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# Classification of Irreducible $\mathbb{Z}_+$-Modules of a $\mathbb{Z}_+$-Ring Using Matrix Equations

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**Abstract:** This paper aims to investigate and categorize all inequivalent and irreducible $\mathbb{Z}_+$-modules of a commutative unit $\mathbb{Z}_+$-ring $\mathcal{A}$, equipped with set $\{1, x, y, xy\}$ satisfying $x^2 = 1, \quad y^2 = 1$ as a $\mathbb{Z}_+$-basis by using matrix equations, which was part of a call for a Special Issue about matrix inequalities and equations by Symmetry. If the rank of the $\mathbb{Z}_+$-module $n \leq 2$, we prove that there are finitely many inequivalent and irreducible $\mathbb{Z}_+$-modules, respectively, one and three. However, if $n \geq 3$, there is no irreducible $\mathbb{Z}_+$-module.

**Keywords:** $\mathbb{Z}_+$-ring; irreducible $\mathbb{Z}_+$-module; matrix equation; NIM solution

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## 1. Introduction

A tensor category is usually considered as a counterpart of groups and rings, and it is also important in algebraic representation theory. In 2000, Bakalov and Kirillov [1] gave an exposition of the relations between monoidal tensor categories and modular functors. In tensor categories, a $\mathbb{Z}_+$-ring is a quite significant invariant. In 1987, Lusztig [2] first introduced the $\mathbb{Z}_+$-ring. Afterwards, Etingof [3] proved in 1994 that irreducible $\mathbb{Z}_+$-representations of a Verlinde algebra which satisfies some conditions correspond to some special Dynkin diagrams. Davydov [4,5] showed the relation between the Drinfel'd algebras or Galois algebras and finite groups. The definitions and properties relevant to $\mathbb{Z}_+$-rings can be found in [3,6]. Particularly, Grothendieck rings [7–11] and Green rings of Hopf algebras [12–17] are both classic examples. Furthermore, Ostrik and others [3,6] introduced the terminology on $\mathbb{Z}_+$-modules over certain rings or algebras. If a $\mathbb{Z}_+$-module $M$ only has a trivial $\mathbb{Z}_+$-submodule, it is called irreducible. Given a ring or algebra, it is important and meaningful to investigate and categorize all inequivalent and irreducible $\mathbb{Z}_+$-modules over it.

In 2003, Ostrik [6] illustrated that given an arbitrary $\mathbb{Z}_+$-ring whose rank is finite, there are finitely many inequivalent and irreducible $\mathbb{Z}_+$-modules. He stated that there is an upper bound for the rank of any irreducible $\mathbb{Z}_+$-module. Moreover, in general, the upper bound is quite large. Usually, it is quite difficult to estimate the size between the rank of a $\mathbb{Z}_+$-ring and that of an irreducible $\mathbb{Z}_+$-module. In this paper, we categorize all inequivalent and irreducible $\mathbb{Z}_+$-modules of a commutative unit $\mathbb{Z}_+$-ring $\mathcal{A}$, equipped with set $\{1, x, y, xy\}$ satisfying relations $x^2 = 1$, $y^2 = 1$ as a $\mathbb{Z}_+$-basis. In addition, this problem can be transformed to solve all the irreducible NIM (nonnegative integer matrix) solutions to matrix equations as follows:

$$\begin{cases} AB = BA, \\ A^2 = E, \\ B^2 = E, \end{cases} \tag{1}$$

where the matrix $E$ is determined as a unit matrix through the paper. Hence, in what follows, we only need to determine all the irreducible NIM solutions to $(1)$. Particularly, Wang [18,19] made great contributions in matrix equations. In 2022, he derived the formula of a general solution to a class of equations named Sylvester type matrix equation of Hamilton quaternions and solvability conditions. Furthermore, he put forward a modified conjugate residual method to solve a class of equations named generalized coupled Sylvester tensor equations. In addition, Wang [20] investigated the minimum-norm least squares solution to a quaternion tensor system by the method of Moore–Penrose inverses in Symmetry.

In Section 3, we prove that there is no irreducible NIM solution to the matrix equations (1) when order $n \geq 5$, and there is no irreducible $\mathbb{Z}_+$-module of $\mathcal{A}$. In Section 4, when $n \leq 4$, we first calculate all NIM solutions and then find the inequivalent irreducible ones. Furthermore, there are irreducible $\mathbb{Z}_+$-modules of $\mathcal{A}$ when and only when the rank of the $\mathbb{Z}_+$-module is no more than two. If $n = 1$, there exists a unique irreducible NIM solution to (1), which corresponds to a unique irreducible $\mathbb{Z}_+$-module of $\mathcal{A}$. If $n = 2$, there are three inequivalent irreducible NIM solutions to the matrix equations (1), which corresponds to three inequivalent irreducible $\mathbb{Z}_+$-modules of $\mathcal{A}$. If $n = 3$ or $n = 4$, there is no irreducible $\mathbb{Z}_+$-module of $\mathcal{A}$. In fact, this paper deals with semiring theory including semiring modules. This theory is closely related with tropical algebras theory. Particularly, Alexander Mikhalev [21,22] developed semigroup theory and semiring theory.

## 2. Preliminaries

Throughout this paper, all matrices are in $\mathbb{M}_n(\mathbb{N})$, which is the set of matrices of order $n$ over natural numbers, and the element of row $i$ and column $j$ of $A \in \mathbb{M}_n(\mathbb{N})$ is denoted by $a_{ij}$. $A$ is said to be positive if $a_{ij} > 0$ for all $i$, $j$, and denoted by $A > 0$. If $a_{ij} \geqslant 0$ for all $i$, $j$, then $A$ is said to be nonnegative, and denoted by $A \geqslant 0$. Moreover, an $n$-order square matrix $A$ is called a nonnegative integer matrix (NIM) if $A \in \mathbb{M}_n(\mathbb{N})$. An $n$-order square matrix $P$ is called a permutation matrix if every row and every column contains a single one and all the other elements are zero.

**Definition 1** ([23], Chapter XIII)**.** *Let $A$ be an $n$-order matrix. Then, $A$ is said to be reducible, if there is a permutation matrix $P$, satisfying $PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}$, where $A_1$ is a $k$-order square matrix and $A_3$ is an $(n-k)$-order square matrix. Note that the submatrix $O$ is a zero matrix and $PAP^T$ is a lower triangular block matrix.*

**Definition 2.** *Let $A$ and $B$ be solutions to the system of matrix equations* (1)*. The solutions are said to be reducible, if $A$ and $B$ are simultaneously reducible, namely, there is an $n$-order permutation matrix $P$, satisfying*

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \quad PBP^T = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix},$$

*where $A_1$ and $B_1$ are both square matrices with the same order and $A_3$ and $B_3$ are both square matrices with the same order. Otherwise, they are said to be irreducible.*

From now on, let $\mathbb{Z}_+$ represent the set of nonnegative integers. In the following, we will present some basic definitions and properties relevant to a $\mathbb{Z}_+$-ring and a $\mathbb{Z}_+$-module [6].

**Definition 3.** *Let $\mathcal{A}$ be a ring and free as a $\mathbb{Z}$-module. A $\mathbb{Z}$-basis $\mathcal{B} = \{b_\omega\}_{\omega\in\Omega}$ of $\mathcal{A}$ is called a $\mathbb{Z}_+$-basis if it satisfies $b_\alpha b_\beta = \sum_{\omega\in\Omega} c^\omega_{\alpha\beta} b_\omega$, and $c^\omega_{\alpha\beta} \in \mathbb{Z}_+$. A $\mathbb{Z}_+$-ring is a ring equipped with a special $\mathbb{Z}_+$-basis and identity one, which can be expressed by a nonnegative linear combination of it.*

**Definition 4.** *(1) Assume $\mathcal{A}$ is a fixed $\mathbb{Z}_+$-ring equipped with a certain basis $\{b_\omega\}_{\omega\in\Omega}$. An $\mathcal{A}$-module $M$ is called a $\mathbb{Z}_+$-module if it is equipped with a special $\mathbb{Z}$-basis $\{m_\gamma\}_{\gamma\in\Gamma}$ such that all the coefficients $t^\gamma_{\omega\alpha}$ defined by the $\mathbb{Z}$-basis relations $b_\omega m_\alpha = \sum_{\gamma\in\Gamma} t^\gamma_{\omega\alpha} m_\gamma$ are nonnegative.*

*(2) Assume $\mathcal{A}$ is a fixed $\mathbb{Z}_+$-ring equipped with a certain basis $\{b_\omega\}_{\omega\in\Omega}$ such that $b_\alpha b_\beta = \sum_{\omega\in\Omega} c_{\alpha\beta}^{\omega} b_\omega$. For a $\mathbb{Z}_+$-module $M$ over $\mathcal{A}$, it assigns each $b_\alpha$ to a matrix $N_\alpha \in \mathbb{M}_n(\mathbb{N})$ such that $N_\alpha N_\beta = \sum_{\omega\in\Omega} c_{\alpha\beta}^{\omega} N_\omega$, for all $\alpha, \beta, \omega \in \Omega$. In particular, the identity of $\mathcal{A}$ is assigned to the unit matrix. It is also easy to see the rank of a $\mathbb{Z}_+$-module $M$ equals the order of the matrix $N_\omega$.*

**Definition 5.** *Take $\mathbb{Z}_+$-modules $M$, $M'$ over $\mathcal{A}$ equipped with bases $\{m_\gamma\}_{\gamma\in\Gamma}$ and $\left\{m'_\beta\right\}_{\beta\in\Gamma'}$. They are said to be equivalent if there is a bijection $\psi : \Gamma \to \Gamma'$ such that the induced $\mathbb{Z}$-linear map $\tilde{\psi}$ of $\mathbb{Z}$-modules $M$, $M'$ defined by $\tilde{\psi}(m_\gamma) = m'_{\psi(\gamma)}$ is an $\mathcal{A}$-isomorphism. Namely, $\mathbb{Z}_+$-modules $M$ and $M'$ of rank $n$ are equivalent if and only if there is an $n$-order permutation matrix $P$ such that $N'_\omega = PN_\omega P^{-1}$ , for all $\omega \in \Omega$.*

**Definition 6.** *(1) Let $M$ be a $\mathbb{Z}_+$-module equipped with a basis $\{m_\gamma\}_{\gamma\in\Gamma}$ and $N$ be an $\mathcal{A}$-submodule of $M$. Then, $N$ is called a $\mathbb{Z}_+$-submodule if it is spanned by $\{m_\gamma\}_{\gamma\in\Gamma'}$ as a subgroup of $M$, where $\Gamma'$ is a subset of $\Gamma$.*

*(2) A $\mathbb{Z}_+$-module $M$ is said to be irreducible if it only has trivial $\mathbb{Z}_+$-submodule 0 or $M$. Otherwise, it is said to be irreducible. In fact, it is equivalent to the fact that all $N_\omega$ cannot be simultaneously transformed into lower triangular matrices, by Definition 2. Thus, it is called an irreducible NIM solution.*

To prove our main theorems in the next sections, we will need the following statements.

**Theorem 1** ([23], Chapter XIII)**.** *Assume $A = \left(a_{ij}\right)_{n\times n}$ is an $n$-order nonnegative real matrix. Then, $A$ is irreducible if and only if $(E + A)^{n-1} > 0$.*

**Theorem 2** ([23], Chapter XIII)**.** *Assume $A = \left(a_{ij}\right)_{n\times n}$ is an $n$-order nonnegative irreducible real matrix. Then, there is an eigenvalue $\lambda$ of $A$ such that $\lambda$ is a real number and $\lambda > 0$. Moreover, $\lambda$ is a simple root such that $\lambda \geq |\alpha|$, where $\alpha$ is any eigenvalue of $A$. The eigenvalue $\lambda$ is called a Perron–Frobenius eigenvalue.*

**Proposition 1** ([24], Proposition 2.1)**.** *Let $A, B \in \mathbb{M}_n(\mathbb{N})$ and $AB = E$, then $A$ and $B$ are both permutation matrices.*

**Corollary 1.** *If $A \in \mathbb{M}_n(\mathbb{N})$ and $A^k = E$ , $k \geq 1$, then $A$ is a permutation matrix.*

**Proof.** Let $B = A^{k-1}$. Then, $AB = A^k = E$. From Proposition 1, $A$ is a permutation matrix. □

### 3. Irreducible NIM Solutions for $n \geq 5$

Now, we infer that if the rank of $\mathbb{Z}_+$-ring $\mathcal{A}$, $n \geq 5$, equipped with a basis {1, $x$, $y$, $xy$} satisfying $x^2 = 1$, $y^2 = 1$, then there is no irreducible $\mathbb{Z}_+$-module $M$. This question is equivalent to proving that there is no irreducible NIM solution to the following equations:

$$\begin{cases} AB = BA, \\ A^2 = E, \\ B^2 = E, \end{cases} \tag{2}$$

where $A$ and $B \in \mathbb{M}_n(\mathbb{N})$ are NIM solutions to the matrix equations (2). Now, in the following, we let $b = E + A + B + AB$. Then, it is easy to see that $b \in \mathbb{M}_n(\mathbb{N})$.

**Theorem 3.** *If $A$ and $B$ are irreducible NIM solutions, then $b$ is irreducible.*

**Proof.** Assume that $b$ is reducible. Hence, we can get that there exists some permutation matrix $P$, satisfying

$$PbP^T = P(E + A + B + AB)P^T = E + PAP^T + PBP^T + PABP^T = \begin{bmatrix} b_1 & O \\ b_2 & b_3 \end{bmatrix}.$$

Since $A$, $B$ and $AB$ are nonnegative integer matrices, we have

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \quad PBP^T = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix}, \quad PABP^T = \begin{bmatrix} C_1 & O \\ C_2 & C_3 \end{bmatrix}.$$

In this case, $A$ and $B$ are reducible solutions, which is a contradiction. Therefore, $b$ is irreducible. □

From Theorem 3, we can get the following theorem.

**Theorem 4.** *If $A$ and $B$ are irreducible NIM solutions, then $b > 0$.*

**Proof.** From Theorem 3, we obtain that $b$ is an irreducible nonnegative matrix. Thus, by Theorem 1, we can get $(E + b)^{n-1} > 0$. According to the binomial expansion and the system of matrix equations (2), we have

$$(E + b)^{n-1} = m_1 E + m_2 A + m_3 B + m_4 AB > 0, \quad m_1, m_2, m_3, m_4 \in \mathbb{Z}_+.$$

Now, we infer that $b = E + A + B + AB > 0$. Assume that there is an element $b_{ij}$ of matrix $b$ such that $b_{ij} = 0$. Since $A$, $B$ and $AB$ are nonnegative integer matrices, the elements of row $i$ and column $j$ for each component $E$, $A$, $B$ and $AB$ are all 0, which is a contradiction. Therefore, we can get $b = E + A + B + AB > 0$. □

Furthermore, we can deduce the following theorem.

**Theorem 5.** *If $b > 0$, then $b$ is irreducible and* $\mathrm{tr}(b) = 4$.

**Proof.** From Theorem 1, $b$ is irreducible. According to the system of matrix equations (2), we have $b^2 = 4b$. Assume that $\lambda$ is an eigenvalue of $b$. Then, we have $\lambda^2 = 4\lambda$, that is $\lambda = 4$ or 0. Moreover, we claim that 4 is an eigenvalue of $b$. Otherwise, the trace of $b$, $\mathrm{tr}(b) = 0$, which contradicts $b > 0$ and $\mathrm{tr}(b) > 0$. Considering that $b$ is irreducible and $b > 0$, by Theorem 2, we infer that 4 is the unique nonzero simple root of $b$. Therefore, we can get $\mathrm{tr}(b) = 4$. □

In what follows, we give our prime theorem in this chapter.

**Theorem 6.** *Let $A, B \in \mathbb{M}_n(\mathbb{N})$ be solutions to the matrix equations (2) and $n \geq 5$. Then, $A$ and $B$ are reducible solutions.*

**Proof.** Assume that there are irreducible NIM solutions $A$ and $B$ to the matrix equations (2). By Theorem 4, we get $b > 0$. Therefore, $\mathrm{tr}(b) \geq \mathrm{tr}(E) = n \geq 5$, which contradicts the fact that $\mathrm{tr}(b) = 4$ by Theorem 5. Hence, the NIM solutions are always reducible. □

## 4. Irreducible NIM Solutions for $n \leq 4$

We have already illustrated that there is no irreducible NIM solution to the matrix equations (2) for $n \geq 5$. In the following, we describe all the inequivalent and irreducible NIM solutions to (2), for $n \leq 4$.

- If $n = 1$, there is a unique irreducible NIM solution:

$$A = 1, \quad B = 1.$$

- If $n = 2$, according to the fact that $A^2 = E$ and $A \in \mathbb{M}_n(\mathbb{N})$, we have that $A$ or $B$ is one of the two matrices as follows:

$$(1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since there is a bijection between permutation matrices $P_2$ and elements of permutation group $S_2$, we can denote the matrices above by (1) and (12), respectively. We also know that $B$ is one of the two matrices. By the commutativity of $A$ and $B$, there are four possible NIM solutions to the matrix equations (2). Furthermore, there are a total of three irreducible NIM solutions to the matrix equations (2), except that both $A$ and $B$ are unit matrices:
**Case(1.1):** $A_1 = (1)$, $B_1 = (12)$.
**Case(1.2):** $A_2 = (12)$, $B_2 = (1)$ or $B_3 = (12)$.

- If $n = 3$, since $A^2 = E$ and $A \in \mathbb{M}_n(\mathbb{N})$, we infer that $A$ is one of the following four matrices:

$$(1)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}(2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}(3)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}(4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there is a bijection between permutation matrices $P_3$ and elements of permutation group $S_3$, we can denote the matrices above by (12), (23), (13) and (1), respectively. Thus, $B$ is one of the four matrices. According to the commutativity of $A$ and $B$, we get the following NIM solutions to the matrix equations (2):
**Case(2.1):** $A_1 = (12)$, then $B_1 = (1)$ or $B_2 = (12)$.
**Case(2.2):** $A_2 = (23)$, then $B_3 = (1)$ or $B_4 = (23)$.
**Case(2.3):** $A_3 = (13)$, then $B_5 = (1)$ or $B_6 = (13)$.
**Case(2.4):** $A_4 = (1)$, then $B_7 = (1)$ or $B_8 = (12)$ or $B_9 = (13)$ or $B_{10} = (23)$.

Furthermore, we find that all the NIM solutions above are reducible. Hence, if $n = 3$, then there is no irreducible NIM solution to the matrix equations (2).

- If $n = 4$, since $A^2 = E$ and $A \in \mathbb{M}_n(\mathbb{N})$, we get that $A$ is one of the following ten matrices:

$$(1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}(2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}(3)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}(4)\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$(5)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}(6)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}(7)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}(8)\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$(9)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}(10)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there is a bijection between permutation matrices $P_4$ and elements of permutation group $S_4$, we can denote the matrices above by (23), (24), (13), (12), (34), (14)(23), (14), (12)(34), (13)(24) and (1), respectively. It is easy to see that $B$ is one of the ten matrices. By the commutativity of $A$ and $B$, we get the following NIM solutions to the matrix equations (2):
**Case(3.1):** If $A_1 = (23)$, then $B_1 = (1)$ or $B_2 = (14)$ or $B_3 = (23)$ or $B_4 = (14)(23)$.
**Case(3.2):** If $A_2 = (24)$, then $B_5 = (1)$ or $B_6 = (24)$ or $B_7 = (13)$ or $B_8 = (13)(24)$.
**Case(3.3):** If $A_3 = (13)$, then $B_9 = (1)$ or $B_{10} = (13)$ or $B_{11} = (24)$ or $B_{12} = (13)(24)$.
**Case(3.4):** If $A_4 = (12)$, then $B_{13} = (1)$ or $B_{14} = (34)$ or $B_{15} = (12)$ or $B_{16} = (12)(34)$.
**Case(3.5):** If $A_5 = (34)$, then $B_{17} = (1)$ or $B_{18} = (34)$ or $B_{19} = (12)$ or $B_{20} = (12)(34)$.
**Case(3.6):** If $A_6 = (14)(23)$, then $B_{21} = (1)$ or $B_{22} = (14)(23)$ or $B_{23} = (14)$ or $B_{24} = (23)$.

**Case(3.7):** If $A_7 = (14)$, then $B_{25} = (1)$ or $B_{26} = (14)$ or $B_{27} = (23)$ or $B_{28} = (14)(23)$.
**Case(3.8):** If $A_8 = (12)(34)$, then $B_{29} = (1)$ or $B_{30} = (12)(34)$ or $B_{31} = (12)$ or $B_{32} = (34)$.
**Case(3.9):** If $A_9 = (13)(24)$, then $B_{33} = (1)$ or $B_{34} = (13)(24)$ or $B_{35} = (13)$ or $B_{36} = (24)$.
**Case(3.10):** If $A_{10} = (1)$, then $B_{37} = (23)$ or $B_{38} = (24)$ or $B_{39} = (13)$ or $B_{40} = (12)$ or $B_{41} = (34)$ or $B_{42} = (14)(23)$ or $B_{43} = (14)$ or $B_{44} = (12)(34)$ or $B_{45} = (13)(24)$ or $B_{46} = (1)$.

Furthermore, we obtain that all the NIM solutions above are reducible. Hence, if $n = 4$, there is no irreducible NIM solution to the matrix equations (2).

According to the discussion above, we give all the irreducible NIM solutions to the matrix equations (2), where $n \leq 4$, and get the following theorem.

**Theorem 7.** *All the irreducible NIM solutions to the matrix equations* (2) *are as follows:*

Now, according to Table 1, we successfully categorize all the irreducible $\mathbb{Z}_+$-modules of $\mathcal{A}$ under the equivalence by Definition 5, and in conclusion we have the main theorem below.

**Table 1.** All the irreducible NIM solutions to (2).

| **All the Irreducible NIM Solutions to (2)** | | |
|---|---|---|
| $n$ | $A$ | $B$ |
| 1 | 1 | 1 |
| 2 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| $\geq 3$ | None | None |

**Theorem 8.** *All the inequivalent and irreducible $\mathbb{Z}_+$-modules of $\mathbb{Z}_+$-ring $\mathcal{A}$ are as follows (see Table 2):*

**Table 2.** All the inequivalent and irreducible $\mathbb{Z}_+$-modules of $\mathbb{Z}_+$-ring $\mathcal{A}$.

| **All the Inequivalent and Irreducible $\mathbb{Z}_+$-Modules of $\mathcal{A}$** | | |
|---|---|---|
| $n$ | $A$ | $B$ |
| 1 | 1 | 1 |
| 2 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| $\geq 3$ | None | None |

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