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Approximation by Operators for the Sheffer–Appell Polynomials

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Abstract: In this paper, we introduce a generalization of the Kantrovich–Stancu-type Szasz operator asymmetry with hybrid families of special polynomials. Additionally, we construct certain positive linear operators together with the Sheffer–Appell polynomial sequences and then obtain the properties of convergence and the order of convergence, which is symmetric to these operators. For applications, we consider certain explicit examples including mixed-type special polynomials.

Keywords: Kanotrovich–Stancu-type generalization of Szasz operators; modulus of continuity; Sheffer–Appell polynomials

MSC: 33C45; 33E20; 41A10; 41A25; 41A36

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1. Introduction and Preliminaries

In recent years, approximation theory has contributed to the development of different computational techniques as it provides a crucial link between pure and applied mathematics. It deals with the process of approximating the functions in the best way with much simpler or amenable functions and methods depending on using recent approximation processes. In this theory, positive approximation techniques play a fundamental role and emerge in a very natural way in many problems relating to the approximation of the continuous functions, especially when one needs further qualitative properties, such as monotonicity, convexity, shape preservation, symmetry, and so on.

The positive estimate processes given by Korovkin [1] play out as an essential technique in order to determine several related practical and symphonic investigations, measure hypotheses, PDEs, and probability hypotheses. In 1953, P. P. Korovkin [1] discovered perhaps the most powerful and, at the same time, the simplest criterion in order to decide whether a given sequence $(K_n)_{n\in\mathbb{N}}$ of positive linear operators on the space $C[0,1]$ is an approximation process, i.e., $K_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$. Following this outcome, many mathematicians have extended Korovkin's theorem to other function spaces or, more generally, to abstract spaces, such as Banach lattices, Banach algebras, Banach spaces, and so on. Korovkin's work, in fact, delineated a new theory that may be called the Korovkin-type approximation theory. The Korovokin-type approximation properties and convergence rates have been inspected by numerous scientists, e.g., see [1–6]. Some interesting contributions to approximation theories can be found in the following studies [7,8].

The Szasz operators [5] are popular examples of positive linear operators.

Szasz [5] proposed the following positive linear operators:

$$S_\eta(\check{f};t) := e^{-\eta t}\sum_{\kappa=0}^{\infty}\frac{(\eta t)^\kappa}{\kappa!}f\left(\frac{\kappa}{\eta}\right), \tag{1}$$

where $t \geq 0$ and $\check{f} \in C[0,\infty)$ once the sum (1) converges. Many researchers have used the generalization of Szasz operators, see for example [3,6,8,9].

In 1880, Appell [10] established and studied η^{th}-degree sequences of polynomials $A_\eta(t),\ \eta = 0,1,2,\cdots$. These polynomials satisfy the recurrence relation

$$\frac{d}{dt}A_\eta(t) = \eta A_{\eta-1}(t),\ \ \eta = 0,1,2,\cdots, \tag{2}$$

and have the generating function as follows:

$$\mathcal{A}(x)e^{tx} = \sum_{\kappa=0}^{\infty} A_\kappa(t)x^\kappa, \tag{3}$$

where

$$\mathcal{A}(x) = \sum_{\kappa=0}^{\infty} \theta_\kappa x^\kappa,\ \ \theta_0 \neq 0. \tag{4}$$

Jakimovski and Leviatan [11], by virtue of the Appell polynomials $A_\kappa(t)$, constructed a generalization of Szasz operators.

Under the following assumptions:

$$\begin{aligned} &(i)\ A_\kappa(t) \geq 0,\ \ t \in [0,\infty);\\ &(ii)\ A(1) \neq 0,\\ &(iii)\ \text{generating function (3) and the power series (4) converge for } |x| < \Re\ (\Re > 1), \end{aligned} \tag{5}$$

The positive linear operators $P_\eta(\check{f};t)$ and their approximation properties are introduced by Jakimovski and Leviatan via

$$P_\eta(\check{f};t) := \frac{e^{-\eta t}}{\mathcal{A}(1)} \sum_{\kappa=0}^{\infty} A_\kappa(\eta t)\check{f}\left(\frac{\kappa}{\eta}\right) \tag{6}$$

After that, by taking the help of Sheffer polynomials $s_\kappa(t)$, Ismail [9] introduced a generalization of Szasz and Jakimovski–Leviatan operators. The Sheffer sequences $s_\kappa(t)$ [12] are an important class of polynomial sequences and appear in various problems of applied mathematics, approximation theory, and multiple other mathematical areas.

The exponential generating function of $s_\kappa(t)$ is given by:

$$A(x)e^{tH(x)} = \sum_{\kappa=0}^{\infty} s_\kappa(t)x^k, \tag{7}$$

where

$$A(x) = \sum_{\kappa=0}^{\infty} a_\kappa x^\kappa,\ \ a_0 \neq 0, \tag{8}$$

$$H(x) = \sum_{\kappa=0}^{\infty} h_\kappa x^\kappa,\ \ h_1 \neq 0. \tag{9}$$

Here we assume the following:

$$\begin{aligned} &(i)\ s_\kappa(t) \geq 0,\ \ t \in [0,\infty);\\ &(ii)\ A(1) \neq 0,\ \ \ H'(1) = 1,\\ &(iii)\ generating\ function\ (7)\ and\ power\ series\ (8)\ and\ (9)\ converge\ for\ |x| < \Re\ (\Re > 1). \end{aligned} \tag{10}$$

The properties of approximation of the positive linear operators investigated by Ismail [9] are given as

$$T_\eta(\check{f};t) := \frac{e^{-\eta t H(1)}}{A(1)} \sum_{\kappa=0}^{\infty} s_\kappa(\eta t)\check{f}\left(\frac{\kappa}{\eta}\right),\ \ \eta \in \gtrsim. \tag{11}$$

Additonally, Kantorovich investigated the approximation properties of positive linear operators defined by [13]

$$K_\eta(\check{f};t) := \eta e^{-\eta t} \sum_{\kappa=0}^{\eta} \frac{(\eta t)^\kappa}{\kappa!} \int_{\kappa/\eta}^{(\kappa+1)/\eta} \check{f}(x)dx. \tag{12}$$

Furthermore, many authors in [6,14–16] studied the approximation properties of the Szasz–Mirakyan–Kantorovich operators and their various propagation.

The polynomials defined in terms of the discrete convolution of known polynomials are applied to analyze new classes of special polynomials. Very recently, Khan and Riyasat [17] introduced and studied the discrete Appell convolution of the Sheffer polynomials $s_k(x)$, called the Sheffer–Appell polynomials. These polynomials are denoted by ${}_sA_\kappa(t)$ and are given with the help of the following generating function:

$$\mathcal{A}(x)A(x)e^{tH(x)} = \sum_{\kappa=0}^{\infty} {}_sA_\kappa(t)x^\kappa, \tag{13}$$

where $\mathcal{A}(x)$, $A(x)$, and $H(x)$ are given by Equation (4), Equation (8), and Equation (9), respectively.

It is important to note that the Sheffer–Appell polynomials are actually the Sheffer polynomials, since their generating function is of the type $A^*(t)e^{xH(t)}$, with a suitable choice for $A^*(t)$. That means $A^*(t)$ is the product of two different functions of t, one of which corresponds to the Appell class, while the other should correspond to the Sheffer class.

In recent years, there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions, see for example [3,6,8,15,16]. Motivated by the works on generalizations of the Szasz operators, our aim is to construct the generalization of Szasz operators involving the Sheffer–Appell polynomials defined by generating function (13) and to study their approximation properties.

Since generating function (13) involves two different functions, $\mathcal{A}(x)$, $A(x)$, and $H(x)$, therefore, we can define a new sequence of Kantrovich–Stancu-type approximation operators. It is important to note that the Sheffer–Appell polynomials are actually the Sheffer polynomials, since their generating function is of the type $A^*(t)e^{xH(t)}$, with a suitable choice for $A^*(t)$. That means $A^*(t)$ is the product of two different functions of t, one of which corresponds to the Appell class, while the other should correspond to the Sheffer class.

The present work is organized as follows. In Section 2, the Kantrovich–Stancu-type positive linear operators together with the Sheffer–Appell polynomials are constructed, and their qualitative and quantitative results are derived. In Section 3, certain examples are given to demonstrate the results given in Section 2 with the help of the members belonging to the Sheffer–Appell polynomials.

2. Construction of Operators $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$ and Their Approximation Properties

In this section, a generalization of Kantrovich–Stancu operators is obtained with the help of Sheffer–Appell polynomials ${}_sA_\kappa(t)$. These operators are generalizations of Szasz operators (1) and Jakimovski–Leviatan operators (6). The convergence properties of these operators will also be established.

The operators are constructed together with the Sheffer–Appell polynomials ${}_sA_\kappa(t)$ defined by Equation (13), under the following restrictions:

$$\begin{aligned}
&(i)\ {}_sA_\kappa(t) \geq 0,\ \kappa = 0,1,2,\cdots;t \in [0,\infty),\\
&(ii)\ \mathcal{A}(1) \neq 0,\ \ A(1) \neq 0,\ \ H'(1) = 1,\\
&(iii)\ \text{generating function (12) and power series (4), (8), and (9) converge for}\\
&\qquad |x| < \Re\ (\Re > 1).
\end{aligned} \tag{14}$$

Now, under assumptions (14), a generalized form of positive linear operators as well as the Sheffer–Appell polynomials ${}_sA_\kappa(t)$ is as follows:

$$\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) = \frac{\eta+\vartheta}{\mathcal{A}(1)A(1)e^{\eta t H(1)}} \sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \check{f}(x)dx, \tag{15}$$

where δ, ϑ are parameters satisfying $0 \leq \delta \leq \vartheta$.

Here, note that for $\delta = \vartheta = 0$, $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$ reduces (12).

Remark 1. *For $H(x) = x$, (13) yields the generating function for the recently introduced 2-iterated Appell polynomials [18]*

$$\mathcal{A}(x)A(x)e^{tx} = \sum_{\kappa=0}^{\infty} A_\kappa^{[2]}(t)x^\eta. \tag{16}$$

Thus, taking $H(x) = x$ in (15) and then denoting the resulting 2-iterated Appell polynomials in the right-hand side by $A_\kappa^{[2]}(\eta t)$, the following positive linear operators are obtained:

$$\mathcal{R}_\eta^{[2](\delta,\vartheta)}(\check{f};t) = \frac{\eta+\vartheta}{\mathcal{A}(1)A(1)e^{\eta t}} \sum_{\kappa=0}^{\infty} A_\kappa^{[2]}(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \check{f}(x)dx. \tag{17}$$

Remark 2. *For $\mathcal{A}(x) = 1$, (13) provides the generating function for the Sheffer polynomials. Therefore, taking $\mathcal{A}(x) = 1$ in (15) we obtain Ismail operators (11), including the Sheffer polynomials.*

Again, taking $A(x) = 1$ and $H(x) = x$ in (15), we obtain Jakimovski and Leviatan operators (6) including the Appell polynomials.

Further, for $\mathcal{A}(x) = A(x) = 1$ and $H(x) = x$, (15) reduces to Szasz operators (1).

Korovkin [1,2] demonstrated some noticeable outcomes concerning the convergence of sequences $(K_\eta(\check{f},t))_{\eta=1}^{\infty}$, where $K_\eta(\check{f},t)$, signify positive linear operators. For example, if $K_\eta(\check{f},t)$ approaches uniformly to $\check{f}$ in the specific cases viz. $\check{f}(x) \equiv 1$, $\check{f}(x) \equiv x$, $\check{f}(x) \equiv x^2$, then, at that point, it does likewise for each continuous real function $\check{f}$. Once more, if $K_\eta(\check{f},t)$ approaches uniformly to $\check{f}$ for specific cases $\check{f}(x) \equiv 1$, $\cos x$, $\sin x$, then, at that point, it does likewise for each continuous 2π periodic real function $\check{f}$. Shisha and Mond in [19] concluded the rate of convergence sequences $K_\eta(\check{f},t)$ in terms of the moduli of continuity of $\check{f}$.

Our purpose is to obtain the theorem of convergence and the order of convergence of $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$ given by (15).

First, let us mention some useful definitions and results:

Definition 1. *For any function $\check{f} \in \hat{C}[0,\infty)$ and $\sigma > 0$, $w(\check{f};\sigma)$ i.e., the modulus of continuity is defined by*

$$w(\check{f};\sigma) := \sup_{\substack{u,v\in[0,\infty) \\ |u-v|\leq\sigma}} |\check{f}(u) - \check{f}(v)|, \tag{18}$$

where the space of uniformly continuous functions is given by $\hat{C}[0,\infty)$. Note that for any $\sigma > 0$ and each $u \in [0,\infty)$, one can write

$$|\check{f}(u) - \check{f}(v)| \leq w(\check{f};\sigma)\left(\frac{|u-v|}{\sigma} + 1\right). \tag{19}$$

Definition 2. *For any function $\check{f} \in C_B[0,\infty)$, the second modulus of continuity is given by*

$$w_2(\check{f};\sigma) := \sup_{0<x\leq\sigma} ||\check{f}(.+2x) - 2\check{f}(.+x) + \check{f}(.)||_{C_B}, \tag{20}$$

where the family of real-valued bounded and uniformly continuous functions on $[0,\infty)$ *is represented by* $C_B[0,\infty)$, *associated with the norm*

$$||\check{f}||_{C_B} = \sup_{t\in[0,\infty)} |\check{f}(t)|. \tag{21}$$

Definition 3. *Let* $\breve{g} \in C_B[0,\infty)$, *then the Peetre's K-functional of* $\breve{g}$ *is given by*

$$K(\breve{g};\sigma) := \inf_{\check{f}\in C_B^2[0,\infty)} \left\{ ||\breve{g}-\check{f}||_{C_B} + \sigma||\check{f}||_{C_B^2} \right\}, \tag{22}$$

where

$$C_B^2[0,\infty) := \{\check{f} \in C_B[0,\infty) : \check{f}', \check{f}'' \in C_B[0,\infty)\} \tag{23}$$

and associated with the norm

$||\check{f}||_{C_B^2} := ||\check{f}||_{C_B} + ||\check{f}'||_{C_B} + ||\check{f}''||_{C_B}$ *(see [20]).*

Additionally, for all $\sigma > 0$, *the following inequality holds:*

$$K(\breve{g};\sigma) \leqslant N\{w_2(\breve{g},\sigma) + min(1,\sigma)||\breve{g}||_{C_B}\}, \tag{24}$$

where N is a constant independent of $\breve{g}$ *and* σ.

Lemma 1 ([7])**.** *Suppose that we have the sequence of positive linear operators* $\check{f} \in C^2[0,a]$ *and* $(K_\eta)_{\eta\geq 0}$ *with the property* $K_\eta(1;t) = 1$. *Then,*

$$|K_\eta(\check{f};t) - \check{f}(t)| \leq ||\check{f}'||\sqrt{K_\eta((r-t)^2;t)} + \frac{1}{2}||\check{f}''||\, K_\eta((r-t)^2;t). \tag{25}$$

Lemma 2 ([21])**.** *Let* $\check{f}_\ell$ *where* $\check{f} \in C[c,d]$ *and* $\ell \in (0, \frac{c-d}{2})$ *be the second-order Steklov function attached to* $\check{f}$. *Then, the inequalities*

$$(a)\ ||\check{f}_\ell - \check{f}|| \leq \frac{3}{4} w_2(\check{f};\ell), \tag{26}$$

$$(b)\ ||\check{f}_\ell''|| \leq \frac{3}{2\ell^2} w_2(\check{f};\ell). \tag{27}$$

hold well.

In order to establish the theorem of convergence and the rate of convergence of $\mathcal{Y}_\eta^{sA}(\check{f},t)$ including the Sheffer–Appell polynomials, the succeeding results are established:

Lemma 3. *The Kantrovich–Stancu-type operators, defined by* (15) *are linear and positive.*

Lemma 4. *For* $t \in [0,\infty)$, *we have the following properties of the operators* $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$

$$\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t) = 1 \tag{28}$$

$$\mathcal{Y}_\eta^{(\delta,\vartheta)}(s;t) = \frac{\eta}{\eta+\vartheta}t + \frac{1}{\eta+\vartheta}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(1)}{A(1)}\right) + \frac{2\delta+1}{2(\eta+\vartheta)} \tag{29}$$

$$\begin{aligned}\mathcal{Y}_\eta^{(\delta,\vartheta)}(s^2;t) = {} & \left(\frac{\eta}{\eta+\vartheta}\right)^2 t^2 + \frac{\eta}{(\eta+\vartheta)^2}\left(2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\frac{A'(1)}{A(1)} + 2(\delta+1) + H''(1)\right)t \\ & + \frac{1}{(\eta+\vartheta)^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{A''(1)}{A(1)} + 2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(1)}{A(1)}\right) + \frac{\delta^2+\delta+\frac{1}{3}}{(\eta+\vartheta)^2} \\ & + 2\frac{(\delta+1)}{(\eta+\vartheta)^2}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(1)}{A(1)}\right).\end{aligned} \tag{30}$$

Proof. From generating function (13), it follows that

$$\begin{aligned}
\sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) &= \mathcal{A}(1)A(1)e^{\eta t H(1)}, && (31)\\
\sum_{\kappa=0}^{\infty} \kappa\, {}_sA_\kappa(\eta t) &= (\mathcal{A}(1)A'(1)+\mathcal{A}'(1)A(1)+\eta t\mathcal{A}(1)A(1))e^{\eta t H(1)}, && (32)\\
\sum_{\kappa=0}^{\infty} \kappa^2 {}_sA_\kappa(\eta t) &= \Big(\eta^2t^2\mathcal{A}(1)A(1)+\eta t\big(H''(1)\mathcal{A}(1)A(1)+2\mathcal{A}'(1)A(1)+2\mathcal{A}(1)A'(1)\\
&\quad +\mathcal{A}(1)A(1)+\mathcal{A}(1)A(1)\big)+\mathcal{A}(1)A''(1)+\mathcal{A}''(1)A(1)+2\mathcal{A}'(1)A'(1)\\
&\quad +\mathcal{A}(1)A'(1)+\mathcal{A}'(1)A(1)\Big)e^{\eta t H(1)}. && (33)
\end{aligned}$$

Using operator Equation (15) and generating function (13) in Equations (28)–(30), assertions (25)–(27) are obtained. □

Lemma 5. *For $t \in [0,\infty)$, the following identities holds for the operators $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$*

$$\begin{aligned}
\mathcal{Y}_\eta^{(\delta,\vartheta)}(s-t;t) &= \frac{-\vartheta}{\eta+\vartheta}t+\frac{1}{\eta+\vartheta}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(1)}{A(1)}\right)+\frac{2\delta+1}{2(\eta+\vartheta)} && (34)\\
\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t) &= \left(\frac{-\vartheta}{n+\vartheta}\right)^2t^2+\frac{\eta}{(n+\vartheta)^2}\left\{-2\vartheta\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(1)}{A(1)}\right)+2(\delta+1)+H''(1)\right.\\
&\quad \left. -(\delta+1/2)\left(\frac{\eta+\vartheta}{\eta}\right)\right\}t+2\frac{(\delta+1)}{(n+\vartheta)^2}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(1)}{A(1)}\right)\\
&\quad +\frac{1}{(\eta+\vartheta)^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\frac{A''(1)}{A(1)}+2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(1)}{A(1)}\right)+\frac{\delta^2+\delta+\frac{1}{3}}{(\eta+\vartheta)^2}. && (35)
\end{aligned}$$

Proof. In view of the linearity property of $\mathcal{Y}_\eta^{(\delta,\vartheta)}$, it follows that

$$\begin{aligned}
\mathcal{Y}_\eta^{(\delta,\vartheta)}(s-t;t) &= \mathcal{Y}_\eta^{(\delta,\vartheta)}(s;t)-t\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t) && (36)\\
\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t) &= \mathcal{Y}_\eta^{(\delta,\vartheta)}(s^2;t)-2t\mathcal{Y}_\eta^{(\delta,\vartheta)}(s;t)+t^2\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t), && (37)
\end{aligned}$$

which on applying Lemma 4, yields assertions (36) and (37), respectively. □

We denote the set of all continuous functions by $C_E[0,\infty)$, such that $|\check{f}(t)| \leq \gamma\rho^t$ for all $t \geq 0$ and for some positive finite ρ and γ.

The theorem of convergence for the operators $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f},t)$ is obtained by proving the following result:

Theorem 1. *Let $\check{f} \in C_E[0,\infty)$. Then*

$$\lim_{n\to\infty} \mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f},t) = \check{f}(t) \qquad (38)$$

uniformly for every compact subset of $[0,\infty)$.

Proof. With the aid of Lemma 3, we get

$$\lim_{\eta\to\infty} \mathcal{Y}_\eta^{(\delta,\vartheta)}(s^i,t) = t^i, \quad i=0,1,2 \qquad (39)$$

uniformly for every compact subset of $[0,\infty)$. Applying Korovkin's theorem to (36) proves the desired result. □

Next, we determine the convergence rate of $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f},t)$ by using the modulus of continuity in the form of the following result:

Theorem 2. *Suppose $\check{f} \in \hat{C}_E[0,\infty)$. Then the following inequality*

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leqslant 2w(\check{f};\sqrt{\mu_\eta(t)}, \tag{40}$$

holds for $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f},t)$. Here,

$$\mu_\eta(x) := (\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t). \tag{41}$$

Proof. Consider

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - f(x)| = \left| \frac{\eta+\vartheta}{\mathcal{A}(1)A(1)e^{\eta t H(1)}} \sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left(f\left(\frac{\kappa}{\eta}\right) - \check{f}(t) \right) dt \right|$$

Using the fact that $\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t) = 1$ and in view of (15), we have

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leqslant \frac{\eta+\vartheta}{\mathcal{A}(1)A(1)e^{\eta t H(1)}} \sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \check{f}\left(\frac{\kappa}{\eta}\right) - \check{f}(t) \right| dt,$$

which on using Equation (19) gives

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leqslant \left(1 + \frac{\eta+\vartheta}{\mathcal{A}(1)A(1)e^{\eta t H(1)}} \frac{1}{\sigma} \sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right| \right) w(\check{f};\delta)\, dt. \tag{42}$$

By considering the Cauchy-Schwarz inequality for integration, we find

$$\int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right| dt \leqslant \sqrt{\frac{1}{\eta+\vartheta}} \left\{ \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right|^2 \right\}^{\frac{1}{2}}, \tag{43}$$

from this, it follows that

$$\sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right| dt \leqslant \sqrt{\frac{1}{\eta+\vartheta}} \sum_{\kappa=0}^{\infty} {}_sA_\kappa(nt) \left\{ \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right|^2 \right\}^{\frac{1}{2}}. \tag{44}$$

Using Cauchy-Schwarz inequality for summation on the right-hand side of (44), we can write

$$\sum_{\kappa=0}^{\infty} {}_sA_\kappa(\eta t) \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} \left| \frac{\kappa}{\eta} - t \right| dt \leqslant \sqrt{\frac{\mathcal{A}(1)A(1)e^{\eta t H(1)}}{\eta+\vartheta}} \left\{ \frac{\mathcal{A}(1)A(1)e^{\eta t H(1)}}{\eta+\vartheta} \left(\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t) \right)^2 \right\}^{\frac{1}{2}}. \tag{45}$$

From the inequality in (42), we find that

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leqslant \left\{ 1 + \frac{1}{\sigma}\sqrt{\mu_\eta(t)} \right\} w(\check{f};\delta). \tag{46}$$

By taking $\sigma = \sqrt{\mu_\eta(t)}$ in above equation, the assertion in Equation (40) is established. □

Theorem 3. *For $\check{f} \in C[0,\delta]$, the succeeding inequality holds:*

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leq \frac{2}{\delta} ||\check{f}||\, l^2 + \frac{3}{4}(\delta + 2 + l^2) w_2(\check{f};l), \tag{47}$$

where

$$l := l_\eta(t) = \left\{ \mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t) \right\}^{\frac{1}{4}}$$

and $w_2(\check{f};l)$ *is the second order modulus of continuity with norm* $||\check{f}|| = \max_{t\in[a,b]} |\check{f}(t)|$

Proof. Let $\check{f}_l$ be the second-order Steklov function of the function f. So, in view of (28), we have

$$\begin{aligned} |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| &\leq |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f} - \check{f}_l;t)| + |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)| + |\check{f}_l(t) - \check{f}(t)|, \\ &\leq 2||\check{f}_l - \check{f}|| + |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)|, \end{aligned} \tag{48}$$

on using (26), (48) becomes

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leq \frac{3}{2} w_2(\check{f};l) + |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)|. \tag{49}$$

Keeping in view that $\check{f}_l \in C^2[0,\delta]$, from Lemma 2, it follows that

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)| \leq ||\check{f}_l'|| \sqrt{\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t)} + \frac{1}{2} \, ||\check{f}_l''|| \, \mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t), \tag{50}$$

now on on using (27), (50) becomes

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)| \leq ||\check{f}_l'|| \sqrt{\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t)} + \frac{3}{4l^2} w_2(\check{f};l) \, \mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t). \tag{51}$$

Further, the Landau inequality

$$||\check{f}_l'|| \leq \frac{2}{\delta} ||\check{f}_l|| + \frac{\delta}{2} ||\check{f}_l''||,$$

combined with inequality (27) gives

$$||\check{f}_l'|| \leq \frac{2}{\delta} ||\check{f}|| + \frac{3\delta}{4l^2} w_2(\check{f};l). \tag{52}$$

Substituting (52) in (53) and taking $l = \sqrt[4]{\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2);t}$, we find

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}_l;t) - \check{f}_l(t)| \leq \frac{2}{\delta} ||\check{f}|| \, l^2 + \frac{3}{4}(\delta + l^2) w_2(\check{f};l). \tag{53}$$

Making use of (48) in (44), we arrive at (42). □

Theorem 4. *Let* $\check{f} \in C_B^2[0,\infty)$. *If* $\mathcal{Y}_\eta^{(\delta,\vartheta)}$ *is defined by* (15), *then one has*

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) - \check{f}(t)| \leqslant \zeta ||v||_{C_B^2}, \tag{54}$$

where

$$\begin{aligned}\zeta &= \zeta_\eta(t)\\ &= \Bigg[\left(\frac{\vartheta}{\eta+\vartheta}\right)^2 t^2 + \frac{\eta}{(\eta+\vartheta)^2}\Bigg\{2\delta+2+H''(1)-2\vartheta\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(1)}{A(1)}\right)\\ &- \Big(\delta+1/2+\eta\vartheta\Big)(\eta+\vartheta)\Bigg\}t + \frac{2\delta+2+\eta+\vartheta}{(\eta+\vartheta)^2}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(1)}{A(1)}\right)+\frac{2\delta+1}{2(\eta+\vartheta)}\\ &+ \frac{1}{(\eta+\vartheta)^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\frac{A''(1)}{A(1)}+2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(1)}{A(1)}\right)+\frac{\delta^2+\delta+\frac{1}{3}}{(\eta+\vartheta)^2}\Bigg]||\check{f}||_{C_B^2}.\end{aligned} \tag{55}$$

Proof. In view of the linearity property of $\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)$ and $\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t)=1$, the Taylor's expansion of function $\check{f}$ can be written as:

$$\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t) = f(x)\mathcal{Y}_\eta^{(\delta,\vartheta)}(1;t)+\check{f}'(t)\mathcal{Y}_\eta^{(\delta,\vartheta)}(s-t;t)+\frac{1}{2}\check{f}''(\xi)\mathcal{Y}_\eta^{(\delta,\vartheta)}((s-t)^2;t),\ \xi\in(t,s). \tag{56}$$

From Lemma 5 $\mathcal{Y}_\eta^{(\delta,\vartheta)}(s-t;t)\geq 0$ for $s\geq t$. Thus by inserting expressions (36) and (37) in (56) and after simplification we get the required result. □

Theorem 5. *If $\check{f}$ is a function such that $\check{f}\in C_B[0,\infty)$, then*

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)-\check{f}(t)| \leqslant 2N\{w_2(\check{f};\sqrt{\sigma})+min(1,\sigma)||\check{f}||_{C_B}\}, \tag{57}$$

where

$$\sigma := \sigma_\eta(t) = \frac{1}{2}\zeta_\eta(t), \tag{58}$$

where $N\geq 0$ is a constant and is independent of $\check{f}$, δ. Additionally, $\zeta_\eta(t)$ is given in Theorem 4.

Proof. If $\check{g}\in C_B^2[0,\infty)$, then from Theorem 4, we have

$$\begin{aligned}|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)-\check{f}(t)| &\leqslant |\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f}-\check{g};t)|+|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{g};t)-\check{g}(t)|+|\check{g}(t)-\check{f}(t)|\\ &\leqslant 2||\check{f}-\check{g}||_{C_B}+\zeta||\check{g}||_{C_B^2} = 2\Big[||\check{f}-\check{g}||_{C_B}+\delta||\check{g}||_{C_B^2}\Big]\end{aligned} \tag{59}$$

Since the left-hand side of (59) is independent of the function $\check{g}\in C_B^2[0,\infty)$,

$$|\mathcal{Y}_\eta^{(\delta,\vartheta)}(\check{f};t)-\check{f}(t)| \leqslant 2K(\check{f};\delta), \tag{60}$$

where K is Peetre's functional defined by (22). Now, by using (24) in (59), (57) holds. □

Remark 3. *In Theorems 3–5, l_n, ξ_n, $\delta_n\to 0$ when $n\to\infty$.*

In the next section, we consider certain examples in support of the above-derived results.

3. Examples

We establish the positive linear operators including certain members of the Sheffer–Appell family by considering the following examples.

Example 1. *The truncated exponential polynomials $e_\kappa(t)$ [22] are the Appell polynomials for $\mathcal{A}(x)=\frac{1}{1-x}$ and are defined by the generating function:*

$$\frac{1}{(1-x)}e^{tx} = \sum_{\kappa=0}^{\infty} e_\kappa(t)x^\kappa. \tag{61}$$

These polynomials have many applications in optics and quantum mechanics and also perform a key role in the evaluation of integrals having products of special functions.

The Laguerre polynomials $L_\kappa(t)$ [23] are essential members of the Sheffer family for $A(x) = \frac{1}{1-x}$ and $H(x) = \frac{-x}{1-x}$ and are defined by the following generating function:

$$\frac{1}{(1-x)} e^{\left(\frac{-tx}{1-x}\right)} = \sum_{\kappa=0}^{\infty} L_\kappa(t) x^\kappa. \tag{62}$$

The Laguerre polynomials arise in the quantum mechanics of the Morse potential and of the 3D isotropic consonant oscillator, in an outspread piece of the arrangement of the Schrödinger condition for a one-electron iota.

Taking $\mathcal{A}(x) = \frac{1}{1-x}$ of the truncated exponential polynomials and $A(x) = \frac{1}{1-x}$; $H(x) = -\frac{x}{1-x}$ of Laguerre polynomials in generating function (12), the following generating function of the Laguerre-truncated exponential polynomials ${}_Le_\kappa(t)$ is obtained:

$$\frac{1}{(1-x)^2} e^{\left(\frac{-tx}{1-x}\right)} = \sum_{\kappa=0}^{\infty} {}_Le_\kappa(t) x^\kappa. \tag{63}$$

For ensuring restrictions (14), generating function (63) is modified by replacing $x \to \frac{x}{2}$ and $t \to -\frac{t}{2}$ as follows:

$$\frac{1}{(1-\frac{x}{2})^2} e^{\left(\frac{tx}{2(2-x)}\right)} = \sum_{\kappa=0}^{\infty} {}_Le_\kappa(-t/2) \frac{x^\kappa}{2^\kappa}, \tag{64}$$

which yields the following explicit representation of ${}_Le_\kappa(-t/2)$:

$${}_Le_\kappa(-t/2) = \sum_{l,s=0}^{l+s\leq\kappa} \frac{(l+1)(\kappa-l-s)_s\, t^{\kappa-l-s}}{2^{\kappa-l-s}(\kappa-l-s)!s!}. \tag{65}$$

From Equation (65), it follows that

$${}_Le_\kappa(-t/2) \geq 0 \text{ for all } t \in [0,\infty).$$

In view of generating function (64), the positive linear operators including the Laguerre-truncated exponential polynomials are constructed as follows:

$$\mathcal{Y}_\eta^{Le(\delta,\vartheta)}(\check{f};t) = \frac{1}{4} e^{-\eta t/2} \sum_{\kappa=0}^{\infty} \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} {}_Le_\kappa(-\eta t/2) \check{f}\left(\frac{\kappa}{\eta}\right) dt. \tag{66}$$

Example 2. *The polynomials denoted by $g_\kappa^{d+1}(t;h)$ [24] are called Gould–Hopper d-orthogonal polynomial sets [25,26] of Hermite type [14]. For $\mathcal{A}(x) = e^{hx^{d+1}}$ these polynomials are the Appell polynomials given by the generating relation:*

$$e^{hx^{d+1}} e^{tx} = \sum_{\kappa=0}^{\infty} g_\kappa^{d+1}(x;h) x^\kappa. \tag{67}$$

Taking $\mathcal{A}(x) = e^{hx^{d+1}}$ of the Gould–Hopper polynomials and $A(x) = \frac{1}{1-x}$; $H(x) = -\frac{x}{1-x}$ of Laguerre polynomials in generating function (12), the following generating function for the Laguerre–Gould–Hopper polynomials ${}_Lg_\kappa^{d+1}(t;h)$ is obtained:

$$\frac{1}{(1-x)} e^{hx^{d+1}} e^{\left(\frac{-tx}{1-x}\right)} = \sum_{\kappa=0}^{\infty} {}_Lg_\kappa^{d+1}(t;h) x^k. \tag{68}$$

For ensuring restrictions (14), generating function (68) is modified by replacing $t \to \frac{x}{2}$ and $t \to -\frac{t}{2}$ as follows:

$$\frac{1}{(1-\frac{x}{2})} e^{h(\frac{x}{2})^{d+1}} e^{\left(\frac{tx}{2(2-x)}\right)} = \sum_{\kappa=0}^{\infty} {}_L g_\kappa^{d+1}(-t/2;h) \frac{x^\kappa}{2^\kappa}. \tag{69}$$

From Equation (69), the following explicit representation of ${}_L g_\kappa^{d+1}(-t/2;h)$ is obtained:

$${}_L g_\kappa^{d+1}(-t/2;h) = \sum_{l,s,v=0}^{l+(d+1)s+v\leq\kappa} \frac{h^s\ (\kappa-l-(d+1)s-v)_v\ t^{\kappa-l-(d+1)s-v}}{2^{\kappa-l-(d+1)s-v}(\kappa-l-(d+1)s-v)!s!v!}. \tag{70}$$

Again, from Equation (70), it follows that

$${}_L g_\kappa^{d+1}(-t/2;h) \geq 0 \text{ for all } t \in [0,\infty).$$

In view of generating function (69), the positive linear operators together with the Laguerre–Gould–Hopper polynomials are constructed as follows:

$$\mathcal{Y}_\eta^{{}_L g^{d+1}(\delta,\vartheta)}(\check{f};t) = \frac{1}{2} e^{-\frac{1}{2}\left(\eta t + \frac{h}{2^d}\right)} \sum_{\kappa=0}^{\infty} \int_{\frac{\kappa+\delta}{\eta+\vartheta}}^{\frac{\kappa+\delta+1}{\eta+\vartheta}} {}_L e_\kappa(-\eta t/2) \check{f}\left(\frac{\kappa}{\eta}\right) dt. \tag{71}$$

The methodology adopted above can be extended to find the positive linear operators including other members of the Sheffer–Appell family, provided these polynomials obey restrictions (14).

4. Concluding Remarks

In this article, the positive linear operators together with the Sheffer–Appell polynomials are introduced. The convergence theorem and rate of convergence of these operators are also established. The Sheffer–Appell family includes a large number of hybrid-type polynomials as its members. Some examples are also provided to give the importance of the operators including the Sheffer–Appell polynomials.

The error estimation for the approximation with the operators including members of the Sheffer–Appell family can be explored. The approximation of any continuous function $\check{f}(t)$ by positive linear operators $\mathcal{Y}_\eta^{{}_s A}(\check{f};t)$ can also be shown graphically. Finding the Kantrovich, Durrmeyer, and Kantrovich–Stancu-type generalizations of the operators including Sheffer–Appell polynomials will be taken in a forthcoming investigation. This article is a first attempt in the direction of finding generalizations of the Szasz operators involving hybrid-type families of special functions.

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