\alpha \chi.$$

**(4)** If $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$ in Theorem 23, then inequality (22) reduces to

$$\int_a^b \frac{\varpi(\chi)u(\chi)}{\int_a^{\chi} \varpi(\Im)dt} \Psi \left( \frac{\int_a^{\chi} \varpi(\Im)\psi(\Im)dt}{\int_a^{\chi} \varpi(\Im)dt} \right) d\chi \leq \frac{1}{2} \int_a^b \varpi(\chi)\Psi(\psi(\chi)) \left( \int_{\chi}^{b} \frac{u(\Im)}{\int_a^{\Im} \varpi(\chi)d\chi} dt \right) d\chi.$$

**(5)** If $\mathbb{T} = h\mathbb{Z}$ and $\alpha = 1$ in Theorem 23, then inequality (22) reduces to

$$\sum_{\chi=a}^{b-1} \frac{\varpi(h\chi)u(h\chi)}{\sum_{\xi=a}^{\chi-1} \varpi(h\xi)} \Upsilon \left( \frac{\sum_{\xi=a}^{\chi} \varpi(h\xi)\psi(h\xi)}{\sum_{\xi=a}^{\chi} \varpi(h\xi)} \right) \leq \sum_{\chi=a}^{b-1} \varpi(h\chi)\Upsilon(\psi(h\chi)) \left( \sum_{\xi=\chi}^{b-1} \frac{u(h\xi)}{\sum_{\chi=a}^{\xi} \varpi(h\chi) \sum_{\chi=a}^{\xi-1} \varpi(h\chi)} \right). \tag{27}$$

**(6)** If $\mathbb{T} = \mathbb{Z}$, we simply take in (27), then inequality (22) reduces to

$$\sum_{\chi=a}^{b-1} \frac{\varpi(\chi)u(\chi)}{\sum_{\Im=a}^{\chi-1} \varpi(\Im)} \Psi\left(\frac{\sum_{\Im=a}^{\chi} \varpi(\Im)\psi(\Im)}{\sum_{\Im=a}^{\chi} \varpi(\Im)}\right) \leq \sum_{\chi=a}^{b-1} \varpi(\chi)\Psi(\psi(\chi))\left(\sum_{\Im=\chi}^{b-1} \frac{u(\Im)}{\sum_{\chi=a}^{\Im} \varpi(\chi) \sum_{\chi=a}^{\Im-1} \varpi(\chi)}\right).$$

**Theorem 24.** *Suppose that $a, c \in [0,\infty)_{\mathbb{T}}, \psi \in C_{rd}([a,b)_{\mathbb{T}} \times [c,d)_{\mathbb{T}}, \mathbb{R}), \phi \in C_{rd}([c,d)_{\mathbb{T}}, \mathbb{R}_+)$ and $\varphi \in C_{rd}([a,b)_{\mathbb{T}}, \mathbb{R}_+)$. If $\Psi \in C((\theta,\vartheta), \mathbb{R}_+)$ is convex, then*

$$\begin{aligned}
&\int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^{\chi}\int_c^{\eta} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \Psi\left(\frac{\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}\right)\Delta_\alpha\eta\Delta_\alpha\chi \\
&\leq \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta)) \\
&\quad \times \left(\frac{1}{\int_a^{\zeta} \varphi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_a^{b} \varphi(\tau)\Delta_\alpha\tau}\right)\left(\frac{1}{\int_c^{\Im} \phi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_c^{d} \phi(\tau)\Delta_\alpha\tau}\right)\Delta_\alpha\Im\Delta_\alpha\zeta. \qquad (28)
\end{aligned}$$

**Proof.** Using the two-dimensional dynamic Jensen inequality (21) and Fubini's theorem on time scales, we get

$$\begin{aligned}
&\int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^{\chi}\int_c^{\eta} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \Psi\left(\frac{\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}\right)\Delta_\alpha\eta\Delta_\alpha\chi \\
&\leq \int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta \int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \\
&\quad \times \left(\int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Psi\big(\psi(\Im,\zeta)\big)\Delta_\alpha\Im\Delta_\alpha\zeta\right)\Delta_\alpha\eta\Delta_\alpha\chi \\
&= \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta)) \\
&\quad \times \left(\int_{\zeta}^{b}\int_{\Im}^{d} \frac{\phi(\chi)\varphi(\eta)\Delta_\alpha\eta\Delta_\alpha\chi}{\int_a^{\chi}\int_c^{\eta} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta \int_a^{\sigma(\chi)}\int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}\right)\Delta_\alpha\Im\Delta_\alpha\zeta
\end{aligned}$$

$$= \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta))$$
$$\times \left( \frac{1}{\int_a^\zeta \varphi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_a^b \varphi(\tau)\Delta_\alpha\tau} \right)\left( \frac{1}{\int_c^\Im \phi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_c^d \phi(\tau)\Delta_\alpha\tau} \right)\Delta_\alpha\Im\Delta_\alpha\zeta.$$

This concludes the proof. □

**Remark 3.** *If we put $\phi(\Im) = \varphi(\Im) = 1$, and $\alpha = 1$ in Theorem 24, then we recapture ([20], Theorem 3.2).*

**(1)** In Theorem 24, if we take $\Psi(u) = u^\beta$, where $\beta > 1$ is a constant, then we have

$$\int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^\chi \int_c^\eta \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \left( \frac{\int_a^{\sigma(\chi)} \int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^{\sigma(\chi)} \int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \right)^\beta \Delta_\alpha\eta\Delta_\alpha\chi$$
$$\leq \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\psi^\beta(\Im,\zeta)$$
$$\times \left( \frac{1}{\int_a^\zeta \varphi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_a^b \varphi(\tau)\Delta_\alpha\tau} \right)\left( \frac{1}{\int_c^\Im \phi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_c^d \phi(\tau)\Delta_\alpha\tau} \right)\Delta_\alpha\Im\Delta_\alpha\zeta.$$

**(2)** In Theorem 24, if we take $\Psi(u) = \exp(u)$ and replace $\psi$ by $\ln\psi$, then we have

$$\int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^\chi \int_c^\eta \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \exp\left( \frac{\int_a^{\sigma(\chi)} \int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\ln\psi(\Im,\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta}{\int_a^{\sigma(\chi)} \int_c^{\sigma(\eta)} \phi(\Im)\varphi(\zeta)\Delta_\alpha\Im\Delta_\alpha\zeta} \right) \Delta_\alpha\eta\Delta_\alpha\chi$$
$$\leq \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)\left( \frac{1}{\int_a^\zeta \varphi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_a^b \varphi(\tau)\Delta_\alpha\tau} \right)\left( \frac{1}{\int_c^\Im \phi(\tau)\Delta_\alpha\tau} - \frac{1}{\int_c^d \phi(\tau)\Delta_\alpha\tau} \right)\Delta_\alpha\Im\Delta_\alpha\zeta.$$

**(3)** If $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$ in Theorem 24, then inequality (28) reduces to

$$\int_a^b \int_c^d \frac{\phi(\chi)\varphi(\eta)}{\int_a^\chi \int_c^\eta \phi(\Im)\varphi(\zeta)dtds} \Psi\left( \frac{\int_a^\chi \int_c^\eta \phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)dtds}{\int_a^\chi \int_c^\eta \phi(\Im)\varphi(\zeta)dtds} \right) dydx$$
$$\leq \int_a^b \int_c^d \phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta))\left( \frac{1}{\int_a^\zeta \varphi(\tau)d\tau} - \frac{1}{\int_a^b \varphi(\tau)d\tau} \right)\left( \frac{1}{\int_c^\Im \phi(\tau)d\tau} - \frac{1}{\int_c^d \phi(\tau)d\tau} \right) dtds.$$

**(4)** If $\mathbb{T} = h\mathbb{Z}$ and $\alpha = 1$ in Theorem 24, then inequality (28) reduces to

$$\sum_{\chi=a}^{b-1}\sum_{\eta=c}^{d-1}\frac{\phi(\chi)\varphi(\eta)}{\sum\limits_{\zeta=a}^{\chi-1}\sum\limits_{\Im=c}^{\eta-1}\phi(h\Im)\varphi(h\zeta)}\Psi\left(\frac{\sum\limits_{\zeta=a}^{\chi}\sum\limits_{\Im=c}^{\eta}\phi(h\Im)\varphi(h\zeta)\psi(h\Im,h\zeta)}{\sum\limits_{\zeta=a}^{\chi}\sum\limits_{\Im=c}^{\eta}\phi(h\Im)\varphi(h\zeta)}\right)$$
$$\leq\sum_{\zeta=a}^{b-1}\sum_{\Im=c}^{d-1}\phi(h\Im)\varphi(h\zeta)\Psi(\psi(h\Im,h\zeta))\left(\frac{1}{\sum\limits_{\tau=a}^{\zeta-1}\varphi(h\zeta)}-\frac{1}{\sum\limits_{\tau=a}^{b-1}\varphi(h\zeta)}\right)\left(\frac{1}{\sum\limits_{\tau=c}^{\Im-1}\phi(h\zeta)}-\frac{1}{\sum\limits_{\tau=c}^{d-1}\phi(h\zeta)}\right). \quad (29)$$

**(5)** If $\mathbb{T}=\mathbb{Z}$, we simply take $h=1$ in (29), then inequality (28) reduces to

$$\sum_{\chi=a}^{b-1}\sum_{\eta=c}^{d-1}\frac{\phi(\chi)\varphi(\eta)}{\sum\limits_{\zeta=a}^{\chi-1}\sum\limits_{\Im=c}^{\eta-1}\phi(\Im)\varphi(\zeta)}\Psi\left(\frac{\sum\limits_{\zeta=a}^{\chi}\sum\limits_{\Im=c}^{\eta}\phi(\Im)\varphi(\zeta)\psi(\Im,\zeta)}{\sum\limits_{\zeta=a}^{\chi}\sum\limits_{\Im=c}^{\eta}\phi(\Im)\varphi(\zeta)}\right)$$
$$\leq\sum_{\zeta=a}^{b-1}\sum_{\Im=c}^{d-1}\phi(\Im)\varphi(\zeta)\Psi(\psi(\Im,\zeta))\left(\frac{1}{\sum\limits_{\tau=a}^{\zeta-1}\varphi(\tau)}-\frac{1}{\sum\limits_{\tau=a}^{b-1}\varphi(\tau)}\right)\left(\frac{1}{\sum\limits_{\tau=c}^{\Im-1}\phi(\tau)}-\frac{1}{\sum\limits_{\tau=c}^{d-1}\phi(\tau)}\right).$$

Our aim in the following theorem is to establish a dynamic Hardy inequality for several functions.

**Theorem 25.** *Assume that $a\in[0,\infty)_{\mathbb{T}}$ and $\varpi,\psi_1,\psi_2,\ldots,\psi_n\in C_{rd}([a,\infty)_{\mathbb{T}},\mathbb{R}_+)$. Define $\Lambda(\Im):=\int_a^{\Im}\varpi(\zeta)\Delta_\alpha\zeta$ and $F_k(\Im):=\int_a^{\Im}\varpi(\zeta)\psi_k(\zeta)\Delta_\alpha\zeta$ for $k=1,2,\ldots,n$. If $\vartheta\geq\gamma>1$, then*

$$\int_a^{\infty}\varpi(\Im)\frac{\left(F_1^\sigma(\Im)F_2^\sigma(\Im)\cdots F_n^\sigma(\Im)\right)^{\beta/n}}{(\Lambda^\sigma(\Im))^\gamma}\Delta_\alpha\Im$$
$$\leq\left(\frac{\beta}{nq-n}\right)^\beta\int_a^{\infty}\frac{\varpi(\Im)(\Lambda^\sigma(\Im))^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(\Im)}(\psi_1(\Im)+\psi_2(\Im)+\cdots+\psi_n(\Im))^\beta\Delta_\alpha\Im. \quad (30)$$

**Proof.** Utilizing the discrete Jensen inequality, we have

$$(F_1^\sigma(\Im)F_2^\sigma(\Im)\cdots F_n^\sigma(\Im))^{1/n}\leq\frac{\sum\limits_{k=1}^{n}F_k^\sigma(\Im)}{n}$$

and thus

$$(F_1^\sigma(\Im)F_2^\sigma(\Im)\cdots F_n^\sigma(\Im))^{\beta/n}\leq\frac{\left(\sum\limits_{k=1}^{n}F_k^\sigma(\Im)\right)^\beta}{n^\beta}. \quad (31)$$

Multiplying both sides of (31) by $\varpi(\Im)/(\Lambda^\sigma(\Im))^\gamma$ and integrating the resulting inequality over $\Im$ from $a$ to $\infty$ yield

$$\int_a^{\infty}\varpi(\Im)\frac{\left(F_1^\sigma(\Im)F_2^\sigma(\Im)\cdots F_n^\sigma(\Im)\right)^{\beta/n}}{(\Lambda^\sigma(\Im))^\gamma}\Delta_\alpha\Im\leq\frac{1}{n^\beta}\int_a^{\infty}\varpi(\Im)\frac{\left(\sum\limits_{k=1}^{n}F_k^\sigma(\Im)\right)^\beta}{(\Lambda^\sigma(\Im))^\gamma}\Delta_\alpha\Im.$$

Applying inequality (19) to the right-hand side of the last inequality implies

$$\int_a^\infty \varpi(\Im)\frac{\left(F_1^\sigma(\Im)F_2^\sigma(\Im)\cdots F_n^\sigma(\Im)\right)^{\beta/n}}{(\Lambda^\sigma(\Im))^\gamma}\Delta_\alpha\Im$$
$$\le \left(\frac{\beta}{nq-n}\right)^\beta \int_a^\infty \frac{\varpi(\Im)(\Lambda^\sigma(\Im))^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(\Im)}(\psi_1(\Im)+\psi_2(\Im)+\cdots+\psi_n(\Im))^\beta\Delta_\alpha\Im.$$

The proof is complete. □

**Remark 4.** *If we put $\varpi(\Im)=1$ and $\alpha=1$ in Theorem 25, then we recapture [20] (Theorem 1.4).*

Below, we present various applications of Theorem 25.

**(1)** If $\mathbb{T}=\mathbb{R}$ and $\alpha=1$ in Theorem 25, then inequality (30) reduces to

$$\int_a^\infty \varpi(\Im)\frac{(F_1(\Im)F_2(\Im)\ldots F_n(\Im))^{\beta/n}}{\Lambda^\gamma(\Im)}dt$$
$$\le \left(\frac{\beta}{nq-n}\right)^\beta \int_a^\infty \varpi(\Im)\Lambda^{\beta-\gamma}(\Im)(\psi_1(\Im)+\psi_2(\Im)+\cdots+\psi_n(\Im))^\beta dt,$$

where $\Lambda(\Im):=\int_a^\Im \varpi(\zeta)ds$ and $F_k(\Im)=\int_a^\Im \varpi(\zeta)\psi_k(\zeta)ds$ for $k=1,2,\ldots,n$.

**(2)** If $\mathbb{T}=h\mathbb{Z}$ and $\alpha=1$ in Theorem 25, then inequality (30) reduces to

$$\sum_{\Im=a}^\infty \varpi(h\Im)\frac{(F_1(h\Im+1)F_2(h\Im+1)\cdots F_n(h\Im+1))^{\beta/n}}{\Lambda^\gamma(h\Im+1)} \tag{32}$$
$$\le \left(\frac{\beta}{nq-n}\right)^\beta \sum_{\Im=a}^\infty \frac{\varpi(h\Im)(\Lambda(h\Im+1))^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(h\Im)}(\psi_1(h\Im)+\psi_2(h\Im)+\cdots+\psi_n(h\Im))^\beta, \tag{33}$$

where $\Lambda(h\Im):=h\sum_{\zeta=a}^{\Im-1}\varpi(\zeta)$ and $F_k(h\Im)=h\sum_{\zeta=a}^{\Im-1}\varpi(\zeta)\psi_k(\zeta)$ for $k=1,2,\ldots,n$.

**(3)** If $\mathbb{T}=\mathbb{Z}$, we simply take $h=1$ in (32), then inequality (30) reduces to

$$\sum_{\Im=a}^\infty \varpi(\Im)\frac{(F_1(\Im+1)F_2(\Im+1)\cdots F_n(\Im+1))^{\beta/n}}{\Lambda^\gamma(\Im+1)}$$
$$\le \left(\frac{\beta}{nq-n}\right)^\beta \sum_{\Im=a}^\infty \frac{\varpi(\Im)(\Lambda(\Im+1))^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(\Im)}(\psi_1(\Im)+\psi_2(\Im)+\cdots+\psi_n(\Im))^\beta,$$

where $\Lambda(\Im):=\sum_{\zeta=a}^{\Im-1}\varpi(\zeta)$ and $F_k(\Im)=\sum_{\zeta=a}^{\Im-1}\varpi(\zeta)\psi_k(\zeta)$ for $k=1,2,\ldots,n$.

## 3. Conclusions, Discussions, and Future Work

There are several applications for Hardy type inequalities and they are subject to strong research; see [3,7,15,16]. In this manuscript, by employing the dynamic Jensen's inequality and Fubini's theorem on time scales, we extended a number of $\alpha$-conformable Hardy type inequalities to a general time scale. Several new Hardy type inequalities were proved. The results extend several dynamic inequalities known in the literature, being new even in the discrete and continuous domains. In future work, we will generalize these results by using $\alpha$-conformable fractional calculus.

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