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Photonic Topological Insulator Based on Frustrated Total Internal Reflection in Array of Coupled Prism Resonators

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Abstract: Total internal reflection occurs at the interface between two media with different refractive indices during propagation of light rays from a medium with a higher refractive index to a medium with a lower refractive index. If the thickness of the second medium is comparable to a specific light wavelength, then total internal reflection is violated partially or completely. Based on the frustrated total internal reflection, herein we discuss a two-dimensional photonic topological insulator in an array consisting of triangular, quadrangular, or hexagonal transparent prism resonators with a narrow gap between them. An array of prism resonators allows topologically stable edge solutions (eigenwaves) similar to those studied in ring resonators. Moreover, total internal reflection occurs at different angles of incidence of light. This makes it possible to obtain a set of fundamentally new edge solutions. The light is presumably concentrated at the surface; however, in the new solutions it penetrates relatively deep into the photonic topological insulator and excites several layers of prisms positioned beyond the surface. Remarkably, the direction of light propagation is precisely biased, and therefore new solutions exhibit lower symmetry than the resonator array symmetry.

Keywords: geometric optics; topological insulator; frustrated total internal reflection

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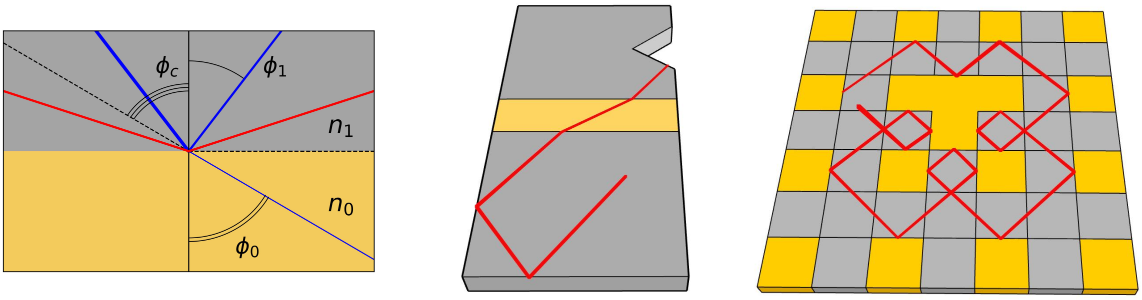
1. Introduction

A photonic topological insulator is a periodic photonic structure that isolates light waves in bulk, but conducts them along the surface (edge) [1,2]. The term topological refers to a special type of stability whereby an arbitrary change of the photonic structure edge does not violate the conduction of light. The role of topology became clear after the discovery of the quantum Hall effect in 1980 in condensed matter physics [3]. The papers [4,5] present theoretical studies of the topology of graphene-like materials. Mathematical aspects of the theory of topological insulators are detailed in [6]. The presence of edge-conducting states is explained using topological invariants in the dispersion diagram of a periodic structure. The formation of band gaps in the frequency spectrum makes the structure an insulator. If the topological invariants on opposite sides of the band gap have different values, then, according to the bulk–edge correspondence, the band gap is closed by means of the states that do not penetrate deep into the structure and are localized at its edge. In gapless insulators, the states in the bulk exist, but do not propagate due to the zero group velocity. This also leads to localization of the conducting states near the structure edge. Perhaps the simplest model of a gapless topological insulator is the Rudner's game. This two-dimensional model was described in [7] in the second quantization language. Owing to its simplicity, this model can also be described in the language of quantum cellular automata [8,9] and is possible in the language of billiards [10]. The model was implemented in arrays of coupled waveguides [11] and ring resonators [12]. Subsequently, an extended theory for ring resonators was developed [13,14].

In this work, we demonstrate the possibility of implementing Rudner's model using the frustrated total internal reflection (FTIR) phenomenon [15,16] (Figure 1a,b) in an array of coupled prism resonators (Figure 1c). It is remarkable that, in ring and prism resonators, the language of light rays in geometrical optics is sufficient to describe the phenomena that are inherently wave-based [12]. Prism resonators, in contrast to ring ones, allow light to travel along the polygonal chain of zero curvature. This difference provides no more than the advantage of simplified perception, while maintaining an identical mathematical structure. Far more important is another difference: total internal reflection occurs at all angles of light incidence greater than the critical angle. This makes it possible to use polygonal lines with several angles and to obtain fundamentally new edge solutions, at which the ray penetrates into deeper edge layers of a photonic topological insulator.



(**a**) (**b**) (**c**)

Figure 1. (**a**) According to the Snell's law, we have $n_1 \sin \phi_1 = n_0 \sin \phi_0$, where n_1 is the refractive index of a medium, e.g., glass (colored in gray), from which a ray falls at angle ϕ_1, and n_0 is the refractive index of a medium, for example, air (colored in orange), into which the ray colored in blue falls, refracting at angle ϕ_0. At $n_0 < n_1$, for angles of incidence greater than the critical angle, i.e., at $\phi > \phi_c = \arcsin n_0/n_1$, the total internal reflection (TIR) effect is observed: the ray colored in red is completely reflected from the interface between the media. (**b**) At an angle of incidence greater than the critical angle, the ray, despite total reflection, can almost completely tunnel from a glass prism to another prism through a thin intermediate layer (air gap). To do this with anti-reflection coating, the gap thickness should be a multiple of half a light wavelength corrected for the angle of incidence [16]. (**c**) Noncontinuous array of glass prisms (coupled prism resonators). The ray propagates through the structure via total internal reflection (FTIR) between the prisms and the TIR at the interfaces with air wells (colored in orange). Black lines represent air gaps between the prisms. It should be noted that the ray trajectory remains qualitatively the same when the air gap has zero thickness.

2. Formulation of the Problem

An FTIR array is a two-dimensional structure of coupled prism resonators with a portion of the prisms removed in a specific and regular (periodic) manner (Figure 1c).

Inside the structure, the light path can be regular or irregular depending on the angle at which the ray penetrates into the FTIR array. For example, Figure 2 shows typical regular polygonal chain paths of the light incident at angles of $\pi/6$, $\pi/4$, and $\pi/3$ in the FTIR insulators with the third-, fourth-, and sixth-order symmetries, respectively. The aforementioned can be formulated in the language of mathematical billiards [10,17,18].

An infinite periodic structure is hereinafter referred to as regular. Let us consider two ways of violating the regularity:

1. Introducing a defect into an infinite FTIR array (creating a hole), Figure 2d.
2. Creating a finite FTIR array by discarding the neighborhood of an infinitely distant point from the structure.

Both methods lead to the appearance of singularities (structure boundaries), on which the topologically stable solutions are sought (Figures 1c and 3a). Let us consider the square FTIR insulators, which, at a reflection angle of $\pi/4$, are equivalent to the Rudner's toy [7].

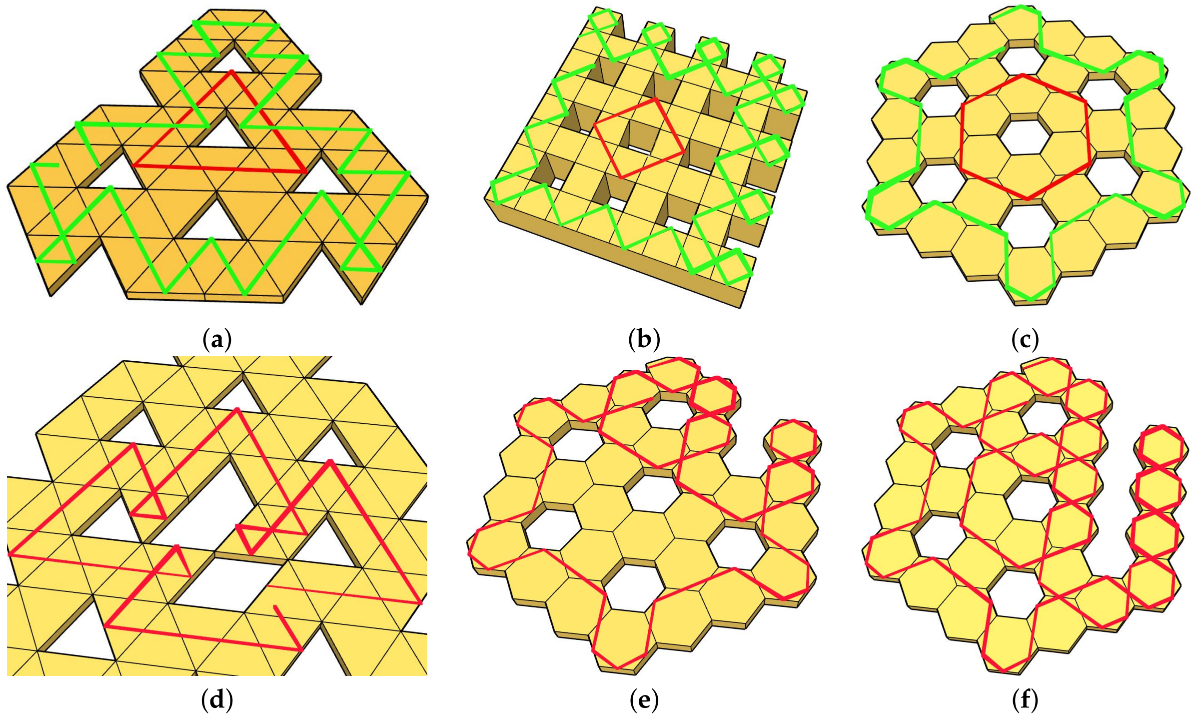


Figure 2. (**a**–**c**) Internal and edge modes on the photonic topological insulators assembled from triangular, quadrangular, and hexagonal glass prisms. (**d**–**f**) Edge modes.

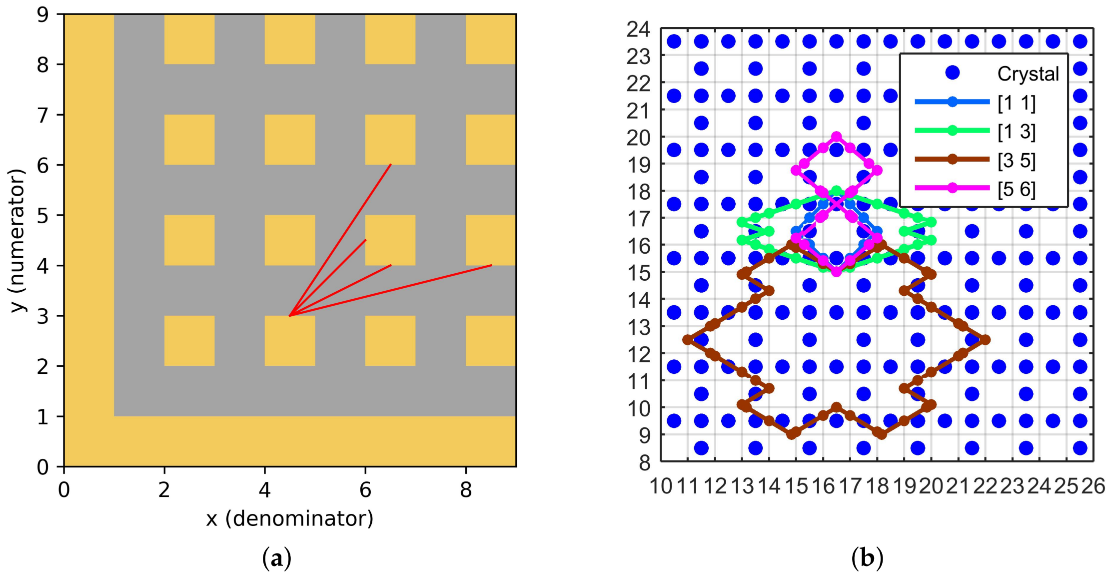


Figure 3. (**a**) Choosing an angle expressed through the rational fraction p/q. Denominator q is plotted on the x-axis and numerator p is plotted on the y-axis from the initial $(4.5, 3)$ position. Red rays correspond to fractions $3/2, 1, 1/2$ and $1/4$. (**b**) Loops (closed light paths) in an infinite resonator array at tangents of angles of incidence of $1, 1/3, 3/5$ and $5/6$. Blue circles correspond to glass prisms and empty white squares correspond to gaps. The symmetry of the loops is described by the second-order dihedral group (D_2), although the array symmetry bears the higher (fourth) order (D_4).

3. Methodology and Limitations

The polygonal chain trajectories of rays in an FTIR insulator are calculated using the MATLAB package. The state of a ray is described by two coordinates of a point at the prism surface and an angle of ray propagation. These three values allow us to calculate the ray state for a point at the next prism surface.

When choosing the initial state we restrict ourselves by rational tangents $\tan\phi = p/q$, where p and q are mutually simple numbers. For rational tangents there are two opportunities. First, the chain is a loop. Second, the chain pattern repeats in the neighboring prism, and the chain is unbounded and periodic. For each initial state, the polygonal chain calculation stops when the state returns to initial (the first case) or translationally the same state (the second case). In both cases, the chain length is of the order of $p + q$. This is why the chain-link calculation process is terminated in finite number of iterations depending on the initial state.

For the ray angle equal to $\pi/4$, the existence of a loop does not depend on the initial position of a point at the prism surface. For some other angles, we have to calculate chains for half of the initial positions along the boundary of the elementary cell.

The suggested model can be realized in the array of precisely positioned prisms. The usage of prisms is limited by surface flatness and high refractive index. Otherwise, the ray scatters on inhomogeneity of the surface and for small angles of incidence the total internal reflection is lost. Both limitations lead to decreased ray brightness for long trajectories. Furthermore, we assume the laser beam has negligible diffraction or it has to be additionally focused inside the resonator array.

4. Results and Discussion

Figure 3a shows several initial links of the polygonal chain trajectories starting in the central point of prism face. The links also finish in the central points. In the theory of mathematical billiards, it can be proved that the polygonal chain trajectories turn out to be repeated not only at $\tan\phi = 1$, but also at all rational tangents [17].

Figure 3b shows sample MATLAB-generated cyclic trajectories with a reflection angle expressed through the rational fraction $p/q = \tan\phi$, where p and q are the arbitrary natural numbers.

Table 1 provides a classification of the polygonal chain trajectories with the rational tangents. The row numbers in the table correspond to the numerator values p, and the column numbers to the denominator values q. Table 1 is filled using the numerical calculations of the edge trajectory for a square array of 59×59 resonators. The array was classified as a conductor (C) for the light propagation directions in which the trajectory went into the bulk of the array. A loop conductor (LC) was classified separately if closed light paths were observed in an infinite array. The array was classified as an insulator (I) if all volumetric solutions turned out to be loops. In this case, the edge solutions either retained the propagation direction (TI) or reversed the direction upon scattering by edge defects (RI). The latter case may be indicative of the existence of higher-order states [19].

Table 2 demonstrates that at $q \leq 13$, insulators were observed only at odd p and q values. Although, loops also appeared at even values. For example, in Figure 3b the loop trajectory is shown for $p = 5$ and $q = 6$.

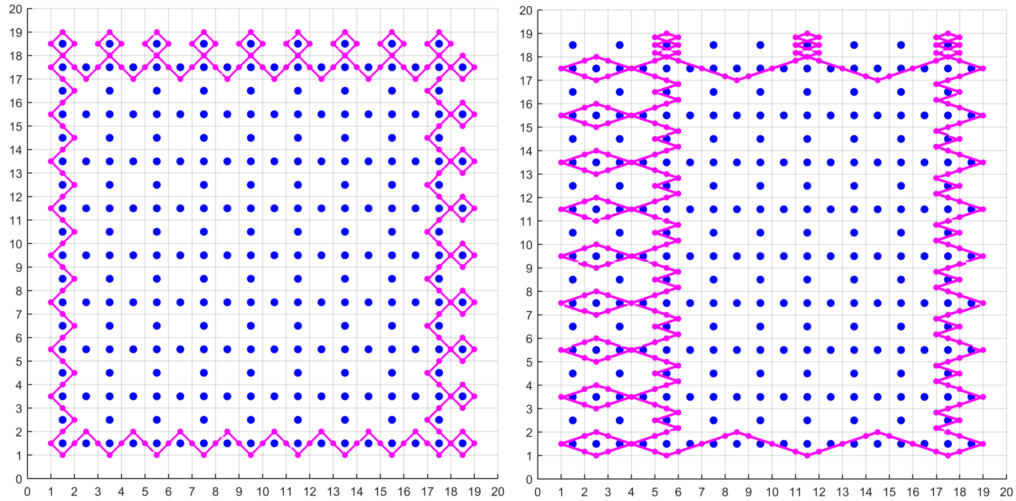
Figure 4 demonstrates two of the simplest rational trajectories $p/q = 1$ (a) and $p/q = 1/3$ (b) yielding topological insulator behavior. The case $p/q = 1/2$ does not produce loops, consequently the trajectory cannot be localized at the edge. The even denominator case $p/q = 5/6$ has an unstable loop (Figure 3b) that depends on the initial point. At the edge, the ray reflection transforms the unstable loop into non-loop trajectory and the ray goes into the bulk without topological insulator behavior.

Table 1. The FTIR insulators were classified using the following designations: topological insulator TI (the light propagates along the edge), conductor C (the trajectory extends to infinity), loop conductor LC (there are trajectories both closed and penetrating deep into the array), and reversible topological insulator RI, in which the edge-wave propagation direction can be reversed. Symbol # denotes the reducible rational fractions accounted for earlier (for lower p and q). The classification table is symmetric with respect to the main diagonal.

p, q	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	TI	C	TI	C	TI	C	TI	C	TI	C	TI	C	TI
2		#	C	#	C	#	C	#	C	#	C	#	C
3			#	C	TI	#	TI	C	#	C	TI	#	TI
4				#	C	#	C	#	C	#	C	#	C
5					#	LC	TI	C	TI	#	TI	C	TI
6						#	LC	#	#	#	C	#	C
7							#	C	TI	C	TI	C	TI
8								#	C	#	C	#	C
9									#	C	RI	#	TI
10										#	C	#	C
11											#	LC	RI
12												#	LC
13													#

Table 2. Topological and reversible FTIR insulators are observed only at odd p and q values taken from Table 1.

p, q	**1**	**3**	**5**	**7**	**9**	**11**	**13**
1	TI	TI	TI	TI	TI	TI	TI
3		#	TI	TI	#	TI	TI
5			#	TI	TI	TI	TI
7				#	TI	TI	TI
9					#	RI	TI
11						#	RI
13							#



(**a**) (**b**)

Figure 4. Finite array of 18 × 18 coupled prism resonators. Blue circles correspond to glass prisms and empty white squares correspond to gaps. For tangents 1 and 1/3 of the angle of incidence, the edge trajectories correspond to a photonic topological insulator. (**a**) Rudner's case $\tan\phi = 1$. (**b**) $\tan\phi = 1/3$, on the left boundary, the trajectory penetrates deep into the insulator extending through five resonator layers and the approximation of the rigid coupling with the nearest neighbors is violated.

5. Conclusions

It was shown that the Rudner's model can be implemented using the frustrated total internal reflection phenomenon in an array of coupled prism resonators (Figure 4a). The inherently wave-like phenomena were described in the language of geometrical optics. The main findings are as follows:

- Beyond Rudner's solutions, fundamentally new edge solutions were found. This new class of solutions is infinite and can be classified by rational numbers, where Rudner's solution is a particular case corresponding to the number 1. Most solutions correspond to ratios with an odd numerator and denominator.
- The ray penetrates into the array crossing several resonator layers. This behavior differs from Rudner's solutions which are completely localized inside the surface layer of resonators. At the same time, the ray cannot penetrate deeper, since only closed trajectories with zero group velocity can occur in an infinite array (the gapless insulator case).
- The new class of trajectories breaks the array symmetry. This behavior is also different to Rudner's solutions that bear the same symmetry as the resonator array.

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Abbreviations

The following abbreviations are used in this manuscript:

FTIR	Frustrated Total Internal Reflection
TI	Topological Insulator
RI	Reversible Topological Insulator
LC	Loop Conductor

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