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Uncertainty Principles for the Two-Sided Quaternion Windowed Quadratic-Phase Fourier Transform

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Abstract: A recent addition to the class of integral transforms is the quaternion quadratic-phase Fourier transform (Q-QPFT), which generalizes various signal and image processing tools. However, this transform is insufficient for addressing the quadratic-phase spectrum of non-stationary signals in the quaternion domain. To address this problem, we, in this paper, study the (two sided) quaternion windowed quadratic-phase Fourier transform (QWQPFT) and investigate the uncertainty principles associated with the QWQPFT. We first propose the definition of QWQPFT and establish its relation with quaternion Fourier transform (QFT); then, we investigate several properties of QWQPFT which includes inversion and the Plancherel theorem. Moreover, we study different kinds of uncertainty principles for QWQPFT such as Hardy's uncertainty principle, Beurling's uncertainty principle, Donoho–Stark's uncertainty principle, the logarithmic uncertainty principle, the local uncertainty principle, and Pitt's inequality.

Keywords: quaternion quadratic-phase Fourier transform; Inversion; Plancherel theorem; uncertainty principle; Donoho–Stark

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1. Introduction

Due to its applications in signal representation, image processing, and quantum mechanics, the theory of parametric time-frequency analysis has attracted the attention in the last few decades [1,2]. The windowed Fourier transform, linear canonical transform, fractional Fourier transform, and Wigner distributions are some well known parametric time-frequency analysis tools.

The quadratic-phase Fourier transform (QPFT), which is the neoteric and most important parametric time-frequency analysis tool introduced by Castro et al. [3], treats both the stationary and non-stationary signals in a simple and insightful way. In the quadratic-phase domain, most of the signals arising in communications like sonar and radar reveal their characteristics better. With a slight modification in [3], the authors in [4] defined the QPFT as

$$\mathbb{Q}_\mu[f](w) = \int_{\mathbb{R}} f(x)\mathcal{Q}_\mu(x,w)dx \tag{1}$$

where $\mathcal{Q}_\mu(x,w)$ is a quadratic-phase kernel represented by

$$\mathcal{Q}_\mu(x,w) = \sqrt{\frac{bi}{2\pi}}e^{-i\left(ax^2+bxw+cw^2+dx+ew\right)} \tag{2}$$

where $a,b,c,d,e \in \mathbb{R}$, $b \neq 0$ are the arbitrary real parameters and have great importance. These can be better used in the analysis of non-transient signals that are involved in radar and other communication systems. With its global kernel and extra degrees of freedom, the QPFT has become one of the efficient tools in solving several problems. To quote some of these, we have science and engineering branches including harmonic analysis, image processing, sampling, reproducing kernel Hilbert spaces, and many more [5]. Although QPFT is the generalization of well known integral transformations, it is inadequate in localizing the quadratic-phase spectrum content of the non-stationary signals. Various authors have come to rescue these shortcomings like Shah et al., who introduced short-time quadratic phase Fourier transform [6], whereas Bhat and Dar [7] studied quadratic phase wave packet transform (QPWPT), wherein they studied the properties of QPWPT and established some of its uncertainty principles (UP).

In the quaternion setting, the generalization of integral transforms from real and complex numbers is the need for the study of higher dimensions like: the quaternion Fourier transform (QFT) [8], the quaternion windowed Fourier transform (QWFT) [9], the quaternion linear canonical transform (QLCT) [10], the fractional quaternion Fourier transform (Fr-QFT) [11], and the quaternion offset linear canonical transform (QOLCT) [12]. Over time, quaternion algebra has proven to be a hot area of research with its applications in image filtering, color image processing, and many more [13,14]. The quaternion Fourier transform (QFT) and its generalizations play a great role in the representation of hypercomplex signals in signal and image processing.

On the other hand, the uncertainty principle (UP) first proposed by German physicist W. Heisenberg in 1927 plays a great role in numerous scientific fields such as quantum physics, mathematics, signal processing, and information theory [15,16]. The UPs like Heisenberg's, Hardy's, and Beurling's related to QFT are discussed in [17–20], and the further extension of UPs in the spectrums of QLCT and QOLCT is discussed in [21–24]. These UPs have numerous applications in the study of optical systems, signal recovery, and many more [25,26]. Recently, Gupta and Verma introduced short-time quadratic phase Fourier transform in quaternion setting and studied some of its associated UPs. Later on, Bhat and Dar [27] introduced quaternion quadratic phase Fourier transform and generalized it to the Gabor quaternion quadratic phase Fourier transform besides studying logarithmic UP and Heisenberg's UP. Thus, there is a need to study the other types of uncertainty principles in a windowed quaternion quadratic phase domain. Thus, motivated by this, we in this paper propose the novel integral transform coined as the two-sided quaternion windowed quadratic-phase Fourier transform (QWQPFT), which provides a unified treatment for several existing classes of signal processing tools. Therefore, it is worthwhile to rigorously study the QWQPFT and associated UPs which can be productive for signal processing theory and applications.

1.1. Paper Contributions

The contributions of this paper are summarized below:

- We introduce the novel integral transform coined as the two-sided quaternion windowed quadratic-phase Fourier transform (QWQPFT);
- We establish the basic relationship between the proposed transform (QWQPFT), the quaternion Fourier transform (QFT) and quaternion quadratic phase Fourier transform (Q-QPFT);
- To study the fundamental properties of the QWQPFT, like the inversion formula, Plancherel formula, and boundedness;
- To examine several classes of uncertainty principles, such as the Hardy's UP, Beurling's UP, Donoho–Stark's UP, the logarithmic UP, and the local UP associated with the proposed transform;
- We explore Pitt's Inequality associated with the QWQPFT.

1.2. Paper Outlines

The paper is organized as follows: In Section 2, we give a brief review of two-sided QFT useful and Q-QPFT, useful in the succeeding sections. In Section 3, we introduce the quaternion windowed quadratic phase Fourier transform and study some of its properties.

In Section 4, we establish some different forms of uncertainty principles (UPs) for the QWQPFT, which includes Hardy's UP, Beurling's UP, Donoho–Stark's UP, the logarithmic UP, the local UP, and Pitt's inequality. Finally, conclusions are drawn in Section 5.

2. Preliminary

In this section, we give a brief review to the two-sided QFT useful and Q-QPFT, which will be needed throughout the paper.

2.1. Quadratic-Phase Fourier Transform

In this subsection, we recall the fundamentals of quadratic-phase Fourier transform (QPFT).

Definition 1 (QPFT [3,4])**.** *For any real parameter set $\mu = (a, b, c, d, e)$, the QPFT of $f \in L^2(\mathbb{R})$ is denoted by $\mathbb{Q}_\mu[f]$ and defined as*

$$\mathbb{Q}_\mu[f](w) = \int_{\mathbb{R}} f(x)\mathcal{Q}_\mu(x, w)dx \tag{3}$$

where $\mathcal{Q}_\mu(x, w)$ is a quadratic-phase kernel and is given by

$$\mathcal{Q}_\mu(x, w) = \sqrt{\frac{bi}{2\pi}}e^{-i\left(ax^2+bxw+cw^2+dx+ew\right)}. \tag{4}$$

The inversion and Parseval's formula for the QPFT are given by

$$f(x) = \int_{\mathbb{R}} \mathbb{Q}_\mu[f](w)\overline{\mathcal{Q}_\mu(x, w)}dw, \tag{5}$$

$$\langle f, g\rangle = \langle \mathbb{Q}_\mu[f], \mathbb{Q}_\mu[g]\rangle, \quad \forall f, g \in L^2(\mathbb{R}). \tag{6}$$

Theorem 1 (QPFT Plancherel [3,4])**.** *For any signal $f \in L^2(\mathbb{R})$, we have*

$$\|f\|^2_{L^2(\mathbb{R})} = \|\mathbb{Q}_\mu[f]\|^2_{L^2(\mathbb{R})}.$$

2.2. Quaternion Algebra

In 1834, W. R. Hamilton introduced quaternion algebra by extension of the complex number to an associative non-commutative 4D algebra, denoted by $\mathbb{H}$ in his honor where every element of $\mathbb{H}$ has a Cartesian form given by

$$\mathbb{H} = \{q|q := [q]_0 + i[q]_1 + j[q]_2 + k[q]_3, [q]_i \in \mathbb{R}, i = 0, 1, 2, 3\} \tag{7}$$

where i, j, k are imaginary units obeying Hamilton's multiplication rules:

$$i^2 = j^2 = k^2 = -1, \tag{8}$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j. \tag{9}$$

Let $[q]_0$ and $q = i[q]_1 + j[q]_2 + k[q]_3$ denote the real scalar part and the vector part of quaternion number $q = [q]_0 + \mathbf{i}[q]_1 + j[q]_2 + k[q]_3$, respectively. Then, the real scalar part has a cyclic multiplication symmetry

$$[pql]_0 = [qlp]_0 = [lpq]_0, \qquad \forall q, p, l \in \mathbb{H}, \tag{10}$$

the conjugate of a quaternion q is defined by $\overline{q} = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3$, and the norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}. \tag{11}$$

It is easy to verify that

$$\overline{pq} = \overline{q}\overline{p}, \qquad |qp| = |q||p|, \quad \forall q, p \in \mathbb{H}. \tag{12}$$

In this paper, we will study the quaternion-valued signal $f : \mathbb{R}^2 \to \mathbb{H}$, f, which can be expressed as $f = f_0 + if_1 + jf_2 + kf_3$, with $f_m \; : \mathbb{R}^2 \to \; \mathbb{R}$ for $m = 0, 1, 2, 3$. The quaternion inner product for quaternion valued signals $f, g \; : \mathbb{R}^2 \; \to \mathbb{H}$, as follows:

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} \tag{13}$$

where $\mathbf{x} = (x_1, x_2)$, $f(\mathbf{x}) = f(x_1, x_2)$, $\mathbf{x} = dx_1dx_2$, and so on.

Hence, the natural norm is given by

$$|f|_2 = \sqrt{< f, f >} = (\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x})^{\frac{1}{2}} \tag{14}$$

and the quaternion module $L^2(\mathbb{R}^2, \; \mathbb{H})$, is given by

$$L^2(\mathbb{R}^2, \; \mathbb{H}) = \{f : \mathbb{R}^2 \; \to \mathbb{H}, \; |f|_2 < \infty\}. \tag{15}$$

We now define the space of rapidly decreasing smooth quaternion function [10].

Definition 2. *For a multi-index $\beta = (\beta_1, \beta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the Schwartz space in $L^2(\mathbb{R}^2, \mathbb{H})$, is defined as*

$$\mathcal{S}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in \mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H});] \sup_{x \in \mathbb{R}^2} \left(1 + |x|^k\right) \left| \frac{\partial^{\beta_1+\beta_2}[f(x)]}{\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}} \right| < \infty \right\}$$

where $\mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H})$ is the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{H}$.

2.3. Quaternion Fourier Transform

Let us begin this part with the QFT. There are three different types of QFT: the left-sided QFT, the two-sided QFT, and the right-sided QFT. Here, our focus will be on two-sided QFT (in the rest of the paper, QFT means two-sided QFT).

Definition 3 (QFT [9])**.** *The two-sided QFT of a quaternion signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is defined by*

$$\mathcal{F}^{\mathbb{H}}[f](\mathbf{w}) = \int_{\mathbb{R}^2} e^{-ix_1w_1} f(\mathbf{x}) e^{-jx_2w_2} d\mathbf{x} \tag{16}$$

and corresponding inverse QFT is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_1w_1} \mathcal{F}^{\mathbb{H}}[f](\mathbf{w}) e^{jx_2w_2} d\mathbf{w} \tag{17}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_1, w_2)$.

Lemma 1 (QFT Parseval [8])**.** *The quaternion product of $f, g \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ and its QFT are related by*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \left\langle \mathcal{F}^{\mathbb{H}}[f], \mathcal{F}^{\mathbb{H}}[g] \right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{18}$$

In particular, if $f = g$, we obtain the quaternion version of the Plancherel formula; that is,

$$\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} = \left\| \mathcal{F}^{\mathbb{H}}[f] \right\|_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{19}$$

Lemma 2 ([24]). *If $1 \le p \le 2$ and letting $\frac{1}{p} + \frac{1}{q} = 1$, for all $f \in L^p(\mathbb{R}^2, \mathbb{H})$, then it holds*

$$\|\mathcal{F}^{\mathbb{H}}\|_q \le (2\pi)^{\frac{1}{q} - \frac{1}{p}} \|f\|_p. \tag{20}$$

2.4. Two-Sided Q-QPFT

In this subsection, we study the two-sided Q-QPFT (for simplicity of notation, we write the Q-QPFT instead of the two-sided Q-QPFT). We recall the definition of Q-QPFT and some of its properties.

Definition 4 (Q-QPFT [4,27]). *Let $\mu_s = (a_s, b_s, c_s, d_s, e_s)$ for $s = 1, 2$; then, the two-sided Q-QPFT of signals $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is denoted by $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]$ and defined as*

$$\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w}) = \int_{\mathbb{R}^2} \mathcal{Q}^i_{\mu_1}(x_1, w_1) f(\mathbf{x}) \mathcal{Q}^j_{\mu_2}(x_2, w_2) d\mathbf{x} \tag{21}$$

where $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathcal{Q}^i_{\mu_1}(x_1, w_1)$ and $\mathcal{Q}^j_{\mu_2}(x_2, w_2)$ are quaternion kernel signals given by

$$\mathcal{Q}^i_{\mu_1}(x_1, w_1) = \sqrt{\frac{b_1 i}{2\pi}} e^{-i\left(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1\right)} \tag{22}$$

$$\mathcal{Q}^j_{\mu_2}(x_2, w_2) = \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left(a_2 x_2^2 + b_2 x_2 w_2 + c_2 w_2^2 + d_2 x_2 + e_2 w_2\right)} \tag{23}$$

where $a_s, b_s, c_s, d_s, e_s \in \mathbb{R}, b_s \neq 0$, and $s = 1, 2$.

Under some suitable conditions, the Q-QPFT above is invertible, and the inversion is given in the following Lemma.

Lemma 3 (Q-QPFT Inversion [4,27]). *Let $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f] \in L^1(\mathbb{R}^2, \mathbb{H})$, then every signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$ can be reconstructed back by the formula*

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} \overline{\mathcal{Q}^i_{\mu_1}(x_1, w_1)} \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](w) \overline{\mathcal{Q}^j_{\mu_2}(x_2, w_2)} d\mathbf{w}. \tag{24}$$

Theorem 2 (Q-QPFT Plancherel [4,27]). *For any signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} = \left\|\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]\right\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{25}$$

3. Quaternionic Windowed Quadratic-Phase Fourier Transform

In this section, we shall formally introduce the notion of the two-sided quaternionic windowed quadratic-phase Fourier transform (QWQPFT) and then establish some properties of the proposed transform.

Definition 5 (QWQPFT). *Let $\mu_s = (a_s, b_s, c_s, d_s, e_s)$, be a matrix parameter such that a_s, b_s, c_s, d_s, $e_s \in \mathbb{R}$, $b_s \neq 0$, for $s = 1, 2$. The two-sided quaternion windowed quadratic-phase Fourier transform of any quaternion valued signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$, with respect window function $\Xi \in L^2(\mathbb{R}^2, \mathbb{H})$ given by*

$$\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w}, \mathbf{u}) = \int_{\mathbb{R}^2} \mathcal{Q}^i_{\mu_1}(x_1, w_1)\, f(\mathbf{x}) \overline{\Xi(\mathbf{x} - \mathbf{u})}\, \mathcal{Q}^j_{\mu_2}(x_2, w_2) d\mathbf{x} \tag{26}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{w} = (w_1, w_2)$, $\mathbf{u} = (u_1, u_2)$, the quaternion kernels $\mathcal{Q}^i_{\mu_1}(x_1, w_1)$, and $\mathcal{Q}^j_{\mu_2}(x_2, w_2)$ are given by Equations (22) and (23), respectively.

Remark 1. *By appropriately choosing parameters in $\mu_s = (a_s, b_s, c_s, d_s, e_s), s = 1,2$, the QWQPFT (26) includes many well-known linear transforms as special cases:*

- *For $\mu_s = (0, -1, 0, 0, 0), s = 1,2$, the QWQPFT (26) boils down to the Quaternion Windowed Fourier Transform [9].*
- *As a special case, when $\mu_s = (a_s, b_s, c_s, 0, 0), s = 1,2$, the QWQPFT (26) can be viewed as the Quaternion Windowed Linear Canonical Transform [28].*
- *For $\mu_s = (\cot\theta, -\csc\theta, \cot\theta, 0, 0), s = 1,2$, the QWQPFT (26) leads to the two-sided Quaternion Fractional Fourier Transform [29].*

Remark 2. *For fixed $\mathbf{u}$, we can see that the relationship between the quaternion windowed quadratic-phase Fourier transform and the quaternion quadratic-phase Fourier transform is given by,*

$$\begin{aligned} \mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) &= \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}\left[f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}\right](\mathbf{w}) \\ &= \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}\left[f^{\Xi}_u(\mathbf{x})\right](\mathbf{w}) \end{aligned} \tag{27}$$

where $f^{\Xi}_u(\mathbf{x})$ is a modified signal.

Now, we give the relationship between the proposed QWQPFT and the QFT.

Theorem 3. *The QWQPFT (26) of a quaternion signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ can be reduced to the QFT (16) as*

$$\begin{aligned} &\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) \\ &\quad = \sqrt{\frac{b_1 i}{2\pi}} e^{-i\left(c_1 w_1^2 + e_1 w_1\right)} \mathcal{F}^{\mathbb{H}}[F](\mathbf{bw},\mathbf{u}) \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left(c_2 w_2^2 + e_2 w_2\right)} \end{aligned} \tag{28}$$

where

$$F(\mathbf{x}) = e^{-i\left(a_1 x_1^2 + d_1 x_1\right)} f^{\Xi}_u(\mathbf{x}) e^{-j\left(a_2 x_2^2 + d_2 x_2\right)} \tag{29}$$

and $\mathbf{b} = (b_1, b_2)$.

Proof. From Definition 5, we obtain

$$\begin{aligned} &\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) \\ &\quad = \int_{\mathbb{R}^2} \sqrt{\frac{b_1 i}{2\pi}} e^{-i\left(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1\right)} f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})} \\ &\quad \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left(a_2 x_2^2 + b_2 x_2 w_2 + c_2 w_2^2 + d_2 x_2 + e_2 w_2\right)} d\mathbf{x} \\ &\quad = \sqrt{\frac{b_1 i}{2\pi}} e^{-i\left(c_1 w_1^2 + e_1 w_1\right)} \left\{ \int_{\mathbb{R}^2} e^{-i x_1 b_1 w_1} \right. \\ &\quad \times \left. \left(e^{-i\left(a_1 x_1^2 + d_1 x_1\right)} f^{\Xi}_u(\mathbf{x}) e^{-j\left(a_2 x_2^2 + d_2 x_2\right)} \right) e^{-j x_2 b_2 w_2} d\mathbf{x} \right\} \\ &\quad \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left(c_2 w_2^2 + e_2 w_2\right)}. \end{aligned}$$

Setting $F(\mathbf{x}) = e^{-i\left(a_1 x_1^2 + d_1 x_1\right)} f^{\Xi}_u(\mathbf{x}) e^{-j\left(a_2 x_2^2 + d_2 x_2\right)}$, we have from the above equation

$$\begin{aligned} &\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) \\ &\quad = \sqrt{\frac{b_1 i}{2\pi}} e^{-i\left(c_1 w_1^2 + e_1 w_1\right)} \mathcal{F}^{\mathbb{H}}[F](\mathbf{bw},\mathbf{u}) \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left(c_2 w_2^2 + e_2 w_2\right)} \end{aligned}$$

where $\mathbf{bw} = (b_1 w_1, b_2 w_2)$. This leads to the desired result. □

Prior to establishing the vital properties of the proposed QWQPFT, we present an explicit example for lucid illustration of the proposed Definition 5:

Example 1. *Consider a 2D Gaussian quaternionic function of the form $f(\mathbf{x}) = \exp\{-(k_1x_1^2 + k_2x_2^2)\}$, for k_1, k_2 are both positive real constants.*
The QWQPFT of a f with respect to the rectangular window function

$$\Xi(\mathbf{x}) = \begin{cases} 1, & \text{if } |x_1| < \lambda, |x_2| < \lambda, \lambda > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

is given by

$$\begin{aligned}
&\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) \\
&= \frac{\sqrt{b_1b_2}}{2\pi}\int_{\mathbb{R}^2} \mathcal{Q}^i_{\mu_1}(x_1,w_1) f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}\mathcal{Q}^j_{\mu_2}(x_2,w_2)d\mathbf{x} \\
&= \frac{\sqrt{b_1b_2}}{2\pi}\int_{u_1-\lambda}^{u_1+\lambda}\int_{u_2-\lambda}^{u_2+\lambda} e^{-i(a_1x_1^2+b_1x_1w_1+c_1w_1^2+d_1x_1+e_1w_1)-k_1x_1^2} \\
&\times e^{-j(a_2x_2^2+b_2x_2w_2+c_2w_2^2+d_2x_2+e_2w_2)-k_2x_2^2}d\mathbf{x} \\
&= \frac{\sqrt{b_1b_2}}{2\pi}e^{-i(c_1w_1^2+e_1w_1)}\int_{u_1-\lambda}^{u_1+1/2} e^{-i((a_1-ik_1)x_1^2+b_1x_1w_1+d_1x_1}dx_1 \\
&\times \int_{u_2-\lambda}^{u_2+\lambda} e^{-j((a_2-jk_2)x_2^2+b_2x_2w_2+d_2x_2}dx_2 \times e^{-j(c_2w_2^2+e_2w_2)}.
\end{aligned} \tag{30}$$

For simplicity, we choose $k_1 = -ia_1$ and $k_2 = -ja_2$, and we obtain from (30)

$$\begin{aligned}
&\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) \\
&= \frac{\sqrt{b_1b_2}}{2\pi}e^{-i(c_1w_1^2+e_1w_1)}\int_{u_1-\lambda}^{u_1+\lambda} e^{-i(b_1w_1+d_1)x_1}dx_1 \\
&\times \int_{u_2-\lambda}^{u_2+\lambda} e^{-j(b_2w_2+d_2)x_2}dx_2 \times e^{-j(c_2w_2^2+e_2w_2)} \\
&= \frac{\sqrt{b_1b_2}e^{-i(b_1w_1u_1+c_1w_1^2+Du_1+e_1w_1)}}{2\pi(b_1w_1+d_1)}\left(e^{-i(b_1w_1+d_1)\lambda} - e^{i(b_1w_1+d_1)\lambda}\right) \\
&\times \left(e^{-j(b_2w_2+d_2)\lambda} - e^{j(b_2w_2+d_2)\lambda}\right)\frac{e^{-j(b_2w_2u_2+c_2w_2^2+Du_2+e_2w_2)}}{(b_2w_2+d_2)}.
\end{aligned}$$

Properties of QWQPFT

In this subsection, we study some properties of the proposed QWQPFT which are useful for signal processing. Some of these have been proved in [27], but we have made a slight modification in the definition of QWQPFT so these properties will change accordingly.

Theorem 4. *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be two quaternion signals and Ξ_1, Ξ_2 be the non zero window functions; then, the QWQPFT satisfies the following properties:*

1. *Linearity:*

$$\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[\alpha f + \beta g](\mathbf{w},\mathbf{u}) = \alpha\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1.\mu_2}[f](\mathbf{w},\mathbf{u}) + \beta\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1.\mu_2}[g](\mathbf{w},\mathbf{u}) \tag{31}$$

   where α and β are in $\mathbb{C}$.

2. *Boundedness*

$$\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1.\mu_2}\right| \leq \frac{|b_1b_2|^{1/2}}{2\pi}\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{32}$$

3. *Anti-linearity:*

$$\mathcal{V}^{\mathbb{H}}_{\alpha\Xi_1+\beta\Xi_2,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u}) = \mathcal{V}^{\mathbb{H}}_{\Xi_1,\mu_1\cdot\mu_2}[f](\mathbf{w},\mathbf{u})\bar{\alpha} + \mathcal{V}^{\mathbb{H}}_{\Xi_2,\mu_1\cdot\mu_2}[f](\mathbf{w},\mathbf{u})\bar{\beta} \tag{33}$$

where α and β are in $\mathbb{C}$.

Proof. It follows from Definition 5 [or see [27]]. □

Theorem 5 (Inversion formula). *Let $\Xi \in L^2(\mathbb{R}^2,\mathbb{H})$ be a quaternion window function; then, every quaternion signal $f \in L^2(\mathbb{R}^2,\mathbb{H})$ can be recovered back from the transformed signal $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})$ by the following formula:*

$$f(\mathbf{x}) = \frac{1}{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \times \overline{\mathcal{Q}^i_{\mu_1}(x_1,w_1)}\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\overline{\mathcal{Q}^j_{\mu_2}(x_2,w_2)}\Xi(\mathbf{x}-\mathbf{u})d\mathbf{w}d\mathbf{u}. \tag{34}$$

Proof. Applying the Inverse QQPFT to (27), we obtain

$$\begin{aligned} f^{\Xi}_u(\mathbf{x}) &= f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})} \\ &= \mathbb{Q}^{-1}_{\mu_1,\mu_2}\left[\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right] \\ &= \int_{\mathbb{R}^2}\overline{\mathcal{Q}^i_{\mu_1}(x_1,w_1)}\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\overline{\mathcal{Q}^j_{\mu_2}(x_2,w_2)}d\mathbf{w}. \end{aligned} \tag{35}$$

Multiplying the above equation both sides from right by $\Xi(\mathbf{x}-\mathbf{u})$ and integrating with respect to du, we obtain

$$\begin{aligned} &f(\mathbf{x})\int_{\mathbb{R}^2}|\Xi(\mathbf{x}-\mathbf{u})|^2du \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\mathcal{Q}^i_{\mu_1}(x_1,w_1)}\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\overline{\mathcal{Q}^j_{\mu_2}(x_2,w_2)}\Xi(\mathbf{x}-\mathbf{u})d\mathbf{w}d\mathbf{u}. \end{aligned}$$

Equivalently, we have

$$f(\mathbf{x}) = \frac{1}{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \times \overline{\mathcal{Q}^i_{\mu_1}(x_1,w_1)}\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\overline{\mathcal{Q}^j_{\mu_2}(x_2,w_2)}\Xi(\mathbf{x}-\mathbf{u})d\mathbf{w}d\mathbf{u}$$

which completes the proof. □

Theorem 6 (QWQPFT Plancherel). *Let $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})$ be the quaternion windowed quadratic-phase Fourier transform of a signal $f \in L^2(\mathbb{R}^2,\mathbb{H})$ with respect to a window function function $\Xi \in L^2(\mathbb{R}^2,\mathbb{H})$, then we have*

$$\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\mathbb{R}^4,\mathbb{H})} = \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{36}$$

Proof. From Remark 2, we have

$$\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]\right\|^2_{L^2(\mathbb{R}^4,\mathbb{H})} = \left\|\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}\left[f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}\right]\right\|^2_{L^2(\mathbb{R}^4,\mathbb{H})}. \tag{37}$$

Applying Theorem 2 to the R.H.S of (37) yields

$$\begin{aligned}\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\mathbb{R}^4,\mathbb{H})} &= \left\|f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}(\mathbf{w})\right\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|f(\mathbf{x})|^2|\Xi(\mathbf{x}-\mathbf{u})|^2 d\mathbf{x}d\mathbf{u} \\ &= \int_{\mathbb{R}^2}|f(\mathbf{x})|^2 d\mathbf{x}\int_{\mathbb{R}^2}|\Xi(\mathbf{y})|^2 d\mathbf{y} \\ &= \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\end{aligned}$$

where we have applied Fubini's theorem in the second to last equation. This completes the proof. □

4. Uncertainty Principles Associated with the Quaternion-QPFT

In this section, we investigate some different forms of UPs associated with QWQPFT including Hardy's UP, Beurling's UP, logarithmic UPs, Donoho–Stark's UP, and Local UP. Let us begin with Hardy's uncertainty principle for the quaternion quadratic-phase Fourier transform (21). We first recall Hardy's uncertainty principle for the QFT.

Lemma 4 (Hardy's UP for the two-sided QFT [17])**.** *Let α and β be positive constants. For $f \in L^2(\mathbb{R}^2,\mathbb{H})$, if*

$$|f(\mathbf{x})| \le ce^{-\alpha|\mathbf{x}|^2} \quad \text{and} \quad |\mathcal{F}^{\mathbb{H}}[f](\mathbf{w})| \le c'e^{-\beta|\mathbf{w}|^2}, \quad \mathbf{u},\mathbf{w}\in\mathbb{R}^2,$$

with some positive constants c, c'. Then, the following three cases can occur:

(1) if $\alpha\beta > \frac{1}{4}$, then $f(\mathbf{x}) \equiv 0$;

(2) if $\alpha\beta = \frac{1}{4}$, then $f(\mathbf{x}) = Ke^{-\alpha|\mathbf{x}|^2}$, for any constant K;

(3) if $\alpha\beta < \frac{1}{4}$, then there are many infinite such functions $f(\mathbf{x})$.

Motivated and inspired by Hardy's UP for the two-sided QFT, we establish Hardy's UP for the Q-QPFT.

Theorem 7 (Hardy's UP for the QWQPFT)**.** *Let α, β be positive constants and $\Xi \in L^2(\mathbb{R}^2,\mathbb{H})$ be a non zero window function. Then, for any signal $f \in L^2(\mathbb{R}^2,\mathbb{H})$ satisfying*

$$|f(\mathbf{x})| \le Ce^{-\alpha|\mathbf{x}|^2}, \quad \text{and} \quad \left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u})\right| \le C'e^{-\beta|\mathbf{w}|^2}, \quad \mathbf{u},\mathbf{w}\in\mathbb{R}^2,$$

with some positive constants C, C', the following three cases can occur:

(1) if $\alpha\beta > \frac{1}{4}$, then $f(\mathbf{x}) \equiv 0$;

(2) if $\alpha\beta = \frac{1}{4}$, then $f(\mathbf{x}) = e^{i(a_1x_1^2+d_1x_1)}\frac{K}{|\overline{\Xi(0)}|}e^{-\alpha|\mathbf{x}|^2}e^{j(a_2x_2^2+d_2x_2)}$, for any constant K;

(3) if $\alpha\beta < \frac{1}{4}$, then there are many infinite such functions $f(\mathbf{x})$.

Proof. Taking $\mathbf{x} = \mathbf{u}$ in (29), we obtain

$$F(\mathbf{x}) = e^{-i(a_1x_1^2+d_1x_1)}f(\mathbf{x})\overline{\Xi(0)}e^{-j(a_2x_2^2+d_2x_2)}.$$

Clearly, $F(\mathbf{x}) \in L^2(\mathbb{R}^2,\mathbb{H})$ and $|\overline{\Xi(0)}|$ is a positive quantity; therefore,

$$\begin{aligned}|F(\mathbf{x})| &= |f(\mathbf{x})||\overline{\Xi(0)}| \\ &\le |\overline{\Xi(0)}|Ce^{-\alpha|\mathbf{x}|^2} \\ &= C_1e^{-\alpha|\mathbf{x}|^2}.\end{aligned}$$

From (28), we obtain

$$\begin{aligned} |\mathcal{F}^{\mathbb{H}}[F](\mathbf{w})| &= \frac{2\pi}{\sqrt{b_1 b_2}}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u})\right| \\ &\leq \frac{2\pi}{\sqrt{b_1 b_2}} C' e^{-\beta|\mathbf{w}|^2} \\ &= C_1' e^{-\beta|\mathbf{w}|^2} \end{aligned}$$

where $C_1 = |\overline{\Xi(0)}|C$ and $C_1' = \frac{2\pi}{\sqrt{b_1 b_2}}C'$.
Thus, by Lemma 4, the following three cases can occur:
(1) if $\alpha\beta > \frac{1}{4}$, then $F(\mathbf{x}) \equiv 0$;
(2) if $\alpha\beta = \frac{1}{4}$, then $F(\mathbf{x}) = Ke^{-\alpha|\mathbf{x}|^2}$, for any constant K;
(3) if $\alpha\beta < \frac{1}{4}$, then there are many infinite such functions $F(\mathbf{x})$.
Equivalently, we have the following conclusions:
(1) if $\alpha\beta > \frac{1}{4}$, then $f(\mathbf{x}) \equiv 0$ for $F(\mathbf{x}) \equiv 0$;
(2) if $\alpha\beta = \frac{1}{4}$, it yields $f(\mathbf{x}) = e^{i\left(a_1 x_1^2 + d_1 x_1\right)} \frac{K}{|\overline{\Xi(0)}|} e^{-\alpha|\mathbf{x}|^2} e^{j\left(a_2 x_2^2 + d_2 x_2\right)}$, where K is a constant;
(3) if $\alpha\beta < \frac{1}{4}$, then it is clear that there are many infinite such functions $f(\mathbf{x})$,
which completes the proof. □

Now, using the relationship between the proposed transform (Q-QPFT) and QFT, we obtain Beurling's uncertainty principle for the Q-QPFT. First, we recall the Beurling's uncertainty principle for the QFT.

Lemma 5 (Beurling's UP for the two-sided QFT [18]). *Let $f(\mathbf{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$ and $d \geq 0$ such that*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|f(\mathbf{x})|\left|\mathcal{F}^{\mathbb{H}}[f](\mathbf{w})\right|}{(1 + |\mathbf{x}| + |\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|} d\mathbf{x} d\mathbf{w} < \infty, \tag{38}$$

then $f(\mathbf{x}) = P(\mathbf{x})e^{-k|\mathbf{x}|^2}$, where $k > 0$ and P is a polynomial of degree $< \frac{d-2}{2}$. In particular, $f = 0$ when $d \leq 2$.

By applying Theorem 3 and Lemma 5, we extend the validity of Beurling's UP for the QWQPFT.

Theorem 8 (Beurling's UP for the QWQPFT). *Let $f(\mathbf{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$ and $d \geq 0$ satisfying*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|f(\mathbf{x})||\overline{\Xi(\mathbf{x}-\mathbf{u})}|\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|}{(1 + |\mathbf{x}| + |\mathbf{b}\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{b}\mathbf{w}|} d\mathbf{x} d\mathbf{w} < \infty,$$

then $f(\mathbf{x}) = e^{i\left(a_1 x_1^2 + d_1 x_1\right)} \frac{P(\mathbf{x})}{\overline{\Xi(\mathbf{x}-\mathbf{u})}} e^{-k|\mathbf{x}|^2} e^{j\left(a_2 x_2^2 + d_2 x_2\right)}$, where $k > 0$ and $P(\mathbf{x})$ is a polynomial of degree $< \frac{d-2}{2}$. In particular, $f = 0$ when $d \leq 2$.

Proof. Taking $f = h(\mathbf{x}, \mathbf{u})$ given in (29), we have from (38)

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|F(\mathbf{x})||\mathcal{F}^{\mathbb{H}}[F](\mathbf{w})|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|} d\mathbf{x}d\mathbf{w} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|f_u^{\Xi}(\mathbf{x})|\left|\frac{2\pi}{\sqrt{b_1b_2}}\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|} d\mathbf{x}d\mathbf{w} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{2\pi}{\sqrt{b_1b_2}} \frac{|f_u^{\Xi}(\mathbf{x})|\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|} d\mathbf{x}d\mathbf{w} \\
&= 2\pi\sqrt{b_1b_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|f(\mathbf{x})||\Xi(\mathbf{x}-\mathbf{u})|\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|}{(1+|\mathbf{x}|+|\mathbf{bw}|)^d} e^{|\mathbf{x}||\mathbf{bw}|} d\mathbf{x}d\mathbf{w} < \infty.
\end{aligned}$$

By Lemma 5, we must have $F(\mathbf{x}) = P(\mathbf{x})e^{-k|\mathbf{x}|^2}$.

Since $F(\mathbf{x}) = e^{-i\left(a_1x_1^2+d_1x_1\right)} f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})} e^{-j\left(a_2x_2^2+d_2x_2\right)}$, which implies $f(\mathbf{x}) = e^{i\left(a_1x_1^2+d_1x_1\right)} \frac{P(\mathbf{x})}{\overline{\Xi(\mathbf{x}-\mathbf{u})}} e^{-k|\mathbf{x}|^2} e^{j\left(a_2x_2^2+d_2x_2\right)}$. In particular, $f = 0$ on account $F(\mathbf{x}) = 0$ when $d \le 2$, which completes the proof. □

In continuation, we establish Donoho–Stark's uncertainty principle for the QWQPFT by considering the relationship between the proposed transform (QWQPFT) and QFT. Let us begin with the definition.

Definition 6 ([30])**.** *A quaternion function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is said to be $\varepsilon-$concentrated on a measurable set $L \subseteq \mathbb{R}^2$, if*

$$\left(\int_{\mathbb{R}^2\setminus L} |f(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} \le \varepsilon \|f\|_2.$$

Lemma 6 (Donoho–Stark's UP for the two-sided QFT [30,31])**.** *Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with $f \neq 0$ be $\varepsilon_{L_1}-$concentrated on $L_1 \subseteq \mathbb{R}^2$ and $\mathcal{F}^{\mathbb{H}}[f]$ be $\varepsilon_{L_2}-$concentrated on $L_2 \subseteq \mathbb{R}^2$. Then,*

$$|L_1||L_2| \ge 2\pi(1-\varepsilon_{L_1}-\varepsilon_{L_2})^2.$$

Theorem 9 (Donoho–Stark's UP for the QWQPFT)**.** *Assuming that non-zero quaternion function $\mathbb{Q}^{-1}_{\mu_1,\mu_2}\left[\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]\right]$ in $L^2(\mathbb{R}^2, \mathbb{H})$ is a $\varepsilon_{L_1}-$concentrated on $L_1 \subseteq \mathbb{R}^2$ and $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]$ is $\varepsilon_{L_2}-$concentrated on $L_2 \subseteq \mathbb{R}^2$. Then,*

$$|L_1||L_2| \ge 2\pi|\mathbf{b}|(1-\varepsilon_{L_1}-\varepsilon_{L_2})^2. \tag{39}$$

Proof. From (29), we have

$$F(\mathbf{x}) = e^{-i\left(a_1x_1^2+d_1x_1\right)} f_u^{\Xi}(\mathbf{x}) e^{-j\left(a_2x_2^2+d_2x_2\right)}. \tag{40}$$

Inserting (35) in (40), we obtain

$$F(\mathbf{x}) = e^{-i\left(a_1x_1^2+d_1x_1\right)} \mathbb{Q}^{-1}_{\mu_1,\mu_2}\left[\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]\right](\mathbf{x},\mathbf{u}) e^{-j\left(a_2x_2^2+d_2x_2\right)}. \tag{41}$$

Since $\mathbb{Q}^{-1}_{\mu_1,\mu_2}\left[\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]\right] \in L^2(\mathbb{R}^2, \mathbb{H})$ is $\varepsilon_{L_1}-$concentrated on $L_1 \subseteq \mathbb{R}^2$, it implies $F(\mathbf{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$ is $\varepsilon_{L_1}-$concentrated on $L_1 \subseteq \mathbb{R}^2$.
On the other hand, we have that $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]$ is $\varepsilon_{L_2}-$concentrated on $L_2 \subseteq \mathbb{R}^2$; thus, by (28), we obtain $\mathcal{F}^{\mathbb{H}}(\mathbf{bw},\mathbf{u})$ is $\varepsilon_{L_2}-$concentrated on $L_2 \subseteq \mathbb{R}^2$, hence $\mathcal{F}^{\mathbb{H}}(\mathbf{w},\mathbf{u})$ is $\varepsilon_{L_2}-$concentrated on $\frac{L_2}{\mathbf{b}} \subseteq \mathbb{R}^2$.

Hence, by Lemma 6, we obtain

$$|L_1|\left|\frac{L_2}{\mathbf{b}}\right| \geq 2\pi(1-\varepsilon_{L_1}-\varepsilon_{L_2})^2,$$

which completes the proof. □

Based on the relation with Quaternion Quadratic-phase Fourier, the logarithmic uncertainty principle for the QWQPFT has been proved in [27]. Here, using a logarithmic uncertainty principle for the QFT, we establish a new version of the logarithmic uncertainty principle for the proposed QWQPFT.

Lemma 7 (Logarithmic uncertainty principle for the QFT [24])**.** *For $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$ [Schwartz space],*

$$\int_{\mathbb{R}^2} \ln|\mathbf{x}||f(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{R}^2} \ln|\mathbf{w}|\left|\mathcal{F}^{\mathbb{H}}[f](\mathbf{w})\right|^2 d\mathbf{w} \geq D\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \tag{42}$$

where $D = \ln(2\pi^2) - 2\psi(1/2)$, $\psi = \frac{d}{dt}(\ln(\Gamma(x)))$ and $\Gamma(x)$ is a Gamma function.

Theorem 10 (Logarithmic UP for the QWQPFT)**.** *Let $\Xi \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$ be a nonzero window function and $\mathcal{V}_{\Xi}^{\mathbb{H}}[f]$ be the quaternion window quadratic-phase Fourier transform of signal $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$. Then, we have the following logarithmic inequality*

$$\begin{aligned}
&\frac{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}{(2\pi)^2}\int_{\mathbb{R}^2} \ln|\mathbf{x}||f(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w} \\
&\geq \frac{1}{(2\pi)^2}(D - \ln|\mathbf{b}|)\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned} \tag{43}$$

Proof. From (28), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w} \\
&= \frac{|\mathbf{b}|}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathcal{F}^{\mathbb{H}}[F](\mathbf{b}\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left|\frac{\mathbf{y}}{\mathbf{b}}\right||\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u}d\mathbf{y} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{y}||\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u}d\mathbf{y} \\
&\quad - \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{b}||\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u}d\mathbf{y}.
\end{aligned}$$

Using Parseval's formula for QFT yields

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{y}||\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u}d\mathbf{y} - \frac{\ln|\mathbf{b}|}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |F(\mathbf{y})|^2 d\mathbf{u}d\mathbf{y} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{y}||\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u}d\mathbf{y} - \frac{\ln|\mathbf{b}|}{(2\pi)^2}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned} \tag{44}$$

As $f, \Xi \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$ implies $F(\mathbf{y}) \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$. Thus, replacing f with F, in the logarithmic uncertainty principle for QFT, we have

$$\int_{\mathbb{R}^2} \ln|\mathbf{x}||F(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{R}^2} \ln|\mathbf{w}|\left|\mathcal{F}^{\mathbb{H}}[F](\mathbf{w})\right|^2 d\mathbf{w} \geq D\int_{\mathbb{R}^2} |F(\mathbf{x})|^2 d\mathbf{x}, \tag{45}$$

multiplying and integrating both sides of (45), with $\frac{1}{(2\pi)^2}$ and $d\mathbf{u}$, respectively. We obtain

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|\mathbf{x}||F(\mathbf{x})|^2 d\mathbf{x}d\mathbf{u}+\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|\mathbf{w}|\left|\mathcal{F}^{\mathbb{H}}[F](\mathbf{w})\right|^2 d\mathbf{w}d\mathbf{u}\\&\geq\frac{1}{(2\pi)^2}D\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|F(\mathbf{x})|^2 d\mathbf{x}d\mathbf{u},\end{aligned}\tag{46}$$

which implies

$$\begin{aligned}&\frac{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}{(2\pi)^2}\int_{\mathbb{R}^2}\ln|\mathbf{x}||f(\mathbf{x})|^2 d\mathbf{x}+\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|\mathbf{w}|\left|\mathcal{F}^{\mathbb{H}}[F](\mathbf{w})\right|^2 d\mathbf{w}d\mathbf{u}\\&\geq\frac{1}{(2\pi)^2}D\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}.\end{aligned}\tag{47}$$

On inserting (44) into the (47), we obtain

$$\begin{aligned}&\frac{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}{(2\pi)^2}\int_{\mathbb{R}^2}\ln|\mathbf{x}||f(\mathbf{x})|^2 d\mathbf{x}+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|\mathbf{w}||\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w}\\&+\frac{\ln|\mathbf{b}|}{(2\pi)^2}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\geq\frac{1}{(2\pi)^2}D\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})},\end{aligned}$$

which simplifies to

$$\begin{aligned}&\frac{\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}}{(2\pi)^2}\int_{\mathbb{R}^2}\ln|\mathbf{x}||f(\mathbf{x})|^2 d\mathbf{x}+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|\mathbf{w}||\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w}\\&\geq\frac{1}{(2\pi)^2}(D-\ln|\mathbf{b}|)\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}.\end{aligned}$$

This completes the proof. □

Theorem 11. *Let Λ be a measurable set $\subset\mathbb{R}^2\times\mathbb{R}^2$ and $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})$ be the quaternion windowed quadratic-phase Fourier transform of any signal $f\in L^2(\mathbb{R}^2,\mathbb{H})$ with $\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}=\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}=1$ such that*

$$\int\int_{\Lambda}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u}\geq 1-\epsilon.\tag{48}$$

Then, we have $\frac{2\pi(1-\epsilon)}{\sqrt{b_1b_2}}\leq m(\Lambda)$, where $m(\Lambda)$ is Lebesgue measure of Λ.

Proof. From Definition 5, we obtain

$$\begin{aligned}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right| &= \left|\int_{\mathbb{R}^2}\mathcal{Q}^i_{\mu_1}(x_1,w_1)\,f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}\,\mathcal{Q}^j_{\mu_2}(x_2,w_2)d\mathbf{x}\right|\\&\leq\frac{\sqrt{b_1b_2}}{2\pi}\int_{\mathbb{R}^2}\left|f(\mathbf{x})\overline{\Xi(\mathbf{x}-\mathbf{u})}\right|d\mathbf{x}.\end{aligned}\tag{49}$$

By virtue of Holder's inequality, we obtain

$$\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|_{L^\infty(\mathbb{R}^2,\mathbb{H})}\leq\frac{\sqrt{b_1b_2}}{2\pi}\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|_{L^2(\mathbb{R}^2,\mathbb{H})}.\tag{50}$$

On inserting (50) in (48), we obtain

$$\begin{aligned} 1-\epsilon &\leq \int\int_{\Lambda}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u} \\ &\leq m(\Lambda)\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|_{L^\infty(\mathbb{R}^2,\mathbb{H})} \\ &\leq m(\Lambda)\frac{\sqrt{b_1b_2}}{2\pi}\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|_{L^2(\mathbb{R}^2,\mathbb{H})} \\ &\leq m(\Lambda)\frac{\sqrt{b_1b_2}}{2\pi}, \end{aligned}$$

which completes the proof. □

Next, we prove Local UP for the QWQPFT which states that, for a non zero quaternion signal f whose QWQPFT $\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f]$ is concentrated on a measurable set Λ satisfying $0<m(\Lambda)<1$, then either $f\equiv 0$ or $\Xi\equiv 0$.

Theorem 12 (Local UP for the QWQPFT)**.** *Let Λ be a measurable subset of $\mathbb{R}^2\times\mathbb{R}^2$ such that $0<m(\Lambda)<1$. Then, for every $f,\Xi\in L^2(\mathbb{R}^2,\mathbb{H})$, the following inequality holds*

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|_{L^2(\mathbb{R}^2,\mathbb{H})} \leq \frac{1}{\sqrt{1-m(\Lambda)}}\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|_{L^2(\Lambda^c,\mathbb{H})}. \tag{51}$$

Proof. Theorem 11 together with (36) yields

$$\begin{aligned} &\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{H})} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u} \\ &= \int\int_{\Lambda}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u} + \int\int_{\Lambda^c}\left|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u} \\ &\leq m(\Lambda)\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} + \left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\Lambda^c,\mathbb{H})}. \end{aligned}$$

Hence,

$$\begin{aligned} &\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\Lambda^c,\mathbb{H})} \\ &\geq \left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{H})} - m(\Lambda)\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \end{aligned}$$

Again, by virtue of (36), we obtain

$$\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|^2_{L^2(\Lambda^c,\mathbb{H})} \geq (1-m(\Lambda))\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

Equivalently, we have

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}\|\Xi\|_{L^2(\mathbb{R}^2,\mathbb{H})} \leq \frac{1}{\sqrt{1-m(\Lambda)}}\left\|\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})\right\|_{L^2(\Lambda^c,\mathbb{H})},$$

which completes the proof. □

Towards the end of this section, we explore Pitt's inequality associated with the QWQPFT. First, we shall state the following Lemma.

Lemma 8 (Pitt's inequality of the QFT [31]). *For $f \in \mathcal{S}(\mathbb{R}^{2t}, \mathbb{H})$,*

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{F}^{\mathbb{H}}[f](\mathbf{w})|^2 d\mathbf{w} \\ &\leq \Delta_\alpha \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 d\mathbf{x}, \end{aligned} \tag{52}$$

where $\Delta_\alpha = \pi^\alpha \left[\Gamma\left(\frac{2t-\alpha}{4}\right)\Gamma\left(\frac{2t+\alpha}{4}\right)\right]$, $0 \leq \alpha \leq 2t$ and Γ is a Gamma function.

According to the above Lemma, we obtain Pitt's inequality of the QWQPFT.

Theorem 13 (Pitt's inequality of the QWQPFT). *For $f \in \mathcal{S}(\mathbb{R}^{2t}, \mathbb{H})$,*

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w} \\ &\leq \frac{|\mathbf{b}|^\alpha}{4\pi^2} \Delta_\alpha \|\Xi\|^2_{L^2(\mathbb{R}^{2t},\mathbb{H})} \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 d\mathbf{x} \end{aligned} \tag{53}$$

where $\Delta_\alpha = \pi^\alpha \left[\Gamma\left(\frac{2t-\alpha}{4}\right)\Gamma\left(\frac{2t+\alpha}{4}\right)\right]$, $0 \leq \alpha \leq 2t$ and Γ is a Gamma function.

Proof. From Theorem 3, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w} \\ &= \frac{|b_1 b_2|}{(2\pi)^2} \int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |w|^{-\alpha} |\mathcal{F}^{\mathbb{H}}[f](\mathbf{bw},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w}. \end{aligned}$$

Setting $\mathbf{bw} = \mathbf{y}$ yields

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{s}} \left|\frac{\mathbf{y}}{\mathbf{b}}\right|^{-\alpha} |\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u} d\mathbf{y} \\ &= \frac{|\mathbf{b}|^\alpha}{(2\pi)^2} \int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{y}|^{-\alpha} |\mathcal{F}^{\mathbb{H}}[F](\mathbf{y},\mathbf{u})|^2 d\mathbf{u} d\mathbf{y}. \end{aligned}$$

Now, by Lemma 8, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w} \\ &\leq \frac{|\mathbf{b}|^\alpha}{(2\pi)^2} \Delta_\alpha \int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |F(\mathbf{x})|^2 d\mathbf{u} d\mathbf{x}. \end{aligned}$$

Currently, using (29) yields

$$\begin{aligned} &\int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{w}|^{-\alpha} |\mathcal{V}^{\mathbb{H}}_{\Xi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u} d\mathbf{w} \\ &\leq \frac{|\mathbf{b}|^\alpha}{(2\pi)^2} \Delta_\alpha \int_{\mathbb{R}^{2t}} \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |f(\mathbf{x}) \overline{\Xi(\mathbf{x}-\mathbf{u})}|^2 d\mathbf{u} d\mathbf{x} \\ &= \frac{|\mathbf{b}|^\alpha}{(2\pi)^2} \Delta_\alpha \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 \left(\int_{\mathbb{R}^{s}} |\overline{\Xi(\mathbf{x}-\mathbf{u})}|^2 d\mathbf{u} \right) d\mathbf{x} \\ &\leq \frac{|\mathbf{b}|^\alpha}{(2\pi)^2} \Delta_\alpha \|\Xi\|^2_{L^2(\mathbb{R}^{2t},\mathbb{H})} \int_{\mathbb{R}^{2t}} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 d\mathbf{x}, \end{aligned}$$

which completes the proof. □

5. Conclusions

In the study, we have accomplished two major objectives: first, we have introduced the definition of the quaternion windowed quadratic-phase Fourier transform (QWQPFT) and established fundamental properties of the proposed transform, including the inversion formula, linearity, boundedness, and Plancherel formula. Secondly, we investigated some different forms of UPs associated with QWQPFT including Hardy's UP, Beurling's UP, Donoho–Stark's, logarithmic UP, and Local UP. In our future works, we shall study Wigner distribution in the quaternion quadratic-phase domain and its relation with the proposed QWQPFT.

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Abbreviations

The following abbreviations are used in this manuscript:

QWQPFT	Quaternion windowed quadratic-phase Fourier transform
QPFT	Quadratic-phase Fourier transform
Q-QPFT	Quaternion quadratic phase Fourier transform

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