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Theoretical Analysis of Boundary Value Problems for Generalized Boussinesq Model of Mass Transfer with Variable Coefficients

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Abstract: A boundary value problem is formulated for a stationary model of mass transfer, which generalizes the Boussinesq approximation in the case when the coefficients in the model equations can depend on the concentration of a substance or on spatial variables. The global existence of a weak solution of this boundary value problem is proved. Some fundamental properties of its solutions are established. In particular, the validity of the maximum principle for the substance's concentration has been proved. Sufficient conditions on the input data of the boundary value problem under consideration, which ensure the local existence of the strong solution from the space H^2, and conditions that ensure the conditional uniqueness of the weak solution with additional property of smoothness for the substance's concentration are established.

Keywords: generalized Oberbeck-Boussinesq model; global solvability; maximum principle; strong solution; local existence; conditional uniqueness

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1. Introduction and Statement of the Boundary Value Problem

Over last several decades, the significance of the study of the boundary and control problems for heat and mass transfer models has only been increasing (see [1–9]). One of the main reasons consists in the search of the effective mechanisms for controlling physical fields in continuous media. At the same time, the area of applications of control problems is only expanding.

Within the framework of the optimization approach to the control problems, some inverse problems of searching for unknown functions entering the equations or boundary conditions of the models under consideration can be reduced using additional information about the solutions of the corresponding boundary value problems (for the correctness of this approach, see [10–12]). In turn, the study of extremum problems is based on the solvability of boundary value problems and a qualitative analysis of their solutions. At the same time, the less restrictions are imposed on the original model, the more opportunities open up for applications of the control problems.

In this article, we study a boundary value problem for a nonlinear mass transfer model, which generalizes the Boussinesq approximation. It is assumed that the leading coefficients of kinematic viscosity and diffusion, as well as the reaction coefficient, depend nonlinearly on concentration, while the reaction coefficient also depends on spatial variables.

Among the papers devoted to the study of various models generalizing the Boussinesq approximation, we note [13–30]. In [13,14] the global solvability of the stationary boundary value problem for nonlinear heat transfer equations is proved in the case, when the viscosity coefficient depends on temperature. Sufficient conditions are established for the input data, at which the maximum principle for temperature is valid. The local existence and conditional uniqueness of a strong solution of the considered boundary value problem is proved.

In [15,16] the solvability of boundary value problems for the stationary Boussinesq equations of a viscous fluid, considered under mixed boundary conditions for velocity, is studied. In [17] boundary value problems are studied for stationary MHD equations for viscous heat-conducting fluid, considered both in the Boussinesq approximation and under its generalisation. In the latter case, it is assumed that the buoyancy force is a decreasing function of temperature. On one hand, it is justified from a physical point of view, and on the other hand, it allows one to prove the global solvability of boundary value problem using the Schauder fixed point theorem.

It should be noted that the cycle of articles by E.S. Baranovskii with co-authors [18–21] are devoted to the study of boundary and extremum problems for stationary models of the dynamics of viscous incompressible fluid. In detail, the model of non-isothermal creeping flows of an incompressible fluid is considered in [18]. It is assumed that the viscosity and the thermal conductivity coefficients depend on temperature. The main result of this paper includes the proof of the solvability of the boundary control problem for the model under consideration. In [19], the model of the flow of non-uniformly heated viscous fluid is studied while considered under slipping boundary conditions. The existence of a weak solution of the considered boundary value problem is proved and its additional properties are established. This article describe the situation when the coefficients of viscosity and thermal conductivity in the model equations together with the slip coefficient in the boundary condition for velocity depend on temperature.

In [20], the control problem for 2D Stokes equations with variable density and viscosity is studied. In [21], the existence of an optimal solution for the problem of boundary control of non-isothermal stationary flows of low-concentration aqueous polymer solutions in a limited three-dimensional domain is proved.

In [22,23], the global solvability of boundary value problems for nonlinear mass transfer equations was proved in the case, when the reaction coefficient depends nonlinearly on the substance's concentration and also depends on spatial variables. In [22] the homogeneous Dirichlet conditions for the velocity and substance's concentration were set on the entire boundary of the considered domain. In [23] the mixed boundary conditions were used for the concentration and the inhomogeneous Dirichlet condition was used for the velocity. Moreover, in cited papers the maximum and minimum principle for the substance's concentration was established.

In [22], the existence and the conditional uniqueness of the solution of the problem of distributed control is proved, while in [23], the multiplicative control problem was studied. In particular, for a specific reaction coefficient and for several types of cost functionals, the conditional stability estimates for optimal solutions with respect to small perturbations of cost functionals were obtained. The global solvability of boundary value problem for the above mentioned mass transfer equations under non-homogeneous Dirichlet condition for the substance concentration was proved firstly in [24]. Let us note the papers [25–30], devoted to the study of non-stationary models, which generalize the Boussinesq approximation, as well as articles [31–35], in which a number of complicated hydrodynamic, including rheological, models was studied.

From the one side, in the current paper, a number of results, regarding the research of boundary value problems for nonlinear mass transfer equations in the framework of the classical Boussinesq approximation, obtained in [2,3] and in [5–7], was generalized. From the other side, we have also generalised some results from the articles [12,13,22,23,36–39], which include the study of boundary value problems for nonlinear mass transfer equations with variable coefficients.

For example, in [38] the reaction-diffusion-convection equation was considered under inhomogeneous mixed boundary conditions for the substance's concentration. It was assumed that the reaction coefficient in the equation and the mass transfer coefficient in the boundary condition depend nonlinearly on the substance's concentration and also depend on spatial variables.

In [39], the boundary value problem for a nonlinear reaction-diffusion-convection equation under inhomogeneous Dirichlet condition was considered. In this case, the nonlinearity, generated by the reaction coefficient, is monotonic only in a certain subdomain of the considered domain, while in the rest subdomain, the reaction coefficient is bounded by the L^p–norm, where $p \geq 5/3$. Since that, for the solvability of the boundary value problem under consideration the Leray-Schauder principle was used instead of the monotonicity of the corresponding operator, as in [38]. In [38,39], the maximum and minimum principle for the substance's concentration was also established.

Finally, we note articles [40–42] close to [12,36–39], devoted to the study of boundary and control problems for the models of complex heat transfer.

In a bounded domain $\Omega \subset \mathbb{R}^3$ with a boundary Γ the following boundary value problem is considered:

$$-\operatorname{div}(\nu(\varphi)\nabla \mathbf{u}) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} + \beta\mathbf{G}\varphi,\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \tag{1}$$

$$-\operatorname{div}(\lambda(\varphi)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + k(\varphi,\mathbf{x})\varphi = f \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \text{ and } \varphi = 0 \text{ on } \Gamma. \tag{3}$$

Here, $\mathbf{u}$ is a velocity vector, function φ represents the concentration of substance, $p = P/\rho$, where P is pressure, $\rho = \text{const}$ is fluid density, $\nu = \nu(\varphi) > 0$ is the coefficient of kinematic viscosity, $\lambda = \lambda(\varphi) > 0$ is the diffusion coefficient, β is the coefficient of mass expansion, $\mathbf{G} = -(0,0,G)$ is the acceleration of gravity, $\mathbf{f}$ or f are volume densities of external forces or external sources of substance, respectively, and the function $k = k(\varphi,\mathbf{x})$ is the reaction coefficient, where $\mathbf{x} \in \Omega$. Below, we will refer to the problem (1)–(3) for given functions $\nu, \lambda, \mathbf{f}, f$ and k as to Problem 1.

In this article, we prove the global existence of a weak solution of Problem 1 in the case, when diffusion, viscosity, and reaction coefficients depend on the concentration of substance. In this case, the reaction coefficient also depends on spatial variables. Under additional conditions on the input data of Problem 1, the maximum principle is established for the concentration φ. Further, for a smoother boundary $\Gamma \in C^2$ of Ω we prove a local existence of a strong solution to Problem 1 and conditional uniqueness of its weak solution with additional property that $\Delta\varphi \in L^2(\Omega)$.

Let us introduce a brief outline of this article below. In the second section, the functional spaces are introduced, auxiliary results are given and the global existence of weak solution of Problem 1 is proved. In Section 3, the maximum principle for the concentration φ is established. In Section 4, the local existence of a strong solution of Problem 1 is obtained. Section 5 includes the sufficient conditions on the input data of Problem 1, which provide conditional uniqueness of the weak solution with additional property that $\Delta\varphi \in L^2(\Omega)$. Section 6 contains a discussion of the prospects for the application of the obtained results in the study of new boundary value and control problems. In the last Section 7, our results are briefly summarized and concluding comments are given.

2. Solvability of the Boundary Value Problem

Below, we will use the Sobolev functional spaces $H^s(D)$, $s \in \mathbb{R}$. Here, D means either a domain Ω or some subset $Q \subset \Omega$, or the boundary Γ. By $\|\cdot\|_{s,Q}$, $|\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$, respectively. The norms and the scalar product in $L^2(Q)$ and $L^2(\Omega)$ will be denoted by $\|\cdot\|_Q$, $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)$, correspondingly. By X^* we will denote the adjoint space of Hilbert space X, while the duality for a pair X and X^* is written as $\langle\cdot,\cdot\rangle_{X^*\times X}$ or simply as $\langle\cdot,\cdot\rangle$.

We will use the following functional spaces:

$$H^0(\operatorname{div},\Omega) = \{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{h} = 0 \text{ in } \Omega\},$$

$$L^p_+(\Omega) = \{k \in L^p(\Omega) : k \geq 0\}, \quad p \geq 3/2,$$

$$L_0^2(\Omega) = \{h \in L^2(\Omega) : (h, 1) = 0\},$$
$$\mathcal{D}(\Omega) = \{\mathbf{v} \in C_0^\infty(\Omega)^3 : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\},$$
$$H \text{ is the closure } \mathcal{D}(\Omega) \text{ in } L^2(\Omega)^3,$$
$$V \text{ is the closure } \mathcal{D}(\Omega) \text{ in } H^1(\Omega)^3.$$

It is well known, see e.g., [43], that for the domain Ω with Lipschitz boundary the spaces H and V are characterized as follows:

$$H = \{\mathbf{v} \in H^0(\operatorname{div}, \Omega) : \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0 \text{ in } H^{-1/2}(\Gamma)\},$$
$$V = \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}.$$

We define the products of the spaces $X = H_0^1(\Omega)^3 \times H_0^1(\Omega)$, $W = V \times H_0^1(\Omega)$ with the norm

$$\|\mathbf{x}\|_X^2 = \|\mathbf{u}\|_{1,\Omega}^2 + \|\varphi\|_{1,\Omega}^2 \quad \forall \mathbf{x} \equiv (\mathbf{u}, \varphi) \in X \text{ (or } (\mathbf{u}, \varphi) \in W)$$

and the space $X^* = (H^{-1}(\Omega)^3)^* \times H^{-1}(\Omega)$ which is the dual of X.

Let the following conditions be satisfied:

2.1. Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$;

2.2. $\mathbf{f} \in H^{-1}(\Omega)^3$, $f \in H^{-1}(\Omega)$, $\mathbf{b} = \beta\mathbf{G} \in L^2(\Omega)^3$;

2.3. for any function $\varphi \in H_0^1(\Omega)$ the embedding $k(\varphi, \cdot) \in L_+^p(\Omega)$ is true, $p \geq 3/2$, where p does not depend on φ; and for any sphere $B_r = \{\varphi \in H_0^1(\Omega) : \|\varphi\|_{1,\Omega} \leq r\}$ of radius r the following inequality takes place:

$$\|k(\varphi_1, \cdot) - k(\varphi_2, \cdot)\|_{L^p(\Omega)} \leq L\|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \quad \forall \varphi_1, \varphi_2 \in B_r.$$

Here, L is the constant, which depends on r, but does not depend on $\varphi_1, \varphi_2 \in B_r$;

2.4. the functions $\nu(\tau)$ and $\lambda(\tau)$ are continuous as $\tau \in \mathbb{R}$, and there are positive constants $\nu_{\min}, \nu_{\max}, \lambda_{\min}$ and $\lambda_{\max}$ such that

$$0 < \nu_{\min} \leq \nu(\tau) \leq \nu_{\max},\ 0 < \lambda_{\min} \leq \lambda(\tau) \leq \lambda_{\max}\ \forall \tau \in \mathbb{R}.$$

Note that the condition **2.3** describes an operator from $H_0^1(\Omega)$ to $L^p(\Omega)$, where $p \geq 3/2$ (see [12,36]). For example,

$$\tilde{k}_1(\varphi, \cdot) = \varphi^2 \text{ (or } \tilde{k}_1(\varphi) = \varphi^2|\varphi|) \text{ in subdomain } Q \subset \Omega \text{ and}$$
$$\tilde{k}_1(\varphi, \mathbf{x}) = k_0(\mathbf{x}) \in L_+^{3/2}(\Omega \setminus \overline{Q}) \text{ in } \Omega \setminus \overline{Q}.$$

Let us consider the function $\mu(\tau)$, where $\tau \in \mathbb{R}$, which satisfies the condition **2.4**, i.e., this function is continuous and satisfies the following condition:

$$0 < \mu_{\min} \leq \mu(\tau) \leq \mu_{\max} < \infty.$$

It is clear that $\mu(h) \in L^\infty(\Omega)$ for any $h \in H^1(\Omega)$, and $|\mu(h)| \leq \mu_{\max}$ a.e. in Ω, $\|\mu(h)\|_{L^\infty(\Omega)} \leq \mu_{\max}$. Besides

$$\mu(h_n) \to \mu(h) \text{ a.e. in } \Omega, \text{ if } h_n \to h \text{ a.e. in } \Omega, \text{ as } n \to \infty.$$

Let $\mu_n = \mu(h_n)$. Since $|\mu(h_n)| \leq \mu_{\max}$ a.e. in Ω, then by the Lebesgue theorem on majorant convergence we obtain that

$$\int_\Omega \mu_n f d\mathbf{x} \to \int_\Omega \mu f d\mathbf{x} \text{ as } n \to \infty\ \forall f \in L^1(\Omega). \tag{4}$$

It follows from the above that

$$\int_{\Omega}(\mu_n-\mu)^2 f d\mathbf{x} \to 0 \text{ as } n\to\infty\ \forall f\in L^1(\Omega). \tag{5}$$

It is the property (5) that will be used to prove the solvability of Problem 1.

Here is an example of a function $\mu(\varphi)$ that satisfies the condition **2.4** and can describe both the diffusion coefficient $\lambda(\varphi)$ and the viscosity coefficient $\nu(\varphi)$:

$$\mu(\varphi)=\frac{1}{1+\varphi^2}+1,\ \ \mu_{\min}=1,\ \ \mu_{\max}=2.$$

Recall that, by the Sobolev embedding theorem, the space $H^1(\Omega)$ embeds into the space $L^s(\Omega)$ continuously for $s\le 6$, and compactly for $s<6$ and with some constant C_s depending on s and Ω, we have the estimate

$$\|\varphi\|_{L^s(\Omega)}\le C_s\|\varphi\|_{1,\Omega}\ \ \forall\varphi\in H^1(\Omega). \tag{6}$$

The following technical lemma holds (see details in [3,5,44,45]).

Lemma 1. *Let the conditions* **2.1** *and* **2.4** *hold and* $k_0\in L^p(\Omega), p\ge 3/2, \mathbf{u}\in V, \mathbf{b}\in L^2(\Omega)^3$. *Then, there exists positive constants* $\delta_0,\delta_1,\gamma_1,\gamma_1',\gamma_2,\gamma_2',\gamma_p,\beta$ *and* β_0, *which depend on* Ω *or depends on* Ω *and* p, *such that the following relations hold:*

$$|(\nu(h)\nabla\mathbf{v},\nabla\mathbf{w})|\le\nu_{\max}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}\ \ \forall\mathbf{v},\mathbf{w}\in H_0^1(\Omega)^3, \tag{7}$$

$$(\nabla\mathbf{v},\nabla\mathbf{v})\ge\delta_0\|\mathbf{v}\|^2_{1,\Omega},$$
$$(\nu(\varphi)\nabla\mathbf{v},\nabla\mathbf{v})\ge\nu_*\|\mathbf{v}\|^2_{1,\Omega}\ \forall\mathbf{v}\in H_0^1(\Omega)^3,\ \varphi\in H_0^1(\Omega),$$
$$\nu_*=\nu_{\min}\delta_0, \tag{8}$$

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})=-((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}),\ ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{v})=0\ \forall\mathbf{v},\mathbf{w}\in H_0^1(\Omega)^3, \tag{9}$$

$$|(\mathbf{b}h,\mathbf{v})|\le\beta_0\|\mathbf{b}\|_{\Omega}\|h\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\ \forall\mathbf{v}\in H_0^1(\Omega)^3, h\in H_0^1(\Omega), \tag{10}$$

$$|((\mathbf{w}\cdot\nabla)\mathbf{h},\mathbf{v})|\le\gamma_1'\|\mathbf{w}\|_{L^4(\Omega)^3}\|\mathbf{h}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\le$$
$$\le\gamma_1\|\mathbf{w}\|_{1,\Omega}\|\mathbf{h}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\ \forall\mathbf{w},\mathbf{h},\mathbf{v}\in H_0^1(\Omega)^3, \tag{11}$$

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^3,\mathbf{v}\ne\mathbf{0}} -(\mathrm{div}\mathbf{v},p)/\|\mathbf{v}\|_{1,\Omega}\ge\beta\|p\|_{\Omega}\ \ \forall p\in L_0^2(\Omega), \tag{12}$$

$$|(\lambda(\varphi)\nabla h,\nabla\eta)|\le\lambda_{\max}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \forall\varphi,h,\eta\in H_0^1(\Omega), \tag{13}$$

$$|(k_0h,\eta)|\le\gamma_p\|k_0\|_{L^p(\Omega)}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \forall h,\eta\in H_0^1(\Omega), \tag{14}$$

$$|(\mathbf{w}\cdot\nabla h,\eta)|\le\gamma_2'\|\mathbf{w}\|_{L^4(\Omega)^3}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}\le$$
$$\le\gamma_2\|\mathbf{w}\|_{1,\Omega}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \forall\mathbf{w}\in H_0^1(\Omega)^3,\ h,\eta\in H_0^1(\Omega), \tag{15}$$

$$(\nabla h,\nabla h)\ge\delta_1\|h\|^2_{1,\Omega},\ \ (\lambda(\varphi)\nabla h,\nabla h)\ge\lambda_*\|h\|^2_{1,\Omega}$$
$$\forall h,\varphi\in H_0^1(\Omega),\ \ \ \lambda_*\equiv\delta_1\lambda_{\min}. \tag{16}$$

$$(\mathbf{u}\cdot\nabla h,h)=0\ \ \ \forall h\in H_0^1(\Omega). \tag{17}$$

From (14) and from condition **2.3**, it follows:

$$|((k(\varphi_1,\cdot)-k(\varphi_2,\cdot))\varphi,\eta)|\le$$
$$\le\gamma_p\|k(\varphi_1,\cdot)-k(\varphi_2,\cdot)\|_{L^p(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega}\le$$

$$\leq \gamma_p L\|\varphi_1 - \varphi_2\|_{L^4(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \forall \varphi, \varphi_1, \varphi_2, \eta \in H_0^1(\Omega),\ p \geq 3/2. \tag{18}$$

We multiply the first equation in (1) by a function $\mathbf{v} \in H_0^1(\Omega)^3$, Equation (2) by a function $h \in H_0^1(\Omega)$ and integrate over Ω using Green's formulae. Then, we obtain the weak formulation of Problem 1. It consists in finding the triple $(\mathbf{u}, \varphi, p) \in H_0^1(\Omega)^3 \times H_0^1(\Omega) \times L_0^2(\Omega)$, satisfying the relations:

$$(\nu(\varphi)\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) =$$

$$= \langle \mathbf{f}, \mathbf{v}\rangle + (\mathbf{b}\varphi, \mathbf{v})\ \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{19}$$

$$(\lambda(\varphi)\nabla\varphi, \nabla h) + (k(\varphi, \cdot)\varphi, h) + (\mathbf{u}\cdot\nabla\varphi, h) = \langle f, h\rangle\ \forall h \in H_0^1(\Omega), \tag{20}$$

$$\operatorname{div}\mathbf{u} = 0 \text{ in } \Omega. \tag{21}$$

The specified triple $(\mathbf{u}, \varphi, p)$, satisfying (19)–(21), will be called a weak solution of Problem 1.

Let us consider the restriction of the identity (19) to the space V:

$$(\nu(\varphi)\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle + (\mathbf{b}\varphi, \mathbf{v})\ \ \forall \mathbf{v} \in V. \tag{22}$$

To prove the existence of a weak solution to Problem 1 it suffices to prove the existence of a solution $(\mathbf{u}, \varphi) \in H_0^1(\Omega)^3 \times H_0^1(\Omega)$ of problem (20)–(22). About pressure recovery see for details in ([43], p. 134, [44], p. 89).

To prove the solvability of the problem (20)–(22), we apply the Schauder fixed-point theorem (see [44]). We set $\mathbf{z} = (\mathbf{s}, c) \in W$ and $\mathbf{y} = (\mathbf{u}, \varphi) \in W$ and construct the operator $F : W \to W$, acting according to the formula: $F(\mathbf{z}) = \mathbf{y}$, where $\mathbf{y} = (\mathbf{u}, \varphi) \in W$ is the solution to the linear problem

$$a_1^{c\,\mathbf{s}}(\mathbf{u}, \mathbf{v}) = (\nu(c)\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{s}\cdot\nabla)\mathbf{u}, \mathbf{v}) =$$

$$= \langle \mathbf{f}, \mathbf{v}\rangle + (\mathbf{b}\,\varphi, \mathbf{v})\ \ \forall \mathbf{v} \in V, \tag{23}$$

$$a_2^{c\,\mathbf{s}}(\varphi, h) = (\lambda(c)\nabla\varphi, \nabla h) + (k(c, \cdot)\varphi, h) + (\mathbf{s}\cdot\nabla\varphi, h) =$$

$$= \langle f, h\rangle\ \ \forall h \in H_0^1(\Omega). \tag{24}$$

From the estimates (14)–(16) and from the equality (17), it follows that for every fixed pair $(\mathbf{s}, c) \in V \times H_0^1(\Omega)$ the form $a_2^{c\,\mathbf{s}} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is continuous and coercive with the constant λ_* defined in (16). Since $f \in H^{-1}(\Omega)$, it follows by virtue of the Lax-Milgram theorem that for any pair $\mathbf{s} \in V$, $c \in H_0^1(\Omega)$ there is a unique solution $\varphi \in H_0^1(\Omega)$ of problem (24) and the following estimate holds

$$\|\varphi\|_{1,\Omega} \leq M_\varphi \equiv C_*\|f\|_{-1,\Omega},\ \ C_* = \lambda_*^{-1} \equiv (\delta_1\lambda_{\min})^{-1}. \tag{25}$$

In turn, from the estimates (8), (11) and from the equality (9) it follows that the form $a_1^{c\,\mathbf{s}} : V \times V \to \mathbb{R}$ is continuous and coercive with constant ν_*. Moreover, $\mathbf{f} \in V^*$. Therefore for any pair $(\mathbf{s}, c) \in V \times H_0^1(\Omega)$ there exists a unique solution $\mathbf{u} \in V$ to problem (23).

Put $\mathbf{v} = \mathbf{u}$ in (23). From (9)–(11) follows the next inequality:

$$\nu_*\|\mathbf{u}\|_{1,\Omega}^2 \leq \|\mathbf{f}\|_{-1,\Omega}\|\mathbf{u}\|_{1,\Omega} + \beta_0\|\mathbf{b}\|_\Omega\|\varphi\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}. \tag{26}$$

From (26), taking into account (25), we deduce the following estimate

$$\|\mathbf{u}\|_{1,\Omega} \leq M_\mathbf{u} = \nu_*^{-1}(\|\mathbf{f}\|_{-1,\Omega} + \beta_0\|\mathbf{b}\|_\Omega M_\varphi). \tag{27}$$

Thus, we have proved that for any pair $(\mathbf{s}, c) \in W \equiv V \times H_0^1(\Omega)$ there is a unique solution $\mathbf{y} = (\mathbf{u}, \varphi) \in W$ of problem (23), (24), for which the following estimate holds:

$$\|\mathbf{y}\|_X \equiv (\|\mathbf{u}\|_{1,\Omega}^2 + \|\varphi\|_{1,\Omega}^2)^{1/2} \leq M_\mathbf{u} + M_\varphi. \tag{28}$$

In the space W, we define the ball $B_r = \{\mathbf{y} \equiv (\mathbf{u}, \varphi) \in W : \|\mathbf{y}\|_X \leq r\}$, where $r = M_{\mathbf{u}} + M_\varphi$. From the construction of the ball B_r and from (28) it follows that the operator F, defined above, maps the ball B_r into itself.

We prove that the operator F is continuous and compact on the ball B_r. Let $\mathbf{z}_n = (\mathbf{s}_n, c_n)$, $n = 1, 2, \ldots$ is an arbitrary sequence from B_r. Due to the reflexivity of the spaces $H^1(\Omega)$ and $H^1(\Omega)^3$ and the compactness of the embeddings $H^1(\Omega) \subset L^4(\Omega)$ and $H^1(\Omega)^3 \subset L^4(\Omega)^3$, there is a subsequence of the sequence $\{\mathbf{z}_n\} = \{(\mathbf{s}_n, c_n)\}$, which we also denote by $\{\mathbf{z}_n\}$, and there is the pair $\mathbf{z} = (\mathbf{s}, c) \in B_r$ such that

$$\mathbf{s}_n \to \mathbf{s} \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^4(\Omega)^3 \text{ as } n \to \infty,$$

$$c_n \to c \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^4(\Omega) \text{ as } n \to \infty. \tag{29}$$

Let $\mathbf{y} = F(\mathbf{z})$, $\mathbf{y}_n = F(\mathbf{z}_n)$. These relations are equivalent to the fact that the element $\mathbf{y} \equiv (\mathbf{u}, \varphi) \in W$ is a solution to the problem (23), (24), and $\mathbf{y}_n = (\mathbf{u}_n, \varphi_n) \in W$ is the solution to the problem

$$(\nu(c_n)\nabla\mathbf{u}_n, \nabla\mathbf{v}) + ((\mathbf{s}_n \cdot \nabla)\mathbf{u}_n, \mathbf{v}) =$$
$$= \langle \mathbf{f}, \mathbf{v}\rangle + (\mathbf{b}\varphi_n, \mathbf{v}) \ \ \forall \mathbf{v} \in V, \tag{30}$$
$$(\lambda(c_n)\nabla\varphi_n, \nabla h) + (k(c_n, \cdot)\varphi_n, h) + (\mathbf{s}_n \cdot \nabla\varphi_n, h) =$$
$$= \langle f, h\rangle \ \ \forall h \in H^1_0(\Omega), \tag{31}$$

which is obtained from (23), (24) by replacing $\mathbf{z} = (\mathbf{s}, c)$ with $\mathbf{z}_n = (\mathbf{s}_n, c_n)$.

Let us show that $\mathbf{y}_n \to \mathbf{y}$ strongly in X or, equivalently,

$$\varphi_n \to \varphi \text{ strongly in } H^1(\Omega) \text{ and } \mathbf{u}_n \to \mathbf{u} \text{ strongly in } H^1(\Omega)^3 \text{ as } n \to \infty.$$

To do this, subtract (23), (24) from (30), (31). Taking into account the following equalities:

$$(k(c_n, \cdot)\varphi_n, h) - (k(c, \cdot)\varphi, h) =$$
$$= (k(c_n, \cdot)(\varphi_n - \varphi), h) + ((k(c_n, \cdot) - k(c, \cdot))\varphi, h),$$
$$(\lambda(c_n)\nabla\varphi_n, \nabla h) - (\lambda(c)\nabla\varphi, \nabla h) =$$
$$(\lambda(c_n)\nabla(\varphi_n - \varphi), h) + ((\lambda(c_n) - \lambda(c))\nabla\varphi, \nabla h),$$
$$(\nu(c_n)\nabla\mathbf{u}_n, \nabla\mathbf{v}) - (\nu(c)\nabla\mathbf{u}, \nabla\mathbf{v}) =$$
$$= (\nu(c_n)\nabla(\mathbf{u}_n - \mathbf{u}), \nabla\mathbf{v}) + ((\nu(c_n) - \nu(c))\nabla\mathbf{u}, \nabla\mathbf{v}),$$

we come to the relations:

$$(\lambda(c_n)\nabla(\varphi_n - \varphi), h) + (k(c_n, \cdot)(\varphi_n - \varphi), h) + (\mathbf{s}_n \cdot \nabla(\varphi_n - \varphi), h) =$$
$$= -((\lambda(c_n) - \lambda(c))\nabla\varphi, \nabla h) - ((\mathbf{s}_n - \mathbf{s}) \cdot \nabla\varphi, h) -$$
$$- ((k(c_n, \cdot) - k(c, \cdot))\varphi, h) \ \ \forall h \in H^1_0(\Omega), \tag{32}$$
$$(\nu(c_n)\nabla(\mathbf{u}_n - \mathbf{u}), \nabla\mathbf{v}) + ((\mathbf{s}_n \cdot \nabla)(\mathbf{u}_n - \mathbf{u}), \mathbf{v}) =$$
$$= -((\nu(c_n) - \nu(c))\nabla\mathbf{u}, \nabla\mathbf{v}) - (((\mathbf{s}_n - \mathbf{s}) \cdot \nabla)\mathbf{u}, \mathbf{v}) +$$
$$+ (\mathbf{b}(\varphi_n - \varphi), \mathbf{v}) \ \forall \mathbf{v} \in V. \tag{33}$$

Using the estimate (18) with $\varphi_1 = c_n$, $\varphi_2 = c$ and estimate (25), we deduce that

$$|((k(c_n, \cdot) - k(c, \cdot))\varphi, h)| \leq$$
$$\leq \gamma_p L M_\varphi \|c_n - c\|_{L^4(\Omega)} \|h\|_{1,\Omega} \to 0 \text{ as } n \to \infty \ \forall h \in H^1_0(\Omega). \tag{34}$$

Substituting $h = \varphi - \varphi_n$ into (32) and using (14)–(17), just like (34), we arrive at the inequality

$$\lambda_* \|\varphi - \varphi_n\|_{1,\Omega} \leq \|(\lambda(c_n) - \lambda(c))\nabla\varphi\|_\Omega +$$

$$+ \gamma_2' M_\varphi \|\mathbf{s}_n - \mathbf{s}\|_{L^4(\Omega)^3} + \gamma_p L M_\varphi \|c_n - c\|_{L^4(\Omega)}. \tag{35}$$

From (35) due to properties (5) and (29) we deduce that $\|\varphi_n - \varphi\|_{1,\Omega} \to 0$ as $n \to \infty$. Setting $\mathbf{v} = \mathbf{u} - \mathbf{u}_n$ in (33), taking into account (9), we obtain that

$$(\nu(c_n)\nabla(\mathbf{u}_n - \mathbf{u}), \nabla(\mathbf{u}_n - \mathbf{u})) =$$

$$= -((\nu(c_n) - \nu(c))\nabla\mathbf{u}, \nabla(\mathbf{u}_n - \mathbf{u})) -$$

$$-(((\mathbf{s}_n - \mathbf{s}) \cdot \nabla)\mathbf{u}, \mathbf{u}_n - \mathbf{u}) + (\mathbf{b}(\varphi_n - \varphi), \mathbf{u}_n - \mathbf{u}). \tag{36}$$

Using the estimates (8), (10), (11), from (36) we obtain the following inequality:

$$\nu_* \|\mathbf{u}_n - \mathbf{u}\|_{1,\Omega} \leq \|(\nu(c_n) - \nu(c))\nabla\mathbf{u}\|_\Omega +$$

$$+ \gamma_1' \|\mathbf{s}_n - \mathbf{s}\|_{L^4(\Omega)^3} \|\mathbf{u}\|_{1,\Omega} + \beta_0 \|\mathbf{b}\|_\Omega \|\varphi_n - \varphi\|_{1,\Omega}. \tag{37}$$

From (37), taking into account the properties (5), (29), and (35), we conclude that $\|\mathbf{u} - \mathbf{u}_n\|_{1,\Omega} \to 0$ as $n \to \infty$.

Therefore, the operator F is continuous and compact. In this case, it follows from the Schauder fixed-point theorem that the operator F has a fixed point $\mathbf{y} = F(\mathbf{y}) \in W$, which is the solution to the problem (20)–(22). By construction, this solution $\mathbf{y} = (\mathbf{u}, \varphi)$ satisfies the estimates (25), (27).

The existence of pressure $p \in L^2_0(\Omega)$, which together with the specified pair $(\mathbf{u}, \varphi)$ satisfies the relation (19), is proved as in ([44], p. 89). It remains to derive an estimate for p. For this purpose, we will use relation (12), according to which for the function p and any (arbitrarily small) number $\delta > 0$ there exists a function $\mathbf{v}_0 \in H^1_0(\Omega)^3$, $\mathbf{v}_0 \neq 0$, such that

$$-(\mathrm{div}\mathbf{v}_0, p) \geq \beta_* \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega, \ \beta_* = (\beta - \delta) > 0.$$

Setting $\mathbf{v} = \mathbf{v}_0$ in (19), taking into account the last inequality and estimates (7), (10), (11), we deduce that

$$\beta_* \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega \leq \nu_{\max} \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} + \gamma_1 \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|^2_{1,\Omega} +$$

$$+ \beta_0 \|\mathbf{b}\|_\Omega \|\varphi\|_{1,\Omega} \|\mathbf{v}_0\|_{1,\Omega} + \|\mathbf{f}\|_{-1,\Omega} \|\mathbf{v}_0\|_{1,\Omega}.$$

Dividing by $\|\mathbf{v}_0\|_{1,\Omega} \neq 0$ and taking into account the estimates (25), (27), we deduce from this that

$$\|p\|_\Omega \leq M_p = \beta_*^{-1}[(\nu_{\max} + \gamma_1 M_\mathbf{u}) M_\mathbf{u} + \|\mathbf{f}\|_{-1,\Omega} + \beta_0 \|\mathbf{b}\|_\Omega M_\varphi]. \tag{38}$$

Let us formulate the obtained result in the form of the following theorem.

Theorem 1. *Let the conditions* **2.1–2.4** *be satisfied. Then, there exists the weak solution* $(\mathbf{u}, \varphi, p) \in H^1_0(\Omega)^3 \times H^1_0(\Omega) \times L^2_0(\Omega)$ *of Problem 1 and the estimates (25), (27) and (38) hold.*

3. Maximum Principle

In this section, we establish sufficient conditions on the input data of Problem 1 under which the maximum principle is valid for the component φ of the solution $(\mathbf{u}, \varphi, p)$ of Problem 1.

Let $f_{\max}$ be a positive number and, in addition to **2.1–2.4**, the following condition is satisfied:

3.1. $f \in L^2(\Omega) : 0 \leq f \leq f_{\max}$ a.e. in Ω;

3.2. the nonlinearity $k(\varphi,\cdot)\varphi$ is monotonic in the following sense:

$$(k(\varphi_1,\cdot)\varphi_1 - k(\varphi_2,\cdot)\varphi_2, \varphi_1 - \varphi_2) \geq 0 \;\forall \varphi_1, \varphi_2 \in H^1(\Omega).$$

We assume that the reaction coefficient has the following form:
3.3. $k(\varphi, \mathbf{x}) = a(\mathbf{x})k_1(\varphi)$, where $k_1(\cdot) : \mathbb{R} \to \mathbb{R}_+$ is a continuous function, $0 < a_{\min} \leq a(\mathbf{x}) \leq a_{\max} < \infty$ a.e. in Ω and the equation

$$k_1(s)s = f_{\max}/a_{\min} \tag{39}$$

has at least one (positive) solution.

Theorem 2. *Let under conditions* **2.1**–**2.3** *and* **3.1**–**3.3**, *the functions* $\nu(\tau)$ *and* $\lambda(\tau)$ *are continuous as* $\tau \in \mathbb{R}$, *and*

$$\nu_{\min} \leq \nu(\tau) < \infty, \; \lambda_{\min} \leq \lambda(\tau) < \infty \;\forall \tau \in \mathbb{R}.$$

Then, for the component φ *of the weak solution* $(\mathbf{u}, \varphi, p) \in H_0^1(\Omega)^3 \times H_0^1(\Omega) \times L_0^2(\Omega)$ *of Problem 1 the maximum principle holds true:*

$$0 \leq \varphi \leq M \text{ a.e. in } \Omega. \tag{40}$$

Here M is the minimum root of the Equation (39).

Proof of Theorem 2. First we prove that $\varphi \leq M$ a.e. in Ω. To this end, we introduce the function $\tilde{\varphi} = \max\{\varphi - M, 0\}$. It is clear that the maximum principle or estimate $\varphi \leq M$ a.e. in Ω is executed if and only if $\tilde{\varphi} = 0$ a.e. in Ω.

Denote by $Q_M \subset \Omega$ an open measurable subset of Ω in which $\varphi > M$. From, ([46], p. 152) and [47] it follows that $\nabla\tilde{\varphi} = \nabla\varphi$ a.e. in Q_M and $\tilde{\varphi} \in H_0^1(\Omega)$.

Then, the following equalities are true:

$$(\lambda(\varphi)\nabla\varphi, \nabla\tilde{\varphi}) = (\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi})_{Q_M} = (\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}),$$

$$(\mathbf{u}\cdot\nabla\varphi, \tilde{\varphi}) = (\mathbf{u}\cdot\nabla\tilde{\varphi}, \tilde{\varphi}) = 0.$$

With this in mind, setting $h = \tilde{\varphi}$ in (20), we obtain that

$$(\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}) + (k(\varphi,\cdot)\varphi, \tilde{\varphi}) = (f, \tilde{\varphi}). \tag{41}$$

From the properties of $\tilde{\varphi}$ the following equalities hold:

$$(k(\varphi,\cdot)\varphi, \tilde{\varphi}) = (k(\varphi,\cdot)\varphi, \tilde{\varphi})_{Q_M} =$$
$$= (k(\tilde{\varphi}+M,\cdot)(\tilde{\varphi}+M), \tilde{\varphi})_{Q_M}.$$

By virtue of **3.2** for the functions $\varphi_1 = \tilde{\varphi} + M$ and $\varphi_2 = M$ from $H^1(\Omega)$ the following equality holds:

$$0 \leq (k(\tilde{\varphi}+M,\cdot)(\tilde{\varphi}+M) - k(M,\cdot)M, \tilde{\varphi}) =$$
$$= (k(\tilde{\varphi}+M,\cdot)(\tilde{\varphi}+M) - k(M,\cdot)M, \tilde{\varphi})_{Q_M}, \tag{42}$$

because $\tilde{\varphi} = 0$ in $\Omega \setminus \overline{Q}_M$.

With this in mind, subtracting $(k(M,\cdot)M, \tilde{\varphi}) \equiv (a(\cdot)k_1(M)M, \tilde{\varphi})_{Q_M}$ from both parts (41), we obtain

$$(\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}) + (k(\tilde{\varphi}+M,\cdot)(\tilde{\varphi}+M) - k(M,\cdot)M, \tilde{\varphi})_{Q_M} =$$
$$= (f(\cdot) - a(\cdot)k_1(M)M, \tilde{\varphi})_{Q_M}. \tag{43}$$

By Lemma 2.1 and (42), from (43) we come to the estimate

$$\lambda_* \|\tilde{\varphi}\|^2_{1,\Omega} \leq (f_{\max} - a_{\min} k_1(M)M, \tilde{\varphi})_{Q_M}.$$

It follows from the last estimate that if M is chosen from the condition (39), then $\tilde{\varphi} = 0$.

To prove the minimum principle, we introduce the function $\tilde{w} = \min\{\varphi, 0\}$. Arguing as for the function $\tilde{\varphi}$, we conclude that $\tilde{w} \in H^1_0(\Omega)$. We will assume that in measurable open set $Q_m \subset \Omega$ the inequality $\varphi < 0$ is valid. Arguing as above, we arrive at the equality

$$(\lambda(\varphi)\nabla\tilde{w}, \nabla\tilde{w}) + (k(\varphi, \cdot)\tilde{w}, \tilde{w})_{Q_m} = (f, \tilde{w})_{Q_m},$$

from which the estimate follows

$$\lambda_* \|\tilde{w}\|^2_{1,\Omega} \leq 0.$$

It is clear that from the last estimate it follows that $\tilde{w} = 0$. $\square$

Remark 1. *For power-law reaction coefficients, the parameter M is easily calculated. For example, for $k(\varphi) = \varphi^2$ we obtain that $M = f^{1/3}_{\max}$.*

4. Existence of Strong Solution

In this section, we will prove the local existence of a strong solution to Problem 1. For this purpose, we will use the equivalence between the L^2-norm of the Laplace operator and the standard norm $\|\cdot\|_{2,\Omega}$ in the space $H^2(\Omega) \cap H^1_0(\Omega)$ for the domain Ω with a boundary $\Gamma \in C^2$ and similar result for spaces of vector-functions (see [43,48]). This equivalence is described by the following inequalities:

$$\|\Delta h\|_\Omega \leq \tilde{C}_1 \|h\|_{2,\Omega}, \quad \|h\|_{2,\Omega} \leq \tilde{C}_2 \|\Delta h\|_\Omega \quad \forall h \in H^2(\Omega) \cap H^1_0(\Omega),$$

$$\|\Delta \mathbf{v}\|_\Omega \leq \tilde{C}_3 \|\mathbf{v}\|_{2,\Omega}, \quad \|\mathbf{v}\|_{2,\Omega} \leq \tilde{C}_4 \|\Delta \mathbf{v}\|_\Omega \quad \forall \mathbf{v} \in H^2(\Omega)^3 \cap H^1_0(\Omega)^3. \tag{44}$$

Here and below $\tilde{C}_i$, $i = 1, 2, \ldots$ are positive constants, which depend on Ω.

Below, we will also use the following estimates:

$$\|\nabla h\|_{L^4(\Omega)^3} \leq \tilde{C}_5 \|\Delta h\|_\Omega \quad \forall h \in H^2(\Omega) \cap H^1_0(\Omega),$$

$$\|\nabla \mathbf{v}\|_{L^4(\Omega)^3} \leq \tilde{C}_6 \|\Delta \mathbf{v}\|_\Omega \quad \mathbf{v} \in H^2(\Omega)^3 \cap H^1_0(\Omega)^3, \tag{45}$$

that are a consequence of the embedding theorem and of estimates (44), and estimates

$$\|h\|_{L^p(\Omega)} \leq B_p \|h\|_{2,\Omega} \quad \forall h \in H^2(\Omega),$$

$$\|\mathbf{v}\|_{L^p(\Omega)^3} \leq \tilde{B}_p \|\mathbf{v}\|_{2,\Omega} \quad \forall \mathbf{v} \in H^2(\Omega)^3 \tag{46}$$

which follow from continuity of the embeddings of $H^2(\Omega)$ to $L^p(\Omega)$ and $H^2(\Omega)^3$ to $L^p(\Omega)^3$, $1 \leq p \leq \infty$. Here, B_p and $\tilde{B}_p$ are the positive constants, which depend on Ω and p.

We will assume that the following conditions are met:

4.1. Ω is a bounded domain in the space $\mathbb{R}^3$ with boundary $\Gamma \in C^2$;

4.2. functions ν and λ belong to the space C^1, and besides

$$\nu_{\min} \leq \nu(s) \leq \nu_{\max}, \quad \nu'_{\min} \leq \nu'(s) \leq \nu'_{\max},$$

$$\lambda_{\min} \leq \lambda(s) \leq \lambda_{\max}, \quad \lambda'_{\min} \leq \lambda'(s) \leq \lambda'_{\max} \quad \forall s \in \mathbb{R},$$

where $\nu_{\min}, \nu_{\max}, \nu'_{\min}, \nu'_{\max}$ and $\lambda_{\min}, \lambda_{\max}, \lambda'_{\min}, \lambda'_{\max}$ are positive constants.

In addition to **2.3**, we will assume that the reaction coefficient $k(\varphi, \cdot)$ also satisfies the condition:

4.3. the conditions **2.3** are satisfied with the parameter $p \geq 2$ (instead of $p \geq 3/2$) and the following estimate holds:

$$\|k(\varphi,\cdot)\|_{L^p(\Omega)} \leq C_k^p \quad \forall \varphi \in H_0^1(\Omega), \quad p \geq 2,$$

where C_k^p is a positive constant;

4.4. $\mathbf{f} \in L^2(\Omega)^3$, $f \in L^2(\Omega)$, $\mathbf{b} \equiv \beta \mathbf{G} \in L^2(\Omega)^3$.

To study a strong solution, we introduce the product of spaces

$$\mathcal{X} = H^2(\Omega)^3 \cap H_0^1(\Omega)^3 \times H^2(\Omega) \cap H_0^1(\Omega)$$

with the norm

$$\|(\mathbf{v},h)\|_{\mathcal{X}}^2 = \|\mathbf{v}\|_{2,\Omega}^2 + \|h\|_{2,\Omega}^2.$$

As in [13], we will use the Stokes operator $\tilde{\Delta}$ defined by:

$$\tilde{\Delta} = -P\Delta : \mathrm{Dom}(\tilde{\Delta}) \subset L^2(\Omega)^3 \to L^2(\Omega)^3,$$

where $\mathrm{Dom}(\tilde{\Delta}) = V \cap H^2(\Omega)^3$ is the domain of $\tilde{\Delta}$. It is well known that for any function $\mathbf{u} \in H^2(\Omega)^3 \cap V$ the following decomposition is valid (see [43]):

$$-\Delta \mathbf{u} = \tilde{\Delta}\mathbf{u} + \nabla q. \tag{47}$$

Here, $q \in H^1(\Omega)$ is a function uniquely determined by the function $\mathbf{u}$, and the following estimates hold [43]:

$$\|q\|_{1,\Omega} \leq \tilde{C}_7 \|\tilde{\Delta}\mathbf{u}\|_{\Omega}, \quad \|\Delta \mathbf{u}\|_{\Omega} \leq (\tilde{C}_7 + 1)\|\tilde{\Delta}\mathbf{u}\|_{\Omega}. \tag{48}$$

Along with the nonlinear Problem 1, we will consider its linear analogue in the form of the following boundary value problem for the triple $(\mathbf{u},\varphi,q)$:

$$-\operatorname{div}(\nu(c)\nabla\mathbf{u}) + (\mathbf{s}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f} + \mathbf{b}\,\varphi, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \tag{49}$$

$$-\operatorname{div}(\lambda(c)\nabla\varphi) + k(c,\cdot)\varphi + \mathbf{s}\cdot\nabla\varphi = f \text{ in } \Omega, \tag{50}$$

$$\mathbf{u} = \mathbf{0}, \quad \varphi = 0 \quad \text{on } \Gamma. \tag{51}$$

Here, $(\mathbf{s},c)$ is a given pair from the space X or $\mathcal{X}$.

The triple $((\mathbf{u},\varphi),q) \in X \times L_0^2(\Omega)$, which satisfies the identity

$$(\nu(c)\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{s}\cdot\nabla)\mathbf{u},\mathbf{v}) - (q,\operatorname{div} v) = (\mathbf{f},\mathbf{v}) + (\mathbf{b}\,\varphi,\mathbf{v}) \;\forall \mathbf{v} \in V \tag{52}$$

and identity (24) from Section 2 will be called a weak solution of problem (49)–(51).

The restriction of (52) to the space V takes the form (23). In Section 2, using the Lax–Milgram theorem, it was shown that for any pair $(\mathbf{s},c) \in W$ a weak solution $(\mathbf{u},\varphi) \in W$ to the problem (23), (24) exists and is unique, and the corresponding a priori estimates (25) and (27) were obtained. The restoration of the function $q \in L_0^2(\Omega)$ by the pair $(\mathbf{u},\varphi) \in X$ is performed similarly to the restoration of the pressure p in Section 2.

Moreover, if $(\mathbf{s},c) \in \mathcal{X}$ and conditions **4.1**–**4.4** are satisfied, then due to the property of elliptic regularity (see [43]), a weak solution $((\mathbf{u},\varphi),q)$ of the problem (49)–(51) is its strong solution from the space $\mathcal{X} \times H^1(\Omega) \cap L_0^2(\Omega)$, satisfying the Equations (49), (50), a.e. in Ω.

Let us formulate the above result in the form of the following lemma.

Lemma 2. *Let the conditions* **4.1**–**4.4** *be satisfied. Then for each pair $(\mathbf{s},c) \in \mathcal{X}$ there exists a strong solution $((\mathbf{u},\varphi),q) \in \mathcal{X} \times H_0^1(\Omega) \cap L_0^2(\Omega)$ of problem (49)–(51), and the Equations (49) and (50) hold a.e. in Ω.*

Our nearest goal is to prove the local existence of a strong solution to Problem 1, by which we mean the triple $((\mathbf{u}, \varphi), p) \in \mathcal{X} \times H^1(\Omega) \cap L^2(\Omega)$ satisfying the Equations (1) and (2) a.e. in Ω. We first formulate an important auxiliary lemma concerning estimates for bilinear and trilinear forms, which we will use when proving the local existence theorem.

Lemma 3. *Let under condition* **4.1** $\mathbf{b} \in L^2(\Omega)^3$, $k_0 \in L^p(\Omega)$, $p \geq 2$. *Then, the following inequalities hold:*

$$|(\nu'(c)\nabla c\,\nabla \mathbf{v}, \Delta \mathbf{w})| \leq \beta_1 \nu'_{\max}\|\Delta c\|_\Omega \|\tilde{\Delta}\mathbf{v}\|_\Omega \|\tilde{\Delta}\mathbf{w}\|_\Omega,$$

$$|(\mathbf{b}h, \Delta \mathbf{w})| \leq \beta_2 \|\mathbf{b}\|_\Omega \|\Delta h\|_\Omega \|\tilde{\Delta}\mathbf{w}\|_\Omega$$

$$\forall c, h \in H^2(\Omega) \cap H^1_0(\Omega),\ \mathbf{w}, \mathbf{v} \in H^2(\Omega)^3 \cap H^1_0(\Omega)^3, \tag{53}$$

$$|((\mathbf{w}\cdot\nabla)\mathbf{v}, \tilde{\Delta}\mathbf{u})| \leq \beta_3 \|\tilde{\Delta}\mathbf{w}\|_\Omega \|\tilde{\Delta}\mathbf{v}\|_\Omega \|\tilde{\Delta}\mathbf{u}\|_\Omega$$

$$\forall \mathbf{w}, \mathbf{v}, \mathbf{u} \in H^2(\Omega)^3 \cap H^1_0(\Omega)^3, \tag{54}$$

$$|(\lambda'(c)\nabla c \cdot \nabla h, \Delta \eta)| \leq \lambda'_{\max}\beta_4 \|\Delta c\|_\Omega \|\Delta h\|_\Omega \|\Delta \eta\|_\Omega,$$

$$|(k_0 h, \Delta\eta)| \leq \beta_5 \|k_0\|_{L^p(\Omega)} \|\Delta h\|_\Omega \|\Delta\eta\|_\Omega\ \forall c, h, \eta \in H^2(\Omega) \cap H^1_0(\Omega), \tag{55}$$

$$|(\mathbf{s}\cdot\nabla h, \Delta\eta)| \leq \beta_6 \|\tilde{\Delta}\mathbf{s}\|_\Omega \|\Delta h\|_\Omega \|\Delta\eta\|_\Omega,$$

$$\forall \mathbf{s} \in H^2(\Omega)^3 \cap H^1_0(\Omega)^3,\ h, \eta \in H^2(\Omega) \cap H^1_0(\Omega). \tag{56}$$

Here and below β_i, $i = 1, 2, \ldots$, are positive constants depending on Ω or on Ω and p.

Proof of Lemma 3. Let us prove, for example, the first inequality in (53). Using the Hölder inequality, estimates (45) and taking into account the condition **4.2** we have

$$|(\nu'(c)\nabla c\,\nabla\mathbf{v}, \Delta\mathbf{w})| \leq \nu'_{\max}\|\nabla c\|_{L^4(\Omega)^3}\|\nabla\mathbf{v}\|_{L^4(\Omega)^3}\|\Delta\mathbf{w}\|_\Omega \leq$$

$$\leq \beta_1\nu'_{\max}\|\Delta c\|_\Omega\|\tilde{\Delta}\mathbf{v}\|_\Omega\|\tilde{\Delta}\mathbf{w}\|_\Omega,\ \ \beta_1 = \tilde{C}_5\tilde{C}_6(\tilde{C}_7+1)^2.$$

The remaining inequalities in Lemma 3 are proved in a similar way. □

Remark 2. *In Lemma 3 and below $\nabla c\,\nabla\mathbf{v}$ denotes the vector field, in which i-th component is given by formula: $[\nabla c\,\nabla\mathbf{v}]_i = \nabla c\cdot\nabla\mathbf{v}_i$, $i = 1, 2, 3$.*

Below, we will use, together with Lemma 3, the following estimate, which was obtained in [13], using (47) and (48):

$$|(\nu(c)\nabla q, \tilde{\Delta}\mathbf{u})| \leq \nu'_{\max}\beta_7\|\Delta c\|_\Omega\|\tilde{\Delta}\mathbf{u}\|^2_\Omega$$

$$\forall c \in H^2(\Omega)\cap H^1_0(\Omega),\ \mathbf{u} \in H^2(\Omega)^3 \cap V. \tag{57}$$

Here, the function $\nu(\cdot) \in C^1$ satisfies the first condition in **4.2**, q is the function related with $\mathbf{u}$ by the formula (47).

Presently, we are able to prove the following main theorem of this section

Theorem 3. *Let the conditions* **4.1–4.4** *and the smallness conditions*

$$2\beta_3(2\beta_2\|\mathbf{b}\|_\Omega(1/\lambda_{\min})\|f\|_\Omega + (1/\nu_{\min})\|\mathbf{f}\|_\Omega)+$$

$$+2\nu'_{\max}(\beta_1+\beta_7)(1/\lambda_{\min})\|f\|_\Omega \leq \nu_{\min}/2,$$

$$2\beta_6(2(\beta_2/\nu_{\min}\|\mathbf{b}\|_\Omega)(1/\lambda_{\min})\|f\|_\Omega + (1/\nu_{\min})\|\mathbf{f}\|_\Omega)+$$

$$+\,2\lambda'_{\max}\beta_4(1/\lambda_{\min})\|f\|_\Omega + \beta_5 C_k^p \leq \lambda_{\min}/2,\ p \geq 2, \tag{58}$$

be satisfied. Then, there exists a strong solution $((\mathbf{u}, \varphi), p) \in \mathcal{X} \times (H^1(\Omega) \cap L_0^2(\Omega))$ *of Problem 1 such that*

$$-\mathrm{div}(\nu(\varphi)\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{b}\varphi,$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ a.e. in } \Omega, \tag{59}$$

$$-\,\mathrm{div}(\lambda(\varphi)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + k(\varphi,\cdot)\varphi = f \text{ a.e. in } \Omega \tag{60}$$

and the following a priori estimates hold:

$$\|\mathbf{u}\|_{2,\Omega} \le M_{\mathbf{u}}^0 \equiv 2(1/\nu_{\min})\tilde{C}_4(\tilde{C}_7+1)[2\beta_2(1/\lambda_{\min})\|\mathbf{b}\|_\Omega\|f\|_\Omega + \|\mathbf{f}\|_\Omega], \tag{61}$$

$$\|\varphi\|_{2,\Omega} \le M_\varphi^0 \equiv (2/\lambda_{\min})\tilde{C}_2\|f\|_\Omega, \tag{62}$$

$$\|p\|_{1,\Omega} \le M_p^0 \equiv M_p + \beta_2 M_\varphi^0\|\mathbf{b}\|_\Omega + \nu_{\max}\tilde{C}_3 M_{\mathbf{u}}^0 + \beta_8(M_{\mathbf{u}}^0)^2 +$$

$$+\,\beta_9\nu'_{\max}M_\varphi^0 M_{\mathbf{u}}^0 + \|\mathbf{f}\|_\Omega,\ \ \beta_8 = \tilde{B}_4\tilde{C}_3\tilde{C}_6,\ \ \beta_9 = \tilde{C}_2\tilde{C}_3\tilde{C}_5\tilde{C}_6. \tag{63}$$

Here, the constants $\tilde{C}_k$, $k = 2,3,\ldots,7$, *are defined in (44), (45), (48) and (57), the constant* M_p *is defined in (38). The constants* β_i, $i = 1,\ldots,7$, *are defined in Lemma 3 and (57),* $\tilde{B}_4$ *is a constant from (46).*

Proof of Theorem 3. To prove Theorem 3, we construct in the space $\mathcal{X}$ mapping G acting according to the formula $G((\mathbf{s},c)) = (\mathbf{u},\varphi)$ for any pair $(\mathbf{s},c) \in \mathcal{X}$. Here, the pair $(\mathbf{u},\varphi) \in \mathcal{X}$ is the strong solution respective component of the problem (49)–(51), satisfying the identities (23), (24). The existence of this strong solution under conditions **4.1–4.4** follows from Lemma 2.

Since the embeddings $V \subset H$ and $H_0^1(\Omega) \subset L^2(\Omega)$ are dense, then (23), (24) imply the identities

$$-\,(\mathrm{div}(\nu(c)\nabla\mathbf{u}),\mathbf{v}) + ((\mathbf{s}\cdot\nabla)\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) + (\mathbf{b}\,\varphi,\mathbf{v})\ \ \forall\mathbf{v}\in H, \tag{64}$$

$$-\,(\mathrm{div}(\lambda(c)\nabla\varphi),h) + (k(c,\cdot)\varphi,h) + (\mathbf{s}\cdot\nabla\varphi,h) = (f,h)\ \ \forall h \in L^2(\Omega). \tag{65}$$

Using the relations

$$\mathrm{div}(\lambda(c)\nabla\varphi) = \lambda(c)\Delta\varphi + \nabla\lambda(c)\cdot\nabla\varphi = \lambda(c)\Delta\varphi + \lambda'(c)\nabla c\cdot\nabla\varphi \text{ in } \Omega,$$

$$\mathrm{div}(\nu(c)\nabla\mathbf{u}) = \nu(c)\Delta\mathbf{u} + \nabla\nu(c)\nabla\mathbf{u} = \nu(c)\Delta\mathbf{u} + \nu'(c)\nabla c\,\nabla\mathbf{u} \text{ in } \Omega, \tag{66}$$

we rewrite (64), (65) in the following form:

$$-(\nu(c)\Delta\mathbf{u},\mathbf{v}) = (\nu'(c)\nabla c\,\nabla\mathbf{u},\mathbf{v}) +$$

$$+\,(\mathbf{b}\,\varphi,\mathbf{v}) + (\mathbf{f},\mathbf{v}) - ((\mathbf{s}\cdot\nabla)\mathbf{u},\mathbf{v})\ \forall\mathbf{v}\in H, \tag{67}$$

$$-(\lambda(c)\Delta\varphi,h) + (k(c,\cdot)\varphi,h) + (\mathbf{s}\cdot\nabla\varphi,h) =$$

$$= (\lambda'(c)\nabla c\cdot\nabla\varphi,h) + (f,h)\ \ \forall h\in L^2(\Omega). \tag{68}$$

Setting $\mathbf{v} = \tilde{\Delta}\mathbf{u}$ in (67) and $h = \Delta\varphi$ in (68), taking into account (47), we have

$$(\nu(c)\tilde{\Delta}\mathbf{u},\tilde{\Delta}\mathbf{u}) = (\nu'(c)\nabla c\,\nabla\mathbf{u},\tilde{\Delta}\mathbf{u}) + (\mathbf{b}\,\varphi,\tilde{\Delta}\mathbf{u}) + (\mathbf{f},\tilde{\Delta}\mathbf{u}) +$$

$$-\,((\mathbf{s}\cdot\nabla)\mathbf{u},\tilde{\Delta}\mathbf{u}) + (\nu(c)\nabla q,\tilde{\Delta}\mathbf{u}), \tag{69}$$

$$-(\lambda(c)\Delta\varphi,\Delta\varphi) = (\lambda'(c)\nabla c\cdot\nabla\varphi,\Delta\varphi) - (k(c,\cdot)\varphi,\Delta\varphi) +$$

$$+\,(\mathbf{s}\cdot\nabla\varphi,\Delta\varphi) + (f,\Delta\varphi). \tag{70}$$

From (69), (70), using Lemma 3 and estimate (57), we arrive at the inequalities:

$$\nu_{\min}\|\tilde{\Delta}\mathbf{u}\|_\Omega^2 \le \beta_1\nu'_{\max}\|\Delta c\|_\Omega\|\tilde{\Delta}\mathbf{u}\|_\Omega^2 + \beta_2\|\mathbf{b}\|_\Omega\|\Delta\varphi\|_\Omega\|\tilde{\Delta}\mathbf{u}\|_\Omega +$$

$$+\beta_3\|\tilde{\Delta}\mathbf{s}\|_\Omega\|\tilde{\Delta}\mathbf{u}\|_\Omega^2+\nu'_{\max}\beta_7\|\Delta c\|_\Omega\|\tilde{\Delta}\mathbf{u}\|_\Omega^2+\|\mathbf{f}\|_\Omega\|\tilde{\Delta}\mathbf{u}\|_\Omega, \quad (71)$$

$$\lambda_{\min}\|\Delta\varphi\|_\Omega^2\le\lambda'_{\max}\beta_4\|\Delta c\|_\Omega\|\Delta\varphi\|_\Omega^2+\beta_5C_k^p\|\Delta\varphi\|_\Omega^2+$$
$$+\beta_6\|\tilde{\Delta}\mathbf{s}\|_\Omega\|\Delta\varphi\|_\Omega^2+\|f\|_\Omega\|\Delta\varphi\|_\Omega. \quad (72)$$

From (71), (72), we derive the estimates

$$\|\tilde{\Delta}\mathbf{u}\|_\Omega\le(1/\nu_{\min})[(\beta_3\|\tilde{\Delta}\mathbf{s}\|_\Omega+\nu'_{\max}(\beta_1+\beta_7)\|\Delta c\|_\Omega)\|\tilde{\Delta}\mathbf{u}\|_\Omega$$
$$+\beta_2\|\mathbf{b}\|_\Omega\|\Delta\varphi\|_\Omega+\|\mathbf{f}\|_\Omega], \quad (73)$$

$$\|\Delta\varphi\|_\Omega\le(1/\lambda_{\min})[(\beta_6\|\tilde{\Delta}\mathbf{s}\|_\Omega+\lambda'_{\max}\beta_4\|\Delta c\|_\Omega+\beta_5C_k^p)\|\Delta\varphi\|_\Omega+\|f\|_\Omega]. \quad (74)$$

Let us show further that under the conditions (58) the operator G maps a bounded convex closed set

$$\mathcal{M}=\{(\mathbf{s},c)\in\mathcal{X}:\|\tilde{\Delta}\mathbf{s}\|_\Omega\le r_1,\ \ \|\Delta c\|_\Omega\le r_2\} \quad (75)$$

into itself for certain values of r_1 and r_2, which will be chosen later.

To this end, we rewrite the system of inequalities (73), (74) in the following form:

$$\|\tilde{\Delta}\mathbf{u}\|_\Omega\le(a_{11}\|\tilde{\Delta}\mathbf{s}\|_\Omega+a_{12}\|\Delta c\|_\Omega)\|\tilde{\Delta}\mathbf{u}\|_\Omega+c_1\|\Delta\varphi\|_\Omega+f_1, \quad (76)$$

$$\|\Delta\varphi\|_\Omega\le(a_{21}\|\tilde{\Delta}\mathbf{s}\|_\Omega+a_{22}\|\Delta c\|_\Omega+c_2)\|\Delta\varphi\|_\Omega+f_2. \quad (77)$$

Here,

$$a_{11}=(1/\nu_{\min})\beta_3,\ \ a_{12}=(1/\nu_{\min})\nu'_{\max}(\beta_1+\beta_7),$$
$$c_1=(\beta_2/\nu_{\min})\|\mathbf{b}\|_\Omega,\ \ f_1=(1/\nu_{\min})\|\mathbf{f}\|_\Omega,$$
$$a_{21}=(1/\lambda_{\min})\beta_6,\ \ a_{22}=(1/\lambda_{\min})\lambda'_{\max}\beta_4,$$
$$c_2=(1/\lambda_{\min})\beta_5C_k^p,\ \ f_2=(1/\lambda_{\min})\|f\|_\Omega. \quad (78)$$

We assume that the pair $(\mathbf{s},c)$ belongs to the set $\mathcal{M}$, in which the values r_1 and r_2 are defined by the formulae

$$r_1=2(2c_1f_2+f_1),\ \ r_2=2f_2. \quad (79)$$

This means that the pair $(\mathbf{s},c)$ satisfies the relations

$$\|\tilde{\Delta}\mathbf{s}\|_\Omega\le r_1=2(2c_1f_2+f_1),\ \ \|\Delta c\|_\Omega\le r_2=2f_2. \quad (80)$$

Let us assume that the following conditions are satisfied:

$$2a_{11}(2c_1f_2+f_1)+2a_{12}f_2\le1/2,$$
$$2a_{21}(2c_1f_2+f_1)+2a_{22}f_2+c_2\le1/2. \quad (81)$$

Taking into account the notation (78), conditions (81) take the form of smallness conditions (58).

Using (80) and (81), from (77), we arrive at the estimate

$$\|\Delta\varphi\|_\Omega\le2f_2. \quad (82)$$

Taking into account (80)–(82), from (76), we obtain

$$\|\tilde{\Delta}\mathbf{u}\|_\Omega\le2(2c_1f_2+f_1). \quad (83)$$

The relations (82), (83) together with (79) mean that under the smallness conditions (58) the operator G maps the set $\mathcal{M}$ defined in (75) with the parameters r_1 and r_2 determined in (79) into itself.

Arguing, as in Section 2 (see also [13]), one can show that the operator G is continuous and compact on the set M. In this case, it follows from the Schauder fix-point theorem that the operator G has a fixed point $(\mathbf{u}, \varphi) = G(\mathbf{u}, \varphi)$ satisfying the inequalities (82) and (83). The indicated point $(\mathbf{u}, \varphi)$ together with the corresponding pressure $p \in H^1(\Omega) \cap L_0^2(\Omega)$ is the required strong solution of Problem 1. From (82), (83) it follows that for the pair $(\mathbf{u}, \varphi)$ the a priori estimates (61), (62) hold.

To prove the theorem, it remains to derive an estimate (63) for the pressure p. To this end, taking into account (66) we rewrite the first equation in (59) in the form

$$\nabla p = \nu(\varphi)\Delta\mathbf{u} + \nu'(\varphi)\nabla\varphi \cdot \nabla\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \mathbf{b}\varphi + \mathbf{f} \text{ a.e. in } \Omega.$$

The last relation implies the estimate

$$\|\nabla p\|_\Omega \le \nu_{\max}\|\Delta\mathbf{u}\|_\Omega + \nu'_{\max}\|\nabla\varphi\|_{L^4(\Omega)^3}\|\nabla\mathbf{u}\|_{L^4(\Omega)^3} +$$
$$+ \|\mathbf{u}\|_{L^4(\Omega)^3}\|\nabla\mathbf{u}\|_{L^4(\Omega)^3} + \|\mathbf{b}\|_\Omega\|\varphi\|_{L^\infty(\Omega)} + \|\mathbf{f}\|_\Omega. \tag{84}$$

Taking into account (44), (46), from (84) we obtain that

$$\|\nabla p\|_\Omega \le \nu_{\max}\tilde{C}_3 M^0_{\mathbf{u}} + \nu'_{\max}\tilde{C}_5\tilde{C}_6\tilde{C}_2\tilde{C}_3 M^0_\varphi M^0_{\mathbf{u}} + \tilde{B}_4\tilde{C}_3\tilde{C}_6 (M^0_{\mathbf{u}})^2 +$$
$$+ \|\mathbf{f}\|_\Omega + B_\infty\|\mathbf{b}\|_\Omega M^0_\varphi. \tag{85}$$

Finally, from (38) and (85) we arrive at the estimate (63). □

5. Conditional Uniqueness of Solution to Problem 1

In this section, we prove the conditional uniqueness of a weak solution $(\mathbf{u}, \varphi, p)$ to Problem 1 which as in [13] possess with the additional property that $\Delta\varphi \in L^2(\Omega)$ under the following condition to the coefficients $\nu(\cdot)$ and $\lambda(\cdot)$:

5.1. the functions ν, λ and λ' are Lipschitz continuous:

$$|\nu(s_1) - \nu(s_2)| \le L_\nu|s_1 - s_2|,$$
$$|\lambda(s_1) - \lambda(s_2)| \le L_\lambda|s_1 - s_2| \text{ and}$$
$$|\lambda'(s_1) - \lambda'(s_2)| \le L'_\lambda|s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}.$$

Previously, by analogy with Section 4, we formulate an auxiliary lemma on estimates for some special bilinear and trilinear forms, which will be used in the proof of the uniqueness theorem.

Lemma 4. *Let the condition* **4.1** *be satisfied. There exist positive constants β_9, β_{10}, depending on Ω, with which the following estimates hold:*

$$|(\varphi_1\nabla\varphi_2 \cdot \nabla\varphi_3, \Delta\varphi_4)| \le \beta_9\|\Delta\varphi_1\|_\Omega\|\Delta\varphi_2\|_\Omega\|\Delta\varphi_3\|_\Omega\|\Delta\varphi_4\|_\Omega,$$
$$|(\varphi_1\Delta\varphi_2, \Delta\varphi_3)| \le \beta_{10}\|\Delta\varphi_1\|_\Omega\|\Delta\varphi_2\|_\Omega\|\Delta\varphi_3\|_\Omega$$
$$\forall\varphi_i \in H^2(\Omega) \cap H^1_0(\Omega),\ i = 1, 2, 3, 4. \tag{86}$$

Suppose that $\lambda(\cdot)$ and $\nu(\cdot)$ satisfy the condition **5.1**. Using Lemma 4, we derive the following estimates for any pair of functions $\varphi_1, \varphi_2 \in H^2(\Omega) \cap H^1_0(\Omega)$ and their difference $\varphi = \varphi_1 - \varphi_2$:

$$|((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 \cdot \nabla\varphi_2, \Delta\varphi)| \le L'_\lambda|(|\varphi||\nabla\varphi_1||\nabla\varphi_2|, |\Delta\varphi|)| \le$$
$$\le L'_\lambda\beta_9\|\Delta\varphi_1\|_\Omega\|\Delta\varphi_2\|_\Omega\|\Delta\varphi\|^2_\Omega, \tag{87}$$
$$|((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \Delta\varphi)| \le L_\lambda|(|\varphi||\Delta\varphi_2|, |\Delta\varphi|)| \le$$

$$\leq \beta_{10} L_\lambda \|\Delta\varphi_2\| \|\Delta\varphi\|^2_\Omega. \tag{88}$$

The following uniqueness result for "small" weak solution holds:

Theorem 4. *Let the conditions* **4.1**–**4.4** *and* **5.1** *be satisfied. There exists* $\varepsilon > 0$*, such that, if there exists a weak solution* $(\mathbf{u}, \varphi, p) \in V \times (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2_0(\Omega)$ *of Problem 1 satisfying*

$$\|\mathbf{u}\|_{1,\Omega} + \|\varphi\|_{2,\Omega} < \varepsilon,$$

then it is unique.

Proof of Theorem 4. Suppose there exist two weak solutions $(\mathbf{u}_i, \varphi_i, p_i) \in V \cap (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2_0(\Omega)$, $i = 1, 2$, of the problem (19)–(21). It is clear that the differences

$$\varphi = \varphi_1 - \varphi_2 \in H^2(\Omega) \cap H^1_0(\Omega), \quad \mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2 \in V$$

satisfy the relations

$$\begin{gathered} -(\mathrm{div}(\lambda(\varphi_1)\nabla\varphi), h) + (k(\varphi_1, \cdot)\varphi, h) + (\mathbf{u}_1 \cdot \nabla\varphi, h) = \\ = (\mathrm{div}\,(\lambda(\varphi_1) - \lambda(\varphi_2))\nabla\varphi_2), h) - (k(\varphi_1, \cdot) - k(\varphi_2, \cdot), \varphi_2 h) - \\ -\,(\mathbf{u} \cdot \nabla\varphi_2, h) \ \forall h \in L^2(\Omega), \end{gathered} \tag{89}$$

$$\begin{gathered} (\nu(\varphi_1)\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}_1 \cdot \nabla)\mathbf{u}, \mathbf{v}) = \\ = -((\nu(\varphi_1) - \nu(\varphi_2))\nabla\mathbf{u}_2, \nabla\mathbf{v}) + (\mathbf{b}\varphi, \mathbf{v}) - ((\mathbf{u} \cdot \nabla)\mathbf{u}_2, \mathbf{v}) \ \forall \mathbf{v} \in V. \end{gathered} \tag{90}$$

Taking into account the equality

$$\begin{gathered} \mathrm{div}(\lambda(\varphi_1)\nabla\varphi) = \nabla\lambda(\varphi_1) \cdot \nabla\varphi + \lambda(\varphi_1)\Delta\varphi = \\ = \lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi + \lambda(\varphi_1)\Delta\varphi \text{ in } \Omega, \end{gathered}$$

and correspondingly

$$\begin{gathered} \mathrm{div}\,((\lambda(\varphi_1) - \lambda(\varphi_2))\nabla\varphi_2) = (\lambda'(\varphi_1)\nabla\varphi_1 - \lambda'(\varphi_2)\nabla\varphi_2) \cdot \nabla\varphi_2 + \\ +(\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2 = \\ = ((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2 + \\ +(\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \end{gathered}$$

we rewrite (89) in the following form:

$$\begin{gathered} -(\lambda(\varphi_1)\Delta\varphi, h) = (\lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi, h) - (k(\varphi_1, \cdot)\varphi, h) - (\mathbf{u}_1 \cdot \nabla\varphi, h) + \\ +((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2, h) + \\ +((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, h) - \\ -\,(k(\varphi_1, \cdot) - k(\varphi_2, \cdot), \varphi_2 h) - (\mathbf{u} \cdot \nabla\varphi_2, h) \ \forall h \in L^2(\Omega). \end{gathered} \tag{91}$$

Setting $h = \Delta\varphi$ in (91) and $\mathbf{v} = \mathbf{u}$ in (90), we obtain that

$$\begin{gathered} -(\lambda(\varphi_1)\Delta\varphi, \Delta\varphi) = (\lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi, \Delta\varphi) - \\ -(k(\varphi_1, \cdot)\varphi, \Delta\varphi) - (\mathbf{u}_1 \cdot \nabla\varphi, \Delta\varphi) + \\ +((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2, \Delta\varphi) + \\ +((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \Delta\varphi) - \end{gathered}$$

$$-(k(\varphi_1,\cdot)-k(\varphi_2,\cdot),\varphi_2\Delta\varphi)-(\mathbf{u}\cdot\nabla\varphi_2,\Delta\varphi), \tag{92}$$

$$(\nu(\varphi_1)\nabla\mathbf{u},\nabla\mathbf{u})=$$
$$=-((\nu(\varphi_1)-\nu(\varphi_2))\nabla\mathbf{u}_2,\nabla\mathbf{u})+(\mathbf{b}\varphi,\mathbf{u})-((\mathbf{u}\cdot\nabla)\mathbf{u}_2,\mathbf{u}). \tag{93}$$

From (92), (93), taking into account Lemmas 1, 3, 4, properties **4.2**, **5.1** and (87), (88), we arrive at the inequalities

$$\lambda_{\min}\|\Delta\varphi\|^2_\Omega \le \lambda'_{\max}\beta_4\tilde{C}_1\|\varphi_1\|_{2,\Omega}\|\Delta\varphi\|^2_\Omega+$$
$$+\beta_5\|k(\varphi_1,\cdot)\|_{L^p(\Omega)}\|\Delta\varphi\|^2_\Omega+\tilde{C}_5C_4\|\mathbf{u}_1\|_{1,\Omega}\|\Delta\varphi\|^2_\Omega+$$
$$+\beta_9L'_\lambda\|\varphi_1\|_{2,\Omega}\|\varphi_2\|_{2,\Omega}\tilde{C}_1^2\|\Delta\varphi\|^2_\Omega+\beta_4\tilde{C}_1\lambda'_{\max}\|\varphi_1\|_{2,\Omega}\|\Delta\varphi\|^2_\Omega+$$
$$+\beta_{10}L_\lambda\tilde{C}_1\|\varphi_1\|_{2,\Omega}\|\Delta\varphi\|^2_\Omega+$$
$$+L\|\varphi_2\|_{2,\Omega}\tilde{C}_2B_4B_\infty\|\Delta\varphi\|^2_\Omega+C_4\tilde{C}_1\tilde{C}_5\|\varphi_1\|_{2,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\Delta\varphi\|_\Omega, \tag{94}$$

$$\nu_*\|\mathbf{u}\|^2_{1,\Omega}\le\tilde{C}_1B_\infty L_\nu\|\mathbf{u}_2\|_{1,\Omega}\|\Delta\varphi\|_\Omega\|\mathbf{u}\|_{1,\Omega}+$$
$$+\beta_0\tilde{C}_1\|\mathbf{b}\|_\Omega\|\Delta\varphi\|_\Omega\|\mathbf{u}\|_{1,\Omega}+\gamma_1\|\mathbf{u}_2\|_{1,\Omega}\|\mathbf{u}\|^2_{1,\Omega}. \tag{95}$$

Using Yang's inequality, we have

$$C_4\tilde{C}_1\tilde{C}_5\|\varphi_1\|_{2,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\Delta\varphi\|_\Omega\le(1/2)C_4\tilde{C}_1\tilde{C}_5\|\varphi_1\|_{2,\Omega}(\|\mathbf{u}\|^2_{1,\Omega}+\|\Delta\varphi\|^2_\Omega),$$

$$\tilde{C}_1B_\infty L_\nu\|\mathbf{u}_2\|_{1,\Omega}\|\Delta\varphi\|_\Omega\|\mathbf{u}\|_{1,\Omega}\le$$
$$\le(1/2)\tilde{C}_1B_\infty L_\nu\|\mathbf{u}_2\|_{1,\Omega}(\|\mathbf{u}\|^2_{1,\Omega}+\|\Delta\varphi\|^2_\Omega),$$

$$\beta_0\tilde{C}_1\|\mathbf{b}\|_\Omega\|\Delta\varphi\|_\Omega\|\mathbf{u}\|_{1,\Omega}\le(1/2)\beta_0\tilde{C}_1\|\mathbf{b}\|_\Omega(\|\mathbf{u}\|^2_{1,\Omega}+\|\Delta\varphi\|^2_\Omega). \tag{96}$$

Adding up the inequalities (94) and (95), taking into account (96), we arrive at the estimate

$$\lambda_{\min}\|\Delta\varphi\|^2_\Omega+\nu_*\|\mathbf{u}\|^2_{1,\Omega}\le\mathcal{A}\|\Delta\varphi\|^2_\Omega+\mathcal{B}\|\mathbf{u}\|^2_{1,\Omega}, \tag{97}$$

where

$$\mathcal{A}=\lambda'_{\max}\beta_4\tilde{C}_1\|\varphi_1\|_{2,\Omega}+\beta_5C_k^p+C_4\tilde{C}_5\|\mathbf{u}_1\|_{1,\Omega}+$$
$$+\beta_9L'_\lambda\tilde{C}_1^2\|\varphi_1\|_{2,\Omega}\|\varphi_2\|_{2,\Omega}+\beta_4\tilde{C}_1\lambda'_{\max}\|\varphi_1\|_{2,\Omega}+\beta_{10}L_\lambda\tilde{C}_1\|\varphi_1\|_{2,\Omega}+$$
$$+L\|\varphi_2\|_{2,\Omega}\tilde{C}_2B_4B_\infty+0.5C_4\tilde{C}_1\tilde{C}_5\|\varphi_1\|_{2,\Omega}+$$
$$+0.5\tilde{C}_1B_\infty L_\nu\|\mathbf{u}_2\|_{1,\Omega}+0.5\beta_0\tilde{C}_1\|\mathbf{b}\|_\Omega, \tag{98}$$

$$\mathcal{B}=\gamma_1\|\mathbf{u}_2\|_{1,\Omega}+0.5\tilde{C}_1B_\infty L_\nu\|\mathbf{u}_2\|_{1,\Omega}+$$
$$+0.5\beta_0\tilde{C}_1\|\mathbf{b}\|_\Omega+0.5C_4\tilde{C}_1\tilde{C}_5\|\varphi_1\|_{2,\Omega}. \tag{99}$$

Here, C_4 is the constant from (6), the meaning of the constants $\tilde{C}_i$, β_i and B_∞ is specified in Section 4.

Let the following smallness conditions be satisfied:

$$\mathcal{A}<\lambda_{\min},\quad\mathcal{B}<\nu_*. \tag{100}$$

Under conditions (100) from the inequality (97) it follows that $\|\Delta\varphi\|_\Omega=0$ and $\|\mathbf{u}\|_{1,\Omega}=0$ or $\varphi_1=\varphi_2$ and $\mathbf{u}_1=\mathbf{u}_2$.

Subtracting (19) for $(\mathbf{u}_2,\varphi_2,p_2)$ from (19) for $(\mathbf{u}_1,\varphi_1,p_1)$ and taking into account that $\mathbf{u}=\mathbf{0}$ and $\varphi=0$, we obtain that the difference $p=p_1-p_2$ satisfies the equality

$$-(p,\operatorname{div}\mathbf{v})=0\quad\forall\mathbf{v}\in H_0^1(\Omega)^3.$$

Then from inf-sup condition (12) it follows that $p=0$ or $p_1=p_2$.

It follows from the above that there cannot be more than one weak solution $(\mathbf{u}, \varphi, p)$ of Problem 1 if its input data are small enough to satisfy the conditions (100). This completes the proof of the theorem. □

6. Discussion

The generalized Boussinesq model, which is considered in the paper in the form of the system (1), (2), plays an important role for the study of mass transfer processes in real liquids. It is caused by the fact that this model takes into account the observed in nature dependence of the leading coefficients of viscosity, diffusion, and the reaction coefficient on the substance's concentration. An even more essential part is played by the usage of the model (1), (2) for establishing more effective mechanisms for controlling the processes of propagation of various kinds of substances in real liquids. This is due to the fact that the model (1), (2) contains several variable coefficients, namely: mentioned viscosity, diffusion and reaction coefficients, which describe different physical properties of the considered viscous incompressible fluid. As a consequence, this model provides more opportunities for choosing more effective mechanisms for control by mass hydrodynamic processes. In mathematical terms, the best choice of the desired control mechanisms is achieved by solving new control problems for the considered model of mass transfer, in which the indicated viscosity, diffusion and reaction coefficients, or some of them, play the role of control variables. The authors intend to devote a separate article to the study of these control problems.

Moreover, the authors plan to devote one more paper to the study of the solvability of the model (1), (2), considered under inhomogeneous boundary conditions for velocity and concentration. It is well known (see [2]) that the main difficulty in the study of inhomogeneous boundary value problems for the heat and mass transfer models is associated with the construction of the liftings of boundary data, which remove their inhomogeneity. The liftings will require the introduction of additional conditions on the problems data. One of the conditions may be the requirement that the flow domain is symmetrical. The implementation of this idea for the construction of relevant liftings, which remove the inhomogeneity of the boundary data, will be the basic of the planned article.

7. Conclusions

In the present paper, the global existence of a weak solution of the boundary value problem for a nonlinear mass transfer model, which generalizes the classical Boussinesq approximation, was proved. It is assumed that the leading coefficients of kinematic viscosity ν and diffusion λ, as well as the reaction coefficient k, depend on the substance's concentration and that the coefficient k can also depend on spatial variables. Besides, in the paper additional conditions for the input data of the boundary value problem under consideration, which ensure the validity of the maximum principle for the substance's concentration, were established.

However, due to the dependence of the leading coefficients ν and λ on the concentration φ, it is not possible to prove the conditional uniqueness of the weak solution, as, for example, it was conducted in [2,3] or in [23] for the model of mass transfer with the leading coefficients independent of the solution. Nevertheless, we have succeeded in proving the local existence of the strong solution of Problem 1 from the H^2 class under some additional conditions on the input data. One of these conditions is the condition of continuous differentiability of the functions $\nu(\cdot)$ and $\lambda(\cdot)$. Under more stringent conditions on $\nu(\cdot)$ and $\lambda(\cdot)$, we also proved the conditional uniqueness of the weak solution to Problem 1 with additional property that $\Delta\varphi \in L^2(\Omega)$.

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