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Inequalities of the Ostrowski Type Associated with Fractional Integral Operators Containing the Mittag–Leffler Function

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Abstract: Integral operators with the Mittag–Leffler function in kernels play a very vital role in generalizing classical integral inequalities. This paper aims to derive Ostrowski-type inequalities for k-fractional integrals containing Mittag–Leffler functions. Several new inequalities can be deduced for various fractional integrals in particular cases. Applications of these inequalities are also given.

Keywords: Ostrowski inequality; Mittag–Leffler function; bounds; fractional integrals

MSC: 26A51; 26A33; 33E12

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1. Introduction

In [1], Ostrowski proved the following inequality, which is well known as the Ostrowski inequality.

Theorem 1. *Let $\mathcal{F}_1 : I \longrightarrow \mathbb{R}$ be a differentiable mapping in I^o, the interior of I, and $u_1, u_2 \in I^o$, $u_1 < u_2$. If $|\mathcal{F}_1'(t)| \leq \mathcal{M}$ for all $t \in [u_1, u_2]$, then for $\lambda \in [u_1, u_2]$, the following inequality holds:*

$$\left|\mathcal{F}_1(\lambda) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \mathcal{F}_1(t)dt\right| \leq \left[\frac{1}{4} + \frac{(\lambda - \frac{u_1+u_2}{2})^2}{(u_2-u_1)^2}\right](u_2 - u_1)\mathcal{M}. \tag{1}$$

The inequality (1) provides the boundedness of the difference between the value $\mathcal{F}_1(\lambda)$ of $\mathcal{F}_1$ at an arbitrary point λ of $[u_1, u_2]$ and its integral mean, $\frac{1}{u_2-u_1}\int_{u_1}^{u_2} \mathcal{F}_1(t)dt$, provided that the derivative $\mathcal{F}_1'$ is bounded. From the point of view of applications, this inequality gives the error bounds of the midpoint and trapezoidal numerical quadrature rules, and its applications to special means can be found; see [2,3].

In the last few years, several authors have studied classical inequalities for various types of fractional integrals by using different kinds of convex functions. For example, Set [4] and Liu [5] proved the Ostrowski-type inequalities for Riemann–Liouville fractional integrals via s-convex and h-convex functions. Kermausuor [6] and Lakhal [7] gave the Ostrowski-type inequalities for Riemann–Liouville k-fractional integrals via strongly (α, m)-convex functions and $k-\beta$-convex functions. In [8], Set et al. proved the Ostrowski-type inequalities for conformable fractional integrals via convex and AG-convex functions. In [9], Gürbüz et al. gave the Ostrowski-type inequalities for Katugampola fractional integrals via p-convex functions. In [10], Basci and Baleanu derived the Ostrowski-type inequalities for ψ-Hilfer fractional integrals. In [11], Faisal et al. established the Hermite–Hadamard–Jensen–Mercer fractional inequalities for convex functions, and similar inequalities for α-type real-valued convex functions were also given in [12]. In [13], Farid et al. derived the Ostrowski-type inequalities for fractional integrals containing an extended generalized Mittag–Leffler function.

Inspired by the above research, our aim in this paper is to derive the Ostrowski-type inequalities for the generalized k-fractional integrals given in Definition 3 that contain the Mittag–Leffler function (8). One can deduce several new and existing Ostrowski-type inequalities. Some applications of the established inequalities are discussed in the penultimate section of this paper.

The Mittag–Leffler function plays an important role in solving fractional differential equations. It is also used in the generalization of fractional integrals. In the literature, several inequalities have been established for various fractional integrals containing the Mittag–Leffler function. The Mittag–Leffler function has been generalized by many authors: For example, Wiman [14], Prabhakar [15], Shukla and Prajapati [16], Salim and Faraj [17], and Rahman et al. [18] have contributed significantly to its generalizations and extensions. Recently, Andrić et al. [19] defined the extended generalized Mittag–Leffler function as follows.

Definition 1. *Let $\rho, \alpha, \phi, \vartheta, \varpi \in \mathbb{C}$, $\Re(\rho), \Re(\alpha), \Re(\phi) > 0$, $\Re(\varpi) > \Re(\vartheta) > 0$ with $p \geq 0$, $\varepsilon > 0$, and $0 < \varsigma \leq \varepsilon + \Re(\rho)$. Then,*

$$E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\vartheta + n\varsigma, \varpi - \vartheta)}{\beta(\vartheta, \varpi - \vartheta)} \frac{(\varpi)_{n\varsigma}}{\Gamma(\rho n + \alpha)} \frac{t^n}{(\phi)_{n\varepsilon}} \tag{2}$$

where β_p is the generalized beta function $\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$ and $(\varpi)_{n\varsigma}$ is the Pochhammer symbol $(\varpi)_{n\varsigma} = \frac{\Gamma(\varpi + n\varsigma)}{\Gamma(\varpi)}$.

One can see that $\beta_p(x,y) = \beta_p(y,x)$ and that $\beta_p(.,.)$ is symmetric with respect to its arguments. Symmetry is an important property; things that have this property look more beautiful and fascinating. Likewise, symmetric functions play a very vital role in the theory of mathematical inequalities. Many classical inequalities for symmetric functions have been studied. For example, real functions that are defined on $[u_1, u_2]$ and are symmetric about $\frac{u_1+u_2}{2}$ satisfy the following generalization of the Hadamard inequality.

Theorem 2. *Let $\mathcal{F}_1 : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and $u_1, u_2 \in I$, where $u_1 < u_2$. If $\mathcal{F}_2$ is a symmetric function about $\frac{u_1+u_2}{2}$, then the following inequality holds:*

$$\mathcal{F}_1\left(\frac{u_1+u_2}{2}\right) \int_{u_1}^{u_2} \mathcal{F}_2(\lambda) d\lambda \leq \int_{u_1}^{u_2} \mathcal{F}_1(\lambda)\mathcal{F}_2(\lambda) d\lambda \leq \frac{\mathcal{F}_1(u_1) + \mathcal{F}_1(u_2)}{2} \int_{u_1}^{u_2} \mathcal{F}_2(\lambda) d\lambda. \tag{3}$$

A version of the Hadamard inequality for convex and symmetric functions about $\frac{u_1+u_2}{2}$ via Riemann–Liouville fractional integrals was given in [20]. Next, we define generalized fractional integrals as follows.

Definition 2 ([19])**.** *Let $\mathcal{F}_1 : [u_1, u_2] \longrightarrow \mathbb{R}$, $0 < u_1 < u_2$, be an integrable function. In addition, let $\rho, \alpha, \phi, \vartheta, \varpi, \delta \in \mathbb{C}$, $\Re(\rho), \Re(\alpha), \Re(\phi) > 0$, and $\Re(\varpi) > \Re(\vartheta) > 0$ with $p \geq 0$, $\varepsilon > 0$, and $0 < \varsigma \leq \varepsilon + \Re(\rho)$. Then, for $\lambda \in [u_1, u_2]$, the generalized fractional integrals are defined by:*

$$\left(\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+} \mathcal{F}_1\right)(\lambda; p) = \int_{u_1}^{\lambda} (\lambda - t)^{\alpha-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi}(\delta(\lambda - t)^{\rho}; p) \mathcal{F}_1(t) dt, \tag{4}$$

$$\left(\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-} \mathcal{F}_1\right)(\lambda; p) = \int_{\lambda}^{u_2} (t - \lambda)^{\alpha-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi}(\delta(t - \lambda)^{\rho}; p) \mathcal{F}_1(t) dt. \tag{5}$$

Zhang et al. [21] introduced the generalized k-fractional integrals involving the Mittag–Leffler function as follows:

Definition 3. *Let $\mathcal{F}_1, \mathcal{F}_2 : [u_1, u_2] \longrightarrow \mathbb{R}$, $0 < u_1 < u_2$, be two functions, such that $\mathcal{F}_1$ is positive and $\mathcal{F}_1 \in L_1[u_1, u_2]$, and $\mathcal{F}_2$ is differentiable and strictly increasing. In addition, let*

$\delta, \alpha, \phi, \vartheta, \varpi \in \mathbb{C}$, $\Re(\alpha), \Re(\phi) > 0$, $\Re(\varpi) > \Re(\vartheta) > 0$ with $p \geq 0$, $\rho, \varepsilon > 0$, $0 < \varsigma \leq \varepsilon + \rho$, and $k > 0$. Then, for $\lambda \in [u_1, u_2]$, the generalized k-fractional integrals are defined by:

$$\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}\mathcal{F}_1\right)(\lambda; p) = \int_{u_1}^{\lambda} (\mathcal{F}_2(\lambda) - \mathcal{F}_2(t))^{\frac{\alpha}{k}-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda) - \mathcal{F}_2(t))^{\frac{\rho}{k}}; p)\mathcal{F}_1(t)\mathcal{F}_2'(t)dt, \quad (6)$$

$$\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}\mathcal{F}_1\right)(\lambda; p) = \int_{\lambda}^{u_2} (\mathcal{F}_2(t) - \mathcal{F}_2(\lambda))^{\frac{\alpha}{k}-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(t) - \mathcal{F}_2(\lambda))^{\frac{\rho}{k}}; p)\mathcal{F}_1(t)\mathcal{F}_2'(t)dt, \quad (7)$$

where $E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(t; p)$ is the modified Mittag–Leffler function defined by:

$$E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\vartheta + n\varsigma, \varpi - \vartheta)}{\beta(\vartheta, \varpi - \vartheta)} \frac{(\varpi)_{n\varsigma}}{k\Gamma_k(\rho n + \alpha)} \frac{t^n}{(\phi)_{n\varepsilon}}. \quad (8)$$

Remark 1. *From fractional integrals (6) and (7), various new fractional integrals containing the Mittag–Leffler function can be deduced (for details, see [21], Remark 1). Further, the fractional integrals (6) and (7) reproduce many already-defined fractional integrals (for details, see [21], Remark 2).*

For a constant function, Zhang et al. [21] proved the following:

$$\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\right)(\lambda; p) = k(\mathcal{F}_2(\lambda) - \mathcal{F}_2(u_1))^{\frac{\alpha}{k}} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha+k,\phi,k}(\delta(\mathcal{F}_2(\lambda) - \mathcal{F}_2(t))^{\frac{\rho}{k}}; p) \quad (9)$$

and

$$\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}1\right)(\lambda; p) = k(\mathcal{F}_2(u_2) - \mathcal{F}_2(\lambda))^{\frac{\alpha}{k}} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha+k,\phi,k}(\delta(\mathcal{F}_2(t) - \mathcal{F}_2(\lambda))^{\frac{\rho}{k}}; p). \quad (10)$$

We have organized this paper as follows: In the upcoming section, we first establish an identity in order to derive Ostrowski-type inequalities. Then, by applying this identity and k-fractional integrals (6) and (7), Ostrowski-type inequalities are established. It is mentioned that several new Ostrowski-type inequalities can be deduced for the well-known fractional integrals compiled in [21] (Remarks 1 and 2). In the last section, some applications of the presented results are given.

2. Main Results

First, we establish the following lemma for the modified Mittag–Leffler function.

Lemma 1. *If $\delta, \alpha, \phi, \vartheta, \varpi \in \mathbb{C}$, $\Re(\alpha), \Re(\phi) > 0$, $\Re(\varpi) > \Re(\vartheta) > 0$ with $p \geq 0$, $\rho, \varepsilon > 0$, $0 < \varsigma \leq \varepsilon + \rho$, and $k > 0$, then*

$$\left(\frac{d}{dt}\right)[(\mathcal{F}_2(t))^{\frac{\alpha}{k}-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta\mathcal{F}_2(t)^{\frac{\rho}{k}}; p)] = \frac{\mathcal{F}_2'(t)\mathcal{F}_2(t)^{\frac{\alpha}{k}-2}}{k} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,l}(\delta\mathcal{F}_2(t)^{\frac{\rho}{k}}; p). \quad (11)$$

Proof. We have

$$\begin{aligned}
&\left(\frac{d}{dt}\right)[(\mathcal{F}_2(t))^{\frac{\alpha}{k}-1} E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta\mathcal{F}_2(t)^{\frac{\rho}{k}}; p)] \quad (12)\\
&= \sum_{n=0}^{\infty} \frac{\beta_p(\vartheta + n\varsigma, \varpi - \vartheta)}{\beta(\vartheta, \varpi - \vartheta)} \frac{(\varpi)_{n\varsigma}}{k\Gamma_k(\rho n + \alpha)} \frac{\delta^n(\frac{\rho n+\alpha-k}{k})(\mathcal{F}_2(t))^{\frac{\rho n}{k}+\frac{\alpha}{k}-2}\mathcal{F}_2'(t)}{(\phi)_{n\varepsilon}}\\
&= \sum_{n=0}^{\infty} \frac{\beta_p(\vartheta + n\varsigma, \varpi - \vartheta)}{\beta(\vartheta, \varpi - \vartheta)} \frac{(\varpi)_{n\varsigma}}{k\Gamma_k(\rho n + \alpha - k)} \frac{\delta^n(\mathcal{F}_2(t))^{\frac{\rho n}{k}+\frac{\alpha}{k}-2}\mathcal{F}_2'(t)}{k(\phi)_{n\varepsilon}}.
\end{aligned}$$

After simple computation, the identity (11) is achieved. □

Next, we give the generalized k-fractional Ostrowski-type inequality containing the modified Mittag–Leffler function.

Theorem 3. *Let $\mathcal{F}_1 : I \longrightarrow \mathbb{R}$ be a differentiable mapping in I^o, the interior of I, and $u_1, u_2 \in I^o$, $u_1 < u_2$. In addition, let $\mathcal{F}_2 : [u_1, u_2] \longrightarrow \mathbb{R}$ be an increasing and differentiable function with $\mathcal{F}_2' \in L[u_1, u_2]$. If $\mathcal{F}_1$ is integrable and $|\mathcal{F}_1'(\mathcal{F}_2(t))| \leq \mathcal{M}$ for all $t \in [u_1, u_2]$, then for $\alpha, \beta \geq k$, the following inequality for fractional integrals (6) and (7) holds:*

$$\begin{aligned}
&\Bigg|k\bigg((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\bigg)\mathcal{F}_1(\mathcal{F}_2(\lambda))-\bigg(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p) \\
&+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\bigg)\Bigg| \leq k\mathcal{M}\bigg((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p) \\
&-\bigg(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\Big)(\lambda;p)\bigg)\bigg).
\end{aligned} \tag{13}$$

Proof. Let $\lambda \in [u_1, u_2]$ and $\alpha \geq k$. Then, for the monotonically increasing function $\mathcal{F}_2$ and the Mittag–Leffler function (8), we can write:

$$\begin{aligned}
&(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t) \\
&\leq (\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t) \quad t \in [u_1,\lambda].
\end{aligned} \tag{14}$$

From the boundedness condition of $\mathcal{F}_1'$ and (14), we have the following inequalities:

$$\begin{aligned}
&\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \\
&\leq (\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\int_{u_1}^{\lambda}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \\
&\leq (\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt.
\end{aligned} \tag{16}$$

First, we consider the inequality (15) as follows:

$$\begin{aligned}
&\mathcal{M}\int_{u_1}^{\lambda}(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \\
&-\int_{u_1}^{\lambda}(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_1'(\mathcal{F}_2(t))\mathcal{F}_2'(t)dt \\
&\leq (\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt.
\end{aligned} \tag{17}$$

The inequality (17) takes the following form after integrating by parts and using (11):

$$\begin{aligned}
&k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda)) \\
&-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p) \leq k\mathcal{M}\bigg((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\
&E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)\bigg).
\end{aligned} \tag{18}$$

Similarly, by using the same technique, from (16), one can achieve the following inequality:

$$\begin{aligned}&\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p)-k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}\\&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda))\leq k\mathcal{M}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}\\&E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\right)(\lambda;p)\Big).\end{aligned}\tag{19}$$

From (18) and (19), the following inequality is achieved:

$$\begin{aligned}&\Big|k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda))\\&-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p)\Big|\leq k\mathcal{M}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}\\&E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\right)(\lambda;p)\Big).\end{aligned}\tag{20}$$

Now, on the other hand, let $\lambda\in[u_1,u_2]$ and $\beta\geq k$. Then, for the monotonically increasing function $\mathcal{F}_2$ and the Mittag–Leffler function (8), we can write:

$$\begin{aligned}&(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)\\&\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)\quad t\in[\lambda,u_2].\end{aligned}\tag{21}$$

From the boundedness condition of $\mathcal{F}_1'$ and (21), we have the following inequalities:

$$\begin{aligned}&\int_\lambda^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt\\&\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_\lambda^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt\end{aligned}\tag{22}$$

and

$$\begin{aligned}&\int_\lambda^{u_2}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt\\&\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_\lambda^{u_2}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt.\end{aligned}\tag{23}$$

Following the same technique as that for (15) and (16), one can obtain from (22) and (23) the following inequality:

$$\begin{aligned}&\Big|k(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda))\\&-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p)\Big|\leq k\mathcal{M}\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}\\&E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\right)(\lambda;p)\Big).\end{aligned}\tag{24}$$

From inequalities (20) and (24), the inequality (13) is obtained. □

Corollary 1. *For $\alpha=\beta$ in (13), the following inequality holds:*

$$\Big|k\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1} \tag{25}$$

$$\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)\mathcal{F}_1(\mathcal{F}_2(\lambda))-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)$$

$$+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\Big)\Big|\le k\mathcal{M}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}$$

$$\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)$$

$$-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}1\Big)(\lambda;p)\Big)\Big).$$

Remark 2. *In Theorem 3, for $k=1$, we attain Theorem 2 from [22]. For $\delta=p=0$, we attain Theorem 7 from [23]. For $\mathcal{F}_2(\lambda)=\lambda$ and $k=1$, we attain Theorem 2.1 from [13]. For $\mathcal{F}_2(\lambda)=\lambda$ and $\delta=p=0$, we attain Theorem 1.2 from [24].*

The next result is a general form of a generalized k-fractional Ostrowski inequality containing the modified Mittag–Leffler function (8).

Theorem 4. *Let $\mathcal{F}_1: I\longrightarrow\mathbb{R}$ be a differentiable mapping in I^o, the interior of I, and $u_1,u_2\in I^o$, $u_1<u_2$. In addition, let $\mathcal{F}_2:[u_1,u_2]\longrightarrow\mathbb{R}$ be an increasing and differentiable function with $\mathcal{F}_2'\in L[u_1,u_2]$. If $\mathcal{F}_1$ is integrable and $\mathcal{N}<\mathcal{F}_1'(\mathcal{F}_2(t))\le\mathcal{M}$ for all $t\in[u_1,u_2]$, then for $\alpha,\beta\ge k$, the following inequalities for fractional integrals (6) and (7) hold:*

$$k\mathcal{F}_1(\mathcal{F}_2(\lambda))\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1} \tag{26}$$

$$\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\Big)-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)$$

$$-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\Big)\le k\mathcal{M}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)$$

$$+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)$$

$$+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\Big)(\lambda;p)\Big)\Big)$$

and

$$k\mathcal{F}_1(\mathcal{F}_2(\lambda))\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1} \tag{27}$$

$$\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)+\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)$$

$$-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\Big)\le -k\mathcal{N}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)$$

$$+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)$$

$$+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\Big)(\lambda;p)\Big)\Big).$$

Proof. From the boundedness condition of $\mathcal{F}_1'$ and (14), we have the following inequalities:

$$\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{28}$$

$$\le(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt,$$

$$\int_{u_1}^{\lambda}(\mathcal{F}_1'(\mathcal{F}_2(t))-\mathcal{N})(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{29}$$
$$\leq(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{F}_1'(\mathcal{F}_2(t))-\mathcal{N})\mathcal{F}_2'(t)dt.$$

From inequalities (28) and (29), after simple computation, one can achieve the following inequalities:

$$k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda))-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha-k,\phi,\delta,u_1^+}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p) \tag{30}$$
$$\leq k\mathcal{M}\left((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha,\phi,\delta,u_1^+}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(\lambda;p)\right),$$

and

$$\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha-k,\phi,\delta,u_1^+}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p)-k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda)) \tag{31}$$
$$\leq -k\mathcal{N}\left((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha,\phi,\delta,u_1^+}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(\lambda;p)\right).$$

Now, on the other hand, from the boundedness condition of $\mathcal{F}_1'$ and (21), we have the following inequalities:

$$\int_{\lambda}^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{32}$$
$$\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_{\lambda}^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt$$

and

$$\int_{\lambda}^{u_2}(\mathcal{F}_1'(\mathcal{F}_2(t))-\mathcal{N})(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{33}$$
$$\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_{\lambda}^{u_2}(\mathcal{F}_1'(\mathcal{F}_2(t))-\mathcal{N})\mathcal{F}_2'(t)dt.$$

From inequalities (32) and (33), after simple computation, one can achieve the following inequalities:

$$\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\beta-k,\phi,\delta,u_2^-}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p)-k(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda)) \tag{34}$$
$$\leq k\mathcal{M}\left((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\beta,\phi,\delta,u_2^-}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(\lambda;p)\right),$$

and

$$k(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(\lambda))-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\beta-k,\phi,\delta,u_2^-}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(\lambda;p) \tag{35}$$
$$\leq -k\mathcal{N}\left((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\beta,\phi,\delta,u_2^-}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(\lambda;p)\right).$$

From inequalities (30) and (34), the inequality (26) is achieved. Further, from inequalities (31) and (35), the inequality (27) is achieved. □

Theorem 5. *Under the assumptions of Theorem 4, the following inequalities hold:*

$$\begin{aligned}
&k\mathcal{F}_1(\mathcal{F}_2(\lambda))\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\Big)-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p) \\
&+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\Big)\le k\mathcal{M}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)\Big) \\
&-k\mathcal{N}\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\Big)(\lambda;p)\Big),
\end{aligned}\tag{36}$$

and

$$\begin{aligned}
&-k\mathcal{F}_1(\mathcal{F}_2(\lambda))\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)+\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p) \\
&+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(\lambda;p)\Big)\le k\mathcal{M}\Big((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_2^-}1\Big)(\lambda;p)\Big) \\
&-k\mathcal{N}\Big((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\Big)(\lambda;p)\Big).
\end{aligned}\tag{37}$$

Proof. This proof is similar to the proof of Theorem 4. From inequalities (30) and (35), the inequality (36) is attained. Further, from inequalities (31) and (34), the inequality (37) is attained. □

Theorem 6. *Under the assumptions of Theorem 3, the following inequality holds:*

$$\begin{aligned}
&\Big|k\Big(\mathcal{F}_1(\mathcal{F}_2(u_2))(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)+\mathcal{F}_1(\mathcal{F}_2(u_1)) \\
&(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,\lambda^+}\mathcal{F}_1\circ\mathcal{F}_2\Big)(u_2;p) \\
&+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,\lambda^-}\mathcal{F}_1\circ\mathcal{F}_2\Big)(u_1;p)\Big)\Big|\le k\mathcal{M}\Big[(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\
&\times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p) \\
&-\Big(\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,\lambda^-}1\Big)(u_1;p)+\Big({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,\lambda^+}1\Big)(u_2;p)\Big)\Big].
\end{aligned}\tag{38}$$

Proof. Let $\lambda\in[u_1,u_2]$ and $\alpha\ge k$. Then, for the monotonically increasing function $\mathcal{F}_2$ and the Mittag–Leffler function (8), we can write:

$$\begin{aligned}
&(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t) \\
&\le(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t),\ t\in[u_1,\lambda].
\end{aligned}\tag{39}$$

From the boundedness condition of $\mathcal{F}_1'$ and (39), we have the following inequalities:

$$\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{40}$$
$$\leq(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt$$

and

$$\int_{u_1}^{\lambda}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{41}$$
$$\leq(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt.$$

First, we consider the inequality (40) as follows:

$$\mathcal{M}\int_{u_1}^{\lambda}(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{42}$$
$$-\int_{u_1}^{\lambda}(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(t)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)\mathcal{F}_1'(\mathcal{F}_2(t))dt$$
$$\leq(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\int_{u_1}^{\lambda}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt.$$

The inequality (42) takes the following form after integrating by parts and using Lemma 1:

$$\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha-k,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)-k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(u_1)) \tag{43}$$
$$\leq k\mathcal{M}\left((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(u_1;p)\right).$$

Similarly, by using the same technique as that from (41), one can achieve

$$k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(u_1))-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha-k,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p) \tag{44}$$
$$\leq k\mathcal{M}\left((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(u_1;p)\right).$$

From (43) and (44), the following inequality is achieved:

$$\left|k(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(u_1))-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha-k,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)\right| \tag{45}$$
$$\leq k\mathcal{M}\left((\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E_{\rho,\alpha,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)-\left({}_{\mathcal{F}_2}^{k}\mathcal{Z}_{\rho,\alpha,\phi,\delta,\lambda^-}^{\vartheta,\varepsilon,\varsigma,\varpi}1\right)(u_1;p)\right).$$

Now, on the other hand, let $\lambda\in[u_1,u_2]$ and $\beta\geq k$. Then, for the monotonically increasing function $\mathcal{F}_2$ and the Mittag–Leffler function (8), we can write:

$$(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t) \tag{46}$$
$$\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t),\ t\in[\lambda,u_2].$$

From the boundedness condition of $\mathcal{F}_1'$ and (46), we have the following inequalities:

$$\int_{\lambda}^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \tag{47}$$
$$\leq(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E_{\rho,\beta,\phi,k}^{\vartheta,\varepsilon,\varsigma,\varpi}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_{\lambda}^{u_2}(\mathcal{M}-\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt$$

and

$$\int_{\lambda}^{u_2}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(t))^{\frac{\rho}{k}};p)\mathcal{F}_2'(t)dt \\ \leq (\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\int_{\lambda}^{u_2}(\mathcal{M}+\mathcal{F}_1'(\mathcal{F}_2(t)))\mathcal{F}_2'(t)dt. \tag{48}$$

Following the same technique as that for (40) and (41), one can attain from (47) and (48) the following inequality:

$$\left|k(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)\mathcal{F}_1(\mathcal{F}_2(u_2))-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,\lambda^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_2;p)\right| \\ \leq k\mathcal{M}\left((\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\beta}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)-\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,\lambda^+}1\right)(u_2;p)\right). \tag{49}$$

From inequalities (45) and (49), the inequality (38) is attained. □

Some direct consequences are given below.

Corollary 2. *For $\alpha=\beta$ in (38), the following inequality holds:*

$$\left|k\Big(\mathcal{F}_1(\mathcal{F}_2(u_2))(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p)+\mathcal{F}_1(\mathcal{F}_2(u_1)) \right. \\ \left.(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)-\Big(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,\lambda^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_2;p) \right. \\ \left.+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,\lambda^-}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)\Big)\right| \leq k\mathcal{M}\Big[(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\ \times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p) \\ -\Big(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,\lambda^-}1\right)(u_1;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,\lambda^+}1\right)(u_2;p)\Big)\Big]. \tag{50}$$

Remark 3. *In Theorem 6, for $k=1$, we attain Theorem 4 from [22]. For $\delta=p=0$, we attain Theorem 9 from [23]. For $\mathcal{F}_2(\lambda)=\lambda$ and $k=1$, we attain Theorem 2.6 from [13]. For $\mathcal{F}_2(\lambda)=\lambda$ and $\delta=p=0$, we attain Theorem 1.4 from [24].*

3. Applications

In this section, we give applications of Theorem 6. By applying Theorem 6 at the endpoints of the interval $[u_1,u_2]$ and adding the resulting inequalities, one can achieve the following results.

Theorem 7. *Under the assumptions of Theorem 6, the following inequality holds:*

$$\left|k\Big(\mathcal{F}_1(\mathcal{F}_2(u_2))(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\beta}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+\mathcal{F}_1(\mathcal{F}_2(u_1)) \right. \\ \left.(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\Big)-\Big(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_2;p) \right. \\ \left.+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)\Big)\right| \leq k\mathcal{M}\Big[(\mathcal{F}_2(\lambda)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}} \\ \times E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)+(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\beta}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(\lambda))^{\frac{\rho}{k}};p) \\ -\Big(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}1\right)(u_1;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\beta,\phi,\delta,u_1^+}1\right)(u_2;p)\Big)\Big]. \tag{51}$$

Proof. For $\lambda=u_1$ and $\lambda=u_2$ in (38), by adding the resulting inequalities, the inequality (51) is obtained. □

Corollary 3. *For $\alpha=\beta$ in (51), the following error bounds of the Hadamard-type inequality hold:*

$$\left|\frac{(\mathcal{F}_1(\mathcal{F}_2(u_2))+\mathcal{F}_1(\mathcal{F}_2(u_1)))}{2}k(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\right. \tag{52}$$
$$\left.-\frac{1}{2}\left(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_2^-}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,u_1^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_2;p)\right)\right|$$
$$\leq k\mathcal{M}\left[(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(\delta(\mathcal{F}_2(u_2)-\mathcal{F}_2(u_1))^{\frac{\rho}{k}};p)\right.$$
$$\left.-\frac{1}{2}\left(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}1\right)(u_1;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}1\right)(u_2;p)\right)\right].$$

By applying Theorem 6 at the midpoint of the interval $[u_1,u_2]$, one can achieve the following result.

Theorem 8. *Under the assumptions of Theorem 6, the following inequality holds:*

$$\left|k\left(\mathcal{F}_1(\mathcal{F}_2(u_2))\left(\mathcal{F}_2(u_2)-\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)\right)^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}\left(\delta\left(\mathcal{F}_2(u_2)-\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)\right)^{\frac{\rho}{k}};p\right)\right.\right. \tag{53}$$
$$\left.+\mathcal{F}_1(\mathcal{F}_2(u_1))\left(\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)-\mathcal{F}_2(u_1)\right)^{\frac{\alpha}{k}-1}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}\left(\delta\left(\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)-\mathcal{F}_2(u_1)\right)^{\frac{\rho}{k}};p\right)\right)$$
$$\left.-\left(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,(\frac{u_1+u_2}{2})^+}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_2;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha-k,\phi,\delta,(\frac{u_1+u_2}{2})^-}\mathcal{F}_1\circ\mathcal{F}_2\right)(u_1;p)\right)\right|$$
$$\leq k\mathcal{M}\left[\left(\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)-\mathcal{F}_2(u_1)\right)^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}\left(\delta\left(\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)-\mathcal{F}_2(u_1)\right)^{\frac{\rho}{k}};p\right)\right.$$
$$+\left(\mathcal{F}_2(u_2)-\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)\right)^{\frac{\alpha}{k}}E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}\left(\delta\left(\mathcal{F}_2(u_2)-\mathcal{F}_2\left(\frac{u_1+u_2}{2}\right)\right)^{\frac{\rho}{k}};p\right)$$
$$\left.-\left(\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,(\frac{u_1+u_2}{2})^-}1\right)(u_1;p)+\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,(\frac{u_1+u_2}{2})^+}1\right)(u_2;p)\right)\right].$$

Proof. For $\alpha=\beta$ and $\lambda=\frac{u_1+u_2}{2}$ in (38), the inequality (53) is obtained. □

Example 1. *Let $\delta=0=p$, $\mathcal{F}_2(t)=t$, $\mathcal{F}_1(t)=t^\gamma$, $\gamma>1$, and $t\in[u_1,u_2]$, $0<u_1<u_2$. Then, we have $E^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,k}(t;p)=\frac{1}{k\Gamma_k(\alpha)}$, $\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_1^+}\mathcal{F}_1\right)(\lambda;p)=\left({}^{k}\mathcal{I}_{\alpha,u_1^+}\mathcal{F}_1\right)(\lambda)$, $\left({}^{k}_{\mathcal{F}_2}\mathcal{Z}^{\vartheta,\varepsilon,\varsigma,\varpi}_{\rho,\alpha,\phi,\delta,u_2^-}\mathcal{F}_1\right)$ $(\lambda;p)=\left({}^{k}\mathcal{I}_{\alpha,u_2^-}\mathcal{F}_1\right)(\lambda)$ and $|\mathcal{F}_1'(t)|\leq\gamma u_2^{p-1}$, where $\left({}^{k}\mathcal{I}_{\alpha,u_1^+}\mathcal{F}_1\right)(\lambda)$ and $\left({}^{k}\mathcal{I}_{\alpha,u_2^-}\mathcal{F}_1\right)(\lambda)$ are k-analogs of the left and right Riemann–Liouville fractional integrals. Hence, the inequality (52) takes the following form:*

$$\left|(u_2^\gamma+u_1^\gamma)(u_2-u_1)^{\frac{\alpha}{k}}-\Gamma_k(\alpha)(u_2-u_1)\left(\left({}^{k}\mathcal{I}_{\alpha-k,u_2^-}t^\gamma\right)(u_1)+\left({}^{k}\mathcal{I}_{\alpha-k,u_1^+}t^\gamma\right)(u_2)\right)\right|$$
$$\leq\frac{2(\alpha-k)}{\alpha}\gamma u_2^{\gamma-1}(u_2-u_1)^{\frac{\alpha}{k}+1}.$$

4. Conclusions

Ostrowski-type inequalities for generalized k-fractional integrals containing the modified Mittag–Leffler function (8) were established. The outcomes of this paper include many new and existing Ostrowski-type inequalities for various types of fractional integrals. Some results are mentioned in the form of corollaries and remarks. An example is also provided.

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