symmetry

Review

Spectral Parameter as a Group Parameter

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Abstract: A large class of integrable non-linear partial differential equations is characterized by the existence of the associated linear problem (in the case of two independent variables, known as a Lax pair) containing the so-called spectral parameter. In this paper, we present and discuss the conjecture that the spectral parameter can be interpreted as the parameter of some one-parameter groups of transformation, provided that it cannot be removed by any gauge transformation. If a non-parametric linear problem for a non-linear system is known (e.g., the Gauss–Weingarten equations as a linear problem for the Gauss–Codazzi equations in the geometry of submanifolds), then, by comparing both symmetry groups, we can find or indicate the integrable cases. We consider both conventional Lie point symmetries and the so-called extended Lie point symmetries, which are necessary in some cases. This paper is intended to be a review, but some novel results are presented as well.

Keywords: Lie point symmetries; extended Lie point symmetries; integrable systems; spectral parameter; Lax pair

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1. Introduction

The key role of the spectral parameter for the integrability of non-linear partial differential equations has been recognized since the seminal work [1], in which the Korteweg–de Vries equation appears as the condition for the isospectrality of the one-dimensional Schrödinger spectral problem. Many methods for the theory of solitons, including the inverse scattering method, the Darboux–Bäcklund transformation, and the algebro-geometric approach, are based on the existence of a linear problem with the spectral parameter [2]. Since the very beginning, the crucial problem consisted of finding such linear systems (often referred to as "Lax pairs"), usually by making some assumptions about their general form and solving the resulting algebraic and analytic constraints [3,4].

The first observation of the connection between spectral parameters and Lie symmetries is due to Sasaki [5]. He noticed that scaling transformations applied to three popular soliton equations, accompanied by an appropriate gauge transformation (if needed), can remove the spectral parameter from the corresponding Lax pairs. Ten years later, Levi and Sym realized that the inverse version of Sasaki's procedure can produce Lax pairs with a true spectral parameter from non-parametric linear systems [6] (see also [7]). A similar idea, expressed in the language of nonlocal coverings, was formulated by Krasil'shchik and Vinogradov (see [8] (Section 3.6)). The infinitesimal version of this approach (in which Lie algebras are used instead of Lie groups) was presented in [9,10].

Lie symmetries of integrable non-linear partial differential equations (PDEs) have been studied very frequently, but this has not concerned symmetries of the corresponding Lax pairs. In this context, it is worth mentioning the interesting papers by Estévez and her collaborators [11–13]. They consider a $2+1$-dimensional system of PDEs and its Lax pair (without a parameter). Reducing the system by Lie symmetries to $1+1$-dimensional equations, they obtain the corresponding Lax pairs with spectral parameters. The final goal is similar (though not identical) to ours, but the starting point is different. We should also mention Marvan's approach [14–16] here, in which a non-removable parameter is introduced by cohomological methods. This approach is related to Sakovich's observations (see, for example, [17]). Important results have been recently obtained by Morozov [18,19]. Both approaches (symmetries and cohomologies) were compared in [20], a work in which many examples are presented.

In this paper, we present the results (including some new ones) and open problems related to the research program of interpreting the spectral parameter as a group parameter that cannot be removed by gauge transformations. If a non-parametric linear problem for a non-linear system is known, which is a typical situation in the geometry of submanifolds (i.e., the Gauss–Weingarten equations as a linear problem for the Gauss–Codazzi equations), then the comparison of the symmetry groups of both systems can lead to finding the integrable cases.

2. Classical Soliton Equations and Scaling Transformations

In three classical cases, Sasaki showed that a simple scaling symmetry (possibly accompanied by a gauge transformation) can remove the spectral parameter from the standard Lax pair [5]. It is natural to reverse this observation and use scaling symmetries to insert a parameter into a given non-parametric linear problem.

The sine-Gordon equation

$$u_{xt} = \sin u, \tag{1}$$

is the compatibility condition for the linear system:

$$\Psi_x = \begin{pmatrix} 1 & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & -1 \end{pmatrix}\Psi, \qquad \Psi_t = \frac{1}{4}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}\Psi. \tag{2}$$

Here, and in the sequel, the partial derivatives with respect to x and t are denoted by the corresponding subscripts.

The scaling symmetry of (1) (the Lorentz transformation in light-cone coordinates)

$$\tilde{x} = \lambda x, \qquad \tilde{t} = t/\lambda \tag{3}$$

inserts the parameter λ into (2)

$$\Psi_x = \begin{pmatrix} \lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & -\lambda \end{pmatrix}\Psi, \qquad \Psi_t = \frac{1}{4\lambda}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}\Psi. \tag{4}$$

The Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5}$$

is the compatibility condition for the linear system:

$$\begin{aligned} \Psi_x &= \begin{pmatrix} 1 & u \\ -1 & -1 \end{pmatrix}\Psi, \\ \Psi_t &= \begin{pmatrix} -(4+2u+u_x) & -(u_{xx}+2u_x+2u^2+4u) \\ 2u+4 & u_x+2u+4 \end{pmatrix}\Psi. \end{aligned} \tag{6}$$

The scaling transformation

$$\tilde{x} = \lambda x, \quad \tilde{t} = \lambda^3 t, \quad \tilde{u} = \frac{u}{\lambda^2} \tag{7}$$

is a symmetry of (5) but is not a symmetry of (6). It inserts the spectral parameter into (6). By also performing the gauge transformation

$$\tilde{\Psi} = \begin{pmatrix} 1/\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}\Psi \tag{8}$$

we obtain the Lax pair:

$$\Psi_x = \begin{pmatrix} \lambda & u \\ -1 & -\lambda \end{pmatrix} \Psi,$$
$$\Psi_t = \begin{pmatrix} -(4\lambda^3 + 2\lambda u + u_x) & -(u_{xx} + 2\lambda u_x + 2u^2 + 4\lambda^2 u) \\ 2u + 4\lambda^2 & u_x + 2\lambda u + 4\lambda^3 \end{pmatrix} \Psi. \tag{9}$$

The modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{10}$$

is the compatibility condition for the linear system:

$$\Psi_x = \begin{pmatrix} 1 & u \\ -u & -1 \end{pmatrix} \Psi,$$
$$\Psi_t = \begin{pmatrix} -(2u^2 + 4) & -(u_{xx} + 2u_x + 2u^3 + 4u) \\ u_{xx} - 2u_x + 2u^3 + 4u & 2u^2 + 4 \end{pmatrix} \Psi. \tag{11}$$

The scaling transformation

$$\tilde{x} = \lambda x, \quad \tilde{t} = \lambda^3 t, \quad \tilde{u} = \frac{u}{\lambda} \tag{12}$$

leaves (10) invariant, and, at the same time, it inserts the spectral parameter into (11). Thus, we obtain the following Lax pair (see [5])

$$\Psi_x = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix} \Psi,$$
$$\Psi_t = \begin{pmatrix} -(2\lambda u^2 + 4\lambda^3) & -(u_{xx} + 2\lambda u_x + 2u^3 + 4\lambda^2 u) \\ u_{xx} - 2\lambda u_x + 2u^3 + 4\lambda^2 u & 2\lambda u^2 + 4\lambda^3 \end{pmatrix} \Psi. \tag{13}$$

3. ZS-AKNS Hierarchy and Scaling Transformations

Surprisingly enough, the problem above has never been studied further in the context of the whole hierarchy of soliton equations. In this section, we will show that the spectral parameter can be interpreted as a parameter of some scaling transformation in the case of the SU(2)-reduction in the Zakharov–Shabat-AKNS hierarchy (this reduction is not essential and is chosen just to simplify the presentation) [2,3].

We consider the following linear problem (or Lax pair):

$$\Psi_x = \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}, \qquad \Psi_t = \begin{pmatrix} i\sum_{k=0}^{N} A_k \lambda^k & \sum_{k=0}^{N} B_k \lambda^k \\ -\sum_{k=0}^{N} \bar{B}_k \lambda^k & -i\sum_{k=0}^{N} A_k \lambda^k \end{pmatrix} \Psi, \tag{14}$$

where $q = q(x,t) \in \mathbb{C}$, $A_k = A_k(x,t) \in \mathbb{R}$, and $B_k = B_k(x,t) \in \mathbb{C}$ are some fields or dependent variables, and the bar denotes a complex conjugate. The compatibility conditions for the Lax pair (14) yield the following system of non-linear equations [3,4]:

$$\begin{aligned} & q_t = B_{0x} + 2iqA_0, \\ & A_{kx} = i(q\bar{B}_k - \bar{q}B_k), \qquad (k = 0, 1, \ldots, N), \\ & B_{k-1} = qA_k - \tfrac{i}{2}B_{kx}, \qquad (k = 1, \ldots, N), \\ & B_N = 0. \end{aligned} \tag{15}$$

This system can be considered as the Nth element of the SU(2)-ZS-AKNS hierarchy. In fact, both variables, A_k and B_k, can be explicitly expressed by q and its derivatives so that only one non-linear equation is left. In particular,

$$\begin{aligned} &A_3 = 4, \quad A_2 = 0, \quad A_1 = -2q^2, \quad A_0 = 0, \\ &B_3 = 0, \quad B_2 = 4q, \quad B_1 = -2iq_x, \quad B_0 = -2q^3 - q_{xx}, \end{aligned} \tag{16}$$

is a special solution of (15) equivalent to the modKdV Equation (10).

The corresponding non-parametric linear problem can be obtained by substituting $\lambda = 1$ into (14):

$$\Psi_x = \begin{pmatrix} i & q \\ -\bar{q} & -i \end{pmatrix}, \qquad \Psi_t = \begin{pmatrix} i\sum\limits_{k=0}^{N} A_k & \sum\limits_{k=0}^{N} B_k \\ -\sum\limits_{k=0}^{N} \bar{B}_k & -i\sum\limits_{k=0}^{N} A_k \end{pmatrix} \Psi. \tag{17}$$

It is easy to show that the parameter λ can be reintroduced into (17) by applying the following scaling symmetry of (15):

$$\tilde{x} = \lambda x, \quad \tilde{t} = \lambda^N t, \quad \tilde{q} = \frac{q}{\lambda}, \quad \tilde{A}_k = \lambda^{k-N} A_k, \quad \tilde{B}_k = \lambda^{k-N} B_k, \tag{18}$$

for $k = 0, 1, \ldots, N$. Indeed, suppose that we start from (17), in which we replaced all variables with their tilde counterparts. Then, applying the transformation (18), we obtain the exact Lax pair (14). The modKdV equation, presented earlier, is a special case of this procedure (note that the spectral parameter is renamed: $i\lambda \mapsto \lambda$).

4. Galilean Transformation

The non-parametric Lax pairs (6) and (11) can be obtained by putting $\lambda = 1$ into (9) and (13), respectively. However, one can easily see that more natural (much simpler) non-parametric linear problems can be obtained for $\lambda = 0$:

$$\Psi_x = \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix} \Psi, \qquad \Psi_t = \begin{pmatrix} -u_x & -u_{xx} - 2u^2 \\ 2u & u_x \end{pmatrix} \Psi \tag{19}$$

(for the KdV equation) and

$$\Psi_x = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \Psi, \qquad \Psi_t = \begin{pmatrix} 0 & -u_{xx} - 2u^3 \\ u_{xx} + 2u^3 & 0 \end{pmatrix} \Psi \tag{20}$$

(for the modKdV equation).

Is it possible to insert the spectral parameter into these linear problems using Lie point symmetries? The answer is different in each of these two cases, mainly due to the fact that the symmetry algebras are different.

The Lie algebra of point symmetries of the modKdV Equation (10) is three-dimensional and consists of translations and a scaling:

$$\partial_x, \quad \partial_t, \quad x\partial_x + 3t\partial_t - u\partial_u. \tag{21}$$

All of these transformations leave the linear problem (20) invariant; none of them can insert a parameter.

The Lie algebra of the point symmetries of the KdV equation is four-dimensional:

$$\partial_x, \quad \partial_t, \quad x\partial_x + 3t\partial_t - 2u\partial_u, \quad 6t\partial x + \partial_u. \tag{22}$$

The first three transformations, similar to the modKdV case, are symmetries of the non-parametric linear problem (19). The last vector field generates the famous Galilean transformation:

$$\tilde{t} = t, \quad \tilde{x} = x + 6\varepsilon t, \quad \tilde{u} = u + \varepsilon. \tag{23}$$

Denoting $\varepsilon = -\lambda^2$, we obtain a linear problem with the spectral parameter:

$$\begin{aligned} \Psi_x &= \begin{pmatrix} 0 & u - \lambda^2 \\ -1 & 0 \end{pmatrix} \Psi, \\ \Psi_t &= \begin{pmatrix} -u_x & -u_{xx} - 2u\lambda^2 - 2u^2 + 4\lambda^4 \\ 2u + 4\lambda^2 & u_x \end{pmatrix} \Psi, \end{aligned} \tag{24}$$

which is equivalent to the well-known scalar spectral problem for the KdV equation (including the one-dimensional Schrödinger equation as the first equation) (compare Section 4.2 in Ref. [21]).

Interestingly, the Lax pair (24) can be transformed into (9) by the following gauge transformation

$$\Psi \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \Psi. \tag{25}$$

The special role of two symmetries of the KdV equation and their gauge equivalence were discussed in [20]. The paper [20] contains other interesting examples (including the Burgers equation).

The non-linear Schrödinger (NLS) equation is another equation with a Galilean symmetry. This equation, given by

$$iq_t + q_{xx} + 2q|q|^2 = 0, \tag{26}$$

is the second member ($N = 2$) of the SU(2)-ZS-AKNS hierarchy (but the first one to be non-linear). It can arise as the compatibility conditions for the following linear system:

$$\Psi_x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \Psi, \qquad \Psi_t = \begin{pmatrix} i|q|^2 & iq_x \\ i\bar{q}_x & -i|q|^2 \end{pmatrix} \Psi. \tag{27}$$

The group of Lie point symmetries of the NLS equation is generated by:

$$\partial_t, \quad \partial_x, \quad x\partial_x + 2t\partial_t - q\partial_q, \quad iq\partial_q, \quad 4t\partial_x + 2ixq\partial_q \tag{28}$$

(to be more precise, we should add complex conjugates here, e.g., $iq\partial_q - i\bar{q}\partial_{\bar{q}}$ instead of $iq\partial_q$, which is omitted for clarity and brevity). The last vector field generates the one-parameter Galilean group:

$$\tilde{t} = t, \quad \tilde{x} = x + 4\lambda t, \quad \tilde{q} = q \exp(2i\lambda x + 4i\lambda^2 t). \tag{29}$$

One can easily check that the Galilean transformation inserts a non-removable parameter into the linear problem (27). By also applying the gauge transformation

$$\tilde{\Psi} = \begin{pmatrix} \exp(i\lambda x + i\lambda^2 t) & 0 \\ 0 & \exp(-i\lambda x - i\lambda^2 t) \end{pmatrix} \Psi, \tag{30}$$

we obtain the NLS Lax pair in the standard form:

$$\Psi_x = \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix} \Psi, \qquad \Psi_t = \begin{pmatrix} i|q|^2 - 2i\lambda^2 & iq_x - 2\lambda q \\ 2\lambda\bar{q} + i\bar{q}_x & 2i\lambda^2 - i|q|^2 \end{pmatrix} \Psi. \tag{31}$$

We point out that the third vector field of (28) corresponds to the scaling symmetry

$$\tilde{t} = \lambda^2 t, \qquad \tilde{x} = \lambda x, \qquad \tilde{q} = \lambda^{-1} q. \tag{32}$$

This transformation leaves the linear problem (27) invariant. However, if we consider another non-parametric linear problem (for example, the one obtained by substituting $\lambda = 1$ into (31)), then the scaling (32) can be used to insert the spectral parameter (this is a special case of the procedure described in Section 3).

5. Lie Point Symmetries' Algebras $\mathcal{A}$ and $\mathcal{A}'$

In the first part of this paper, all considered symmetry transformations were, in fact, changes in variables, and it was very easy to find explicit one-parameter groups corresponding to vector fields. In general, the situation can become more complicated, and it is useful to have a precise algorithm to compute the Lie point symmetries of the considered non-linear system, which are not symmetries of the corresponding linear problem [9,10].

Following Olver's monograph [22], we denote the independent variables by x (as a shorthand for $x^1, x^2, \ldots, x^n$, where n is the number of independent variables) and dependent variables by u (as a shorthand for $u^1, u^2, \ldots, u^m$, where m is the number of independent variables). Derivatives of u are denoted by the multi-index J. For instance, $J = \{1,1,2\}$ corresponds to u_{112} which means that u is differentiated twice with respect to x^1 and once with respect to x^2. In the case of $J = \emptyset$ (the empty set), we have no differentiation, i.e., $u_\emptyset \equiv u$. In Section 2, we used the notation: $x^1 = t, x^2 = x$ and $u^1 = u$ (or $u^1 = q$). Therefore, $u_{112} \equiv u_{ttx}$, etc.

We consider a system of partial differential equations, denoted by $F(x, u, u_J) = 0$, which arises as the compatibility conditions for

$$D_k(\Psi) = U_k \Psi, \quad \Longleftrightarrow \quad G(x, u, u_J, \Psi) = 0. \tag{33}$$

where D_k is the total derivative with respect to x^k, and $U_k = U_k(x, u, u_J)$ $(k = 1, \ldots, n)$. $G = 0$ is just a shorthand for the linear system $\Psi_k = U_k \Psi$.

The compatibility conditions for (33) read as follows:

$$D_j(U_k) - D_k(U_j) + [U_k, U_j] = 0 \quad \Longleftrightarrow \quad F(x, u, u_J) = 0. \tag{34}$$

We point out that the pairwise different indices j and k take all values from the set $\{1, 2, \ldots, n\}$. Thus, in general, F is a system of non-linear matrix PDEs. For $n = 2$, we have only one matrix equation.

Infinitesimal point transformations for the variables and their derivatives are denoted as follows:

$$\begin{aligned} \tilde{x}^k &= x^k + \varepsilon \xi^k(x, u) + \ldots, \quad (k = 1, \ldots, n), \\ \tilde{u}^\alpha &= u^\alpha + \varepsilon \eta^\alpha(x, u) + \ldots, \quad (\alpha = 1, \ldots, m), \\ \tilde{u}^\alpha_J &= u^\alpha_J + \varepsilon \Gamma^\alpha_J + \ldots. \end{aligned} \tag{35}$$

We point out that, by assumption, ξ^k and η^α depend only on x and u (i.e., they do not depend on any derivatives u_J). In contrast, the transformations of the derivatives are uniquely determined provided that the transformations of x and u are known. The explicit formula for Γ^α_J reads as follows:

$$\Gamma^\alpha_J = D_J(\eta^\alpha - \xi^k u^\alpha_k) + \xi^k u^\alpha_{Jk} \tag{36}$$

(see, e.g., [22]), where D_J is the total derivative with respect to the variables represented by the multi-index J. In other words, the point transformation can be prolonged on the jet space (the manifold parameterized by all variables and their derivatives: $x, u, u_k, u_{jk}, \ldots, u_J, \ldots$). In particular, the prolongation of the vector field

$$\boldsymbol{v} = \xi^k \partial_k + \eta^\alpha \partial_{u^\alpha} \tag{37}$$

is given by

$$\boldsymbol{v}^{\infty} = \xi^k \partial_k + \eta^\alpha \partial_{u^\alpha} + \Gamma^\alpha_J \partial_{u^\alpha_J} \tag{38}$$

where, in both formulas, the summation of all indices (including the multi-index J) is assumed. The superscript ∞ means that the vector field contains differentiations with respect to all derivatives u^α_J (although, for each concrete case, when acting on F, only a finite number of these terms are used). The alternative notation is $\boldsymbol{v}^\infty = \mathrm{pr}\,\boldsymbol{v}$, but the finite version $\mathrm{pr}^{(m)}\boldsymbol{v}$ (the more precise) is the most popular (compare [22]).

The Lie algebra of the point symmetries of the non-linear system (34) (i.e., $F = 0$), defined in the standard way, is denoted by $\mathcal{A}$:

$$\mathcal{A} = \{\boldsymbol{v} : \boldsymbol{v}^\infty(F)|_{F=0} = 0\}, \tag{39}$$

where the condition $F = 0$ implies all of the differential consequences: $D_J(F) = 0$, as well.

In order to compute the algebra $\mathcal{A}$, one needs to solve the so-called determining equations [22]. The calculations needed to obtain the results of this paper can be completed—and, in fact, were completed—without computer assistance (see, e.g., Appendix A); however, usually, computer assistance, such as in [23,24], is highly welcome.

The Lie point symmetries of the linear system (33) consist of conventional point symmetries and gauge transformations [9,25]:

$$\tilde{\Psi} = G(x,u)\Psi \approx \Psi + \varepsilon M(x,u)\Psi + \ldots \tag{40}$$

The corresponding vector field can be compactly written as

$$\boldsymbol{V} = \boldsymbol{v} + M\Psi\partial_\Psi, \tag{41}$$

where $\boldsymbol{v}$ is of the form (37), and, below, we denote $\pi(\boldsymbol{V}) = \boldsymbol{v}$.

The Lie algebra $\mathcal{A}'$ (introduced in [9,25]) consists of the Lie point transformations (including the gauge transformations with respect to Ψ), which do not change the system (33) (i.e., $G = 0$):

$$\mathcal{A}' = \{\boldsymbol{v} : \boldsymbol{v} = \pi(\boldsymbol{V}) \text{ and } \boldsymbol{V}^\infty(G)|_{G=0} = 0\}, \tag{42}$$

where the condition $G = 0$ implies all its differential consequences: $D_J(G) = 0$, including $F = 0$. We point out that, obviously, $\mathcal{A}' \subset \mathcal{A}$. One can show (see [25]) that the determining equations for the linear system (33) are given by:

$$D_k(M) = [U_k, M] + \boldsymbol{v}^\infty(U_k) + D_k(\xi^j)U_j. \tag{43}$$

Therefore, in order to compute the vector fields belonging to $\mathcal{A}'$, it is sufficient to find vector fields $\boldsymbol{v}$ that satisfy the system (43).

In the case of the Lie algebra generated by the basis $E_1, \ldots, E_N$:

$$M = M^\alpha E_\alpha, \qquad U_k = U^\alpha_k E_\alpha, \tag{44}$$

where M^α and U^α_k ($\alpha = 1, \ldots, N$) are scalar real functions, and the summation of the repeating indices is assumed. Hence,

$$D_k(M^\alpha) = c^\alpha_{\ \beta\gamma} U^\beta_k M^\gamma + \boldsymbol{v}^\infty(U^\alpha_k) + D_k(\xi^j)U^\alpha_j, \tag{45}$$

where $c^\alpha_{\ \beta\gamma}$ are the so-called structure constants of the Lie algebra. In the case of the Lie algebra su(2), we have $N = 3$ and

$$c^\alpha_{\ \beta\gamma} = \varepsilon_{\alpha\beta\gamma}, \tag{46}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi–Civita symbol (in particular, $\varepsilon_{123} = -\varepsilon_{321} = 1$). Indeed, for the Lie algebra su(2), we have $[E_1, E_2] = E_3$ and its cyclic permutations.

6. Geometric Cases

The differential geometry of submanifolds is a source of actual and potential examples of non-parametric linear problems. Indeed, any class of submanifolds immersed in an ambient space can be characterized by the kinematics of the moving frame given by linear Gauss–Weingarten equations and their non-linear compatibility conditions, the Gauss–Codazzi equations (a system of non-linear partial differential equations) (see, e.g., [26,27]).

The Lie point symmetries of the Gauss–Codazzi equations can be used as a tool to insert a parameter into the Gauss–Weingarten equations. If we succeed and this parameter cannot be removed by any gauge transformation, then we conjecture that this parameter is the spectral parameter—the existence of which is crucial for most soliton theory techniques.

The sine-Gordon Equation (1) is the most known case of this kind. It is associated with pseudospherical surfaces immersed in a three-dimensional Euclidean space.

6.1. Constant Mean Curvature Surfaces

In this case, the Gauss–Codazzi equations can be reduced to the elliptic sinh-Gordon equation [28]:

$$\phi_{xx} + \phi_{yy} + 2\sinh(2\phi) = 0, \tag{47}$$

and the Gauss–Weingarten equations can be put in the following form:

$$\begin{aligned} \Psi_x &= (2\cosh\phi\; E_1 - \phi_y E_2)\Psi, \\ \Psi_y &= (2\sinh\phi\; E_3 + \phi_x E_2)\Psi, \end{aligned} \tag{48}$$

where E_1, E_2, and E_3 denote (throughout this section) the standard basis in the Lie algebra su(2) (then, Ψ belongs to the group SU(2)). In particular,

$$[E_1, E_2] = E_3. \tag{49}$$

One can use one of the standard matrix representations (in terms of the Pauli matrices), but it is sufficient and even easier to proceed without any explicit matrix form. The elliptic sinh-Gordon equation is invariant with respect to the rotation

$$\tilde{x} = x\cos\varepsilon + y\sin\varepsilon, \qquad \tilde{y} = y\cos\varepsilon - x\sin\varepsilon, \tag{50}$$

and the linear system (48) transforms into the Lax pair:

$$\begin{aligned} \Psi_x &= \big((e^{\varphi} + e^{-\varphi}\cos 2\varepsilon)E_1 - \varphi_y E_2 + e^{-\varphi}\sin 2\varepsilon, E_3\big)\Psi \\ \Psi_y &= (e^{-\varphi}\sin 2\varepsilon, E_1 + \varphi_x E_2 + (e^{\varphi} - e^{-\varphi}\cos 2\varepsilon)E_3)\Psi, \end{aligned} \tag{51}$$

which can be transformed into the rational form with respect to $\lambda = e^{2i\varepsilon}$ (compare [29]).

The elliptic sinh-Gordon equation also describes the spherical surfaces (surfaces with a positive constant curvature).

6.2. Generalized Bianchi System

Hyperbolic surfaces (i.e., surfaces with a negative Gaussian curvature, immersed in $\mathbb{E}^3$) can always be parameterized by asymptotic coordinates such that the corresponding fundamental forms read as follows:

$$\begin{aligned} I &= \rho^2\big(a^2 dx^2 + 2ab\cos\varphi dxdt + b^2 dt^2\big), \\ II &= 2\rho ab\sin\varphi dxdt, \end{aligned} \tag{52}$$

where the function $\rho = \rho(x,t)$ is related to the Gaussian curvature ($K = \rho^{-2}$), and functions a, b, and φ satisfy the so-called generalized Bianchi system:

$$
\begin{aligned}
&\varphi_{xt} + \frac{\partial}{\partial x}\left(\frac{\rho_x}{2\rho}\frac{b}{a}\sin\varphi\right) + \frac{\partial}{\partial t}\left(\frac{\rho_t}{2\rho}\frac{a}{b}\sin\varphi\right) = ab\sin\varphi, \\
&a_t + \frac{\rho_t}{2\rho}a - \frac{\rho_x}{2\rho}b\cos\varphi = 0, \\
&b_x + \frac{\rho_x}{2\rho}b - \frac{\rho_t}{2\rho}a\cos\varphi = 0.
\end{aligned}
\tag{53}
$$

In particular, $\varphi = \varphi(x,t)$ is the angle between the coordinate lines (see, e.g., [6]). The Gauss–Weingarten equations can be represented in the forms

$$
\begin{aligned}
\Psi_x &= \left(-aE_3 - \left(\varphi_x + \frac{a\rho_t\sin\varphi}{2b\rho}\right)E_2\right)\Psi, \\
\Psi_t &= \left(b(E_3\cos\varphi - E_1\sin\varphi) + \frac{b\rho_x\sin\varphi}{2a\rho}E_2\right)\Psi,
\end{aligned}
\tag{54}
$$

(see [6]). The Lie algebra $\mathcal{A}'$ consists of vector fields of the form [9]:

$$
\xi\partial_x + \tau\partial_t - a\xi'\partial_a - b\dot{\tau}\partial_b \tag{55}
$$

where $\xi = \xi(x)$, $\tau = \tau(t)$, the prime and dot denote the derivatives, and

$$
\xi\rho_x + \tau\rho_t = 2c_0\rho, \tag{56}
$$

where $c_0 = \text{const}$. In the case of $\rho_{xt} \neq 0$, the Lie algebra $\mathcal{A}$ (i.e., the Lie point symmetries of the system (53)) coincides with $\mathcal{A}'$. If $\rho_{xt} = 0$, then $\mathcal{A}' \neq \mathcal{A}$. In the generic case, if $\rho = h(x) + g(t)$, $h'(x) \neq 0$, and $\dot{g} \neq 0$, the following vector field belongs to $\mathcal{A}$ and does not belong to $\mathcal{A}'$:

$$
\frac{h^2}{h_x}\partial_x - \frac{g^2}{g_t}\partial_t - \left(\frac{d}{dx}\left(\frac{h^2}{h_x}\right) - \frac{g+h}{2}\right)a\partial_a + \left(\frac{d}{dt}\left(\frac{g^2}{g_t}\right) - \frac{g+h}{2}\right)b\partial_b. \tag{57}
$$

The one-parameter group of transformations generated by this vector field inserts the spectral parameter into (54):

$$
\begin{aligned}
\Psi_x &= \left(-\lambda aE_3 - \left(\varphi_x + \frac{a\rho_t\sin\varphi}{2b\rho}\right)E_2\right)\Psi, \\
\Psi_t &= \left(\frac{b}{\lambda}(E_3\cos\varphi - E_1\sin\varphi) + \frac{b\rho_x\sin\varphi}{2a\rho}E_2\right)\Psi,
\end{aligned}
\tag{58}
$$

where

$$
\lambda = \sqrt{\frac{1+2\kappa g(t)}{1-2\kappa f(x)}} \tag{59}
$$

and $\kappa = \text{const}$ (compare [6]). We can see that the Lax pair depends in a simple rational way on λ, which is not constant but does depend on x and t. Therefore, we say that this is a non-isospectral Lax pair (actually the parameter κ can be considered a "true" spectral parameter).

Another interesting observation concerning the Bianchi system can be found in [30]. It turns out that the form (59) is the only possibility for any non-isospectral linear problem that depends linearly on λ and $1/\lambda$.

6.3. Isothermic Surfaces

The isothermic surfaces immersed in $\mathbb{R}^3$ are characterized by the property that the curvature lines allow for conformal parameterization (see, e.g., [31,32]). In other words, there exist coordinates with the following fundamental forms:

$$\begin{aligned} I &= e^{2\theta}(dx^2 + dy^2), \\ II &= e^{2\theta}(k_1 dx^2 + k_2 dy^2), \end{aligned} \tag{60}$$

where k_1 and k_2 are the principal curvatures. The Gauss–Codazzi equations read:

$$\begin{aligned} &\theta_{xx} + \theta_{yy} + k_1 k_2 e^{2\theta} = 0, \\ &k_{1,x} + (k_1 - k_2)\theta_x = 0, \\ &k_{2,y} + (k_2 - k_1)\theta_y = 0. \end{aligned} \tag{61}$$

The Gauss–Weingarten equations can be represented in the forms:

$$\Psi_x = (\theta_y E_3 - k_2 e^{\theta} E_2)\Psi, \qquad \Psi_y = (-\theta_x E_3 + k_1 e^{\theta} E_1)\Psi. \tag{62}$$

The Lie algebra $\mathcal{A}$ (symmetries of the system (61)) is four-dimensional:

$$\partial_x, \quad \partial_y, \quad \partial_\theta - k_1\partial_{k_1} - k_2\partial_{k_2}, \quad x\partial_x + y\partial_y - k_1\partial_{k_1} - k_2\partial_{k_2} \tag{63}$$

Unfortunately, the Lie algebra $\mathcal{A}'$ coincides with $\mathcal{A}$: each transformation (63) leaves the linear problem (62) invariant.

However, it is possible to find a linear problem with the spectral parameter. We just have to start from a different non-parametric linear problem (see [31]):

$$\begin{aligned} \Psi_x &= \tfrac{1}{2}\boldsymbol{e}_1(-\theta_y \boldsymbol{e}_2 - k_2 e^{\theta}\boldsymbol{e}_3 + \boldsymbol{e}_4 \sinh\theta + \boldsymbol{e}_5 \cosh\theta)\Psi, \\ \Psi_y &= \tfrac{1}{2}\boldsymbol{e}_2(-\theta_x \boldsymbol{e}_1 - k_1 e^{\theta}\boldsymbol{e}_3 + \boldsymbol{e}_4 \cosh\theta + \boldsymbol{e}_5 \sinh\theta)\Psi, \end{aligned} \tag{64}$$

Here, in order to simplify notation (the matrix representation is rather awkward), Clifford numbers are used. They are characterized by the properties: $\boldsymbol{e}_j\boldsymbol{e}_k = -\boldsymbol{e}_k\boldsymbol{e}_j$ (for $k \neq j$) and $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \boldsymbol{e}_3 = \boldsymbol{e}_4 = -\boldsymbol{e}_5 = 1$, which are fully sufficient to perform all calculations. In particular, we can easily compute the compatibility conditions for the system (64) and show that they are equivalent to the Gauss–Codazzi Equation (61).

Note that we can identify

$$E_1 = \frac{1}{2}\boldsymbol{e}_3\boldsymbol{e}_2, \quad E_2 = \frac{1}{2}\boldsymbol{e}_1\boldsymbol{e}_3, \quad E_3 = \frac{1}{2}\boldsymbol{e}_2\boldsymbol{e}_1, \tag{65}$$

which yields standard commutation relations for the su(2) Lie algebra, including (49). Actually, the linear problem (64) takes the values in the Lie algebra spin(4,1) (which is isomorphic to so(4,1)).

Symmetries of the linear problem (64) are given by:

$$\partial_x, \quad \partial_y, \quad \partial_\theta - k_1\partial_{k_1} - k_2\partial_{k_2}, \quad x\partial_x + y\partial_y - k_1\partial_{k_1} - k_2\partial_{k_2} - \frac{1}{2}c_3\boldsymbol{e}_4\boldsymbol{e}_5\Psi\partial_\Psi \tag{66}$$

By projecting them onto the space (x, y, θ, k_1, k_2), we obtain the Lie algebra $\mathcal{A}'$:

$$\partial_x, \quad \partial_y, \quad \partial_\theta - k_1\partial_{k_1} - k_2\partial_{k_2}, \quad x\partial_x + y\partial_y - k_1\partial_{k_1} - k_2\partial_{k_2}. \tag{67}$$

By comparing (67) and (63), we see that the spectral parameter is inserted, again, by the scaling transformation:

$$x\partial_x + y\partial_y - k_1\partial_{k_1} - k_2\partial_{k_2}. \tag{68}$$

The linear problem (64) is transformed into

$$\begin{aligned} \Psi_x &= \tfrac{1}{2}\boldsymbol{e}_1(-\theta_y\boldsymbol{e}_2 - k_2 e^{\theta}\boldsymbol{e}_3 + \lambda\boldsymbol{e}_4\sinh\theta + \lambda\boldsymbol{e}_5\cosh\theta)\Psi, \\ \Psi_y &= \tfrac{1}{2}\boldsymbol{e}_2(-\theta_x\boldsymbol{e}_1 - k_1 e^{\theta}\boldsymbol{e}_3 + \lambda\boldsymbol{e}_4\cosh\theta + \lambda\boldsymbol{e}_5\sinh\theta)\Psi. \end{aligned} \tag{69}$$

This form of the spectral problem first appeared in [31], motivated by geometric considerations that can be traced back to the works of Darboux and Bianchi. The Lie point symmetries were mentioned, in this context, in [33] and discussed in detail in [34].

6.4. Open Problem: Chebyshev and Semi-Geodesic Coordinates

Krasil'shchik and Marvan found Lax pairs with non-removable parameters for two large classes of surfaces immersed in $\mathbb{R}^3$ [35] by considering the Chebyshev coordinates:

$$\begin{aligned} I &= dx^2 + 2\cos\varphi dx dt + dt^2, \\ II &= b_{11}dx^2 + 2b_{12}dxdt + b_{22}dt^2, \end{aligned} \tag{70}$$

and the semi-geodesic coordinates:

$$\begin{aligned} I &= dx^2 + f(x,y)dy^2, \\ II &= b_{11}dx^2 + 2b_{12}dxdt + b_{22}dt^2, \end{aligned} \tag{71}$$

and applying the covering theory approach. They obtained several integrable cases (characterized by some restrictions on the coefficients of fundamental forms), including linear Weingarten surfaces (where the Gaussian curvature is a linear function of the mean curvature). An interesting open problem consists of performing an analysis of the Lie symmetries for all of these cases.

7. Spectral Parameter as a Group Parameter—Recent Results

In this section, we consider three multidimensional equations with scalar linear problems. They arise as reductions of a large family of equations proposed by Bogdanov and Pavlov [36]. In a recent paper, Morozov obtained their Lax representations using non-central extensions of the symmetry algebras generated by exotic cohomology groups [19]. We are going to show that the spectral parameters in these Lax pairs are group parameters and cannot be removed by gauge transformations. Detailed calculations, presented in the Appendix A, can be also considered as a demonstration of our approach.

7.1. Hyper-CR Equation for Einstein–Weyl Structures

The hyper-Cauchy–Riemann equation for Einstein–Weyl structures, introduced in [37] (see also [19,38]), reads:

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \tag{72}$$

It has the following very simple, non-parametric spectral problem:

$$v_t = -u_y v_x, \quad v_y = -u_x v_x, \tag{73}$$

In Appendix A.1, we found infinitesimal symmetries both for the system (72) and for (73). The Lie algebra $\mathcal{A}$ of the point symmetries of (72) is spanned by

$$\begin{aligned} &A\partial_t + (\dot{A}x + \tfrac{1}{2}\ddot{A}y^2)\partial_x + (\dot{A}y)\partial_y + (\dot{A}u + \ddot{A}xy + \tfrac{1}{6}\dddot{A}y^3)\partial_u, \\ &\dot{B}y\partial_x + B\partial_y + (\dot{B}x + \tfrac{1}{2}\ddot{B}y^2)\partial_u, \qquad C\partial_x + \dot{C}y\partial_u, \qquad D\partial_u, \\ &t\partial_t - x\partial_x - 2u\partial_u, \qquad \partial_t, \qquad y\partial_x + 2x\partial_u, \end{aligned} \tag{74}$$

where A, B, C, and D are arbitrary functions of t. The Lie algebra $\mathcal{A}'$ for the linear system (73) is spanned by the same vector fields, except for the last one (see Appendix A.1).

Therefore, the spectral parameter λ is associated with the one-parameter group generated by the vector field $y\partial_x + 2x\partial_u$, which generates the following one-parameter group of transformations:

$$\tilde{x} = x + \lambda y, \quad \tilde{y} = y, \qquad \tilde{t} = t, \qquad \tilde{u} = u + 2\lambda x + \lambda^2 y. \tag{75}$$

By prolonging this action on the derivatives (note that $\partial_{\tilde{y}} = \partial_y - \lambda\partial_x$) and substituting them into (73), we obtain a linear problem with the spectral parameter λ:

$$v_t = (\lambda^2 - \lambda u_x - u_y)v_x, \quad v_y = (\lambda - u_x)v_x, \tag{76}$$

where the tilde is omitted.

7.2. Four-Dimensional Bogdanov–Pavlov Equation

Bogdanov and Pavlov considered a class of "quasi-classical" self-dual Yang–Mills equations and their reductions [36]. One of them is given by the following $3+1$-dimensional equation (compare also [19]):

$$u_{zz} = u_{tx} + u_z u_{xy} - u_x u_{yz}. \tag{77}$$

The corresponding non-parametric linear problem reads as follows:

$$v_t = -u_z v_y, \quad v_z = -u_x v_y. \tag{78}$$

In Appendix A.2, we found infinitesimal symmetries both for the system (77) and for (78). The Lie algebra $\mathcal{A}$ of point symmetries of (77) is spanned by

$$\begin{aligned} &A\partial_u, \quad \mu\partial_y + (\mu_y u + \mu_t z)\partial_u, \quad 2t\partial_t - u\partial_u, \quad 2x\partial_x + z\partial_z - u\partial_u, \\ &\partial_t, \quad \partial_x, \quad \partial_z, \quad z\partial_x + 2t\partial_z \end{aligned} \tag{79}$$

where $A = A(t, y)$ and $\mu = \mu(t, y)$. The Lie algebra $\mathcal{A}'$ for the linear system (78) is spanned by the same vector fields, except for the last one (see Appendix A.2).

Therefore, the spectral parameter λ is associated with the one-parameter group generated by the vector field $z\partial_x + 2t\partial_z$, which generates the following one-parameter group of transformations:

$$\tilde{x} = x + \lambda z + \lambda^2 t, \qquad \tilde{z} = z + 2\lambda t \tag{80}$$

We prolong this action on the derivatives:

$$\partial_{\tilde{t}} = \partial_t - 2\lambda\partial_z + \lambda^2\partial_x, \quad \partial_{\tilde{x}} = \partial_x, \quad \partial_{\tilde{z}} = \partial_z - \lambda\partial_x, \tag{81}$$

and transform the linear problem (78) into the following linear problem with the spectral parameter λ:

$$v_t = \lambda^2 v_x - (\lambda u_x + u_z)v_y, \qquad v_z = \lambda v_x - u_x v_y, \tag{82}$$

where the tilde is omitted.

7.3. Martínez Alonso–Shabat Equation

For the last example, we consider one of the hydrodynamic-type equations belonging to the hierarchies introduced in [39] (see also [40,41]):

$$u_{ty} = u_z u_{xy} - u_y u_{xz} \tag{83}$$

One can easily verify that

$$v_y = u_y v_x, \quad v_z = u_z v_x - v_t, \tag{84}$$

is a linear problem for (83).

The Lie point symmetries of Equation (83), computed in Appendix A.3, have the following generators:

$$\xi\partial_x + (\xi_x u - \xi_t z)\partial_u, \quad \mu\partial_y, \quad \alpha\partial_u, t\partial_t + z\partial_z, \quad \partial_t, \quad \partial_z, \quad z\partial_z + u\partial_u \tag{85}$$

where $\alpha = \alpha(t, x)$, $\xi = \xi(t, x)$, and $\mu = \mu(y, z)$ are the arbitrary functions of two variables.

The Lie algebra $\mathcal{A}'$, corresponding to the Lie point symmetries of the linear problem (84), is spanned by the same vector fields, except for the last one. The one-parameter group generated by the field $z\partial_z + u\partial_u$ is a simple scaling transformation ($\tilde{z} = \lambda z, \tilde{u} = \lambda u$). It transforms (84) into the following Lax pair:

$$v_y = \lambda u_y v_x, \quad v_z = \lambda(u_z v_x - v_t). \tag{86}$$

The scaling transformation above can easily be guessed, but the calculations performed in Appendix A.3 also show that the obtained spectral parameter cannot be removed by a gauge transformation.

8. Extended Lie Point Symmetries

Extended Lie point symmetries are defined as symmetries of a family of equations, and the family is parameterized by a set of functions $f^1, \ldots, f^N$ (which are prescribed functions, treated as parameters in the considered equations).

The vector field corresponding to the extended point transformation contains derivatives with respect to these parameters

$$\boldsymbol{v} = \xi^j \frac{\partial}{\partial x^j} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \Phi^N \frac{\partial}{\partial f^N}. \tag{87}$$

The prolongation of the field $\boldsymbol{v}$ is computed in the standard way:

$$\mathrm{pr}\,\boldsymbol{v} = \xi^j \frac{\partial}{\partial x^j} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \Gamma^\alpha_j \frac{\partial}{\partial u^\alpha_j} + \Lambda^N_J \frac{\partial}{\partial f^N_J} \tag{88}$$

where, as usual, the summation of the repeating indices (j, α) and multi-indices (J) is assumed.

The key assumption is that we consider the transformations that are standard Lie point transformations with respect to the variables ξ^j and η^α, but we assume nothing about the transformation of the variables f^N. Therefore, in the process of solving the determining equations, we do not treat the functions f^N and their derivatives as independent variables in the jet space. We point out that other authors have recently begun to use a very similar approach (although on the level of Lie group rather than Lie algebras) and have described it in terms of Lie groupoids [42].

In the case of the extended Lie point symmetries, one can define the Lie algebras $\mathcal{A}$ and $\mathcal{A}'$ in a full analogy to the case of the conventional Lie point symmetries discussed in Section 5.

8.1. Non-Homogeneous Non-Linear Schrödinger System

The geometrical considerations starting from semi-geodesic coordinates (71) (see [43]) lead to an interesting generalization of the cubic non-linear Schrödinger equation:

$$iq_t + (fq)_{xx} + 2qR = 0, \qquad R_x = (f|q|^2)_x + f_x|q|^2. \tag{89}$$

The corresponding non-parametric linear problem (motivated by the Gauss–Weingarten equations) is given by:

$$\Psi_x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}\Psi, \qquad \Psi_t = \begin{pmatrix} iR & i(fq)_x \\ i(f\bar{q})_x & -iR \end{pmatrix}\Psi. \tag{90}$$

The system (89) is closely related (gauge equivalent) to the classical continuum non-homogeneous Heisenberg Ferromagnet Model $\vec{S}_t = \vec{S} \times (f\vec{S}_x)_x$ (where $\vec{S} \in \mathbb{R}^3$ and $(\vec{S})^2 = 1$) [44] (see also [10]).

If $f_{xx} \neq 0$, then the extended Lie point symmetries of the non-homogeneous non-linear Schrödinger system (89) are generated by

$$\partial_x, \quad x\partial_x - q\partial_q + 2f\partial_f, \quad \tau\partial_t - \dot{\tau}(R\partial_R + f\partial_f), \quad 2i\mu q\partial_q + \dot{\mu}\partial_R, \tag{91}$$

where $\tau = \tau(t)$ and $\mu = \mu(t)$ are arbitrary functions. In this case, algebras of the extended Lie point symmetries of (89) and (90) are identical: $\mathcal{A}' = \mathcal{A}$ [9,25].

If $f_{xx} = 0$, i.e., $f = ax + b$, where $a = a(t)$ and $b = b(t)$, then the algebra $\mathcal{A}'$ of the extended Lie point symmetries of the system (90) is spanned by:

$$\partial_x - a\partial_b, \quad x\partial_x - q\partial_q + a\partial_a + 2b\partial_b, \quad \tau\partial_t - \dot{\tau}(R\partial_R + a\partial_a + b\partial_b), \quad 2i\mu q\partial_q + \dot{\mu}\partial_R, \tag{92}$$

where $\tau = \tau(t)$ and $\mu = \mu(t)$ are arbitrary functions. In this case, however, the Lie algebra $\mathcal{A}$ is larger and also contains the following "nonlocal" vector field [45]:

$$2x(x\textstyle\int a + \int b)\partial_x + (ix - 2\int a)q\partial_q + 2a\int a\partial_a + (4b\int a - 2a\int b)\partial_b. \tag{93}$$

Surprisingly enough, this vector field can be explicitly integrated (see [46]):

$$\begin{aligned}
&\tilde{x} = \frac{x}{(1-\kappa\int a)^2} + \int\frac{2\kappa b}{(1-\kappa\int a)^3}, \qquad \tilde{t} = t, \qquad \tilde{R} = R, \\
&\tilde{q} = q(1-\kappa\textstyle\int a)^2 \exp\left(\frac{ikx}{1-\kappa\int a} + \int\frac{i\kappa^2 b}{(1-\kappa\int a)^2}\right), \\
&\tilde{a} = \frac{a}{(1-\kappa\int a)^2}, \qquad \tilde{b} = \frac{b}{(1-\kappa\int a)^4} - \frac{a}{(1-\kappa\int a)^2}\int\frac{2\kappa b}{(1-\kappa\int a)^3}.
\end{aligned} \tag{94}$$

This transformation can be considered a generalization of the Galilean symmetry for the non-homogeneous case. By applying it to the linear system (90), we obtain the following non-isospectral Lax pair:

$$\Psi_x = \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}\Psi, \qquad \Psi_t = \begin{pmatrix} i|q|^2 - 2if\lambda^2 & i(fq)_x - 2\lambda fq \\ 2\lambda f\bar{q} + i(f\bar{q})_x & 2if\lambda^2 - i|q|^2 \end{pmatrix}\Psi, \tag{95}$$

where

$$\lambda = \frac{\kappa}{2 + 2\kappa\int a}. \tag{96}$$

This example is especially interesting because the extended Lie point symmetries seem to be necessary in order to introduce a spectral parameter. By using the standard Lie point symmetries, we lose some cases. The Lie point symmetries cannot insert the spectral parameter for any linear functions in x, but rather only for linear functions of the following special form:

$$f(x,t) = \dot{\alpha}(x + c_1 + c_2\alpha) \quad \text{or} \quad f(x,t) = b(t), \tag{97}$$

where c_1 and c_2 are constants, and $\alpha = \alpha(t)$ and $b = b(t)$ are the given functions of one variable.

8.2. Generalized Bianchi System

In this case, the Lie point symmetries are fully sufficient to isolate integrable cases (see Section 6.2), but, interestingly enough, the application of the extended Lie point symmetries yields the same result in an alternative way. In the case of $\rho_{xt} \neq 0$, we have $\mathcal{A} = \mathcal{A}'$, while in the case of $\rho = \rho = h(x) + g(t)$, there exists one extended Lie point symmetry of (53), which does not belong to $\mathcal{A}'$:

$$(f+g)(a\partial_a - b\partial_b) + 2f^2\partial_f - 2g^2\partial_g. \tag{98}$$

The corresponding one-parameter group

$$\tilde{h} = \frac{h}{1-2\kappa h}, \quad \tilde{g} = \frac{g}{1+2\kappa g}, \quad \tilde{a} = a\sqrt{\frac{1+2\kappa g}{1-2\kappa h}}, \quad \tilde{b} = b\sqrt{\frac{1-2\kappa h}{1+2\kappa g}}, \tag{99}$$

transforms (54) into the Lax pair (58).

Comparing the vector fields (57) and (98), which insert the spectral parameter, we can say that although both lead to the same result, in this case, the extended Lie point symmetries seem to be computationally simpler and more elegant.

9. Conclusions

In this paper, we presented results supporting the conjecture that the spectral parameters of the soliton theory can be interpreted as non-removable (by gauge transformations) parameters of explicitly given one-parameter groups. In Sections 2 and 3, we underlined the important role of simple scaling transformations. In many cases, including the ZS-AKNS class of integrable systems, the spectral parameter is related to scaling. The Galilean transformation (see Section 4), although less common, introduces the spectral parameter into the simplest non-parametric linear problems in two very important cases: the Korteweg–de Vries equation and the non-linear Schrödinger equation.

The geometric models presented in Section 6 are related to the geometry of the surfaces in $\mathbb{R}^3$. Therefore, they have su(2)-valued linear problems (su(2) is isomorphic to so(3)). One case, namely the isothermic surfaces, is especially interesting because it is not possible to introduce a parameter into the su(2)-valued linear problem. In order to obtain a Lax pair with the spectral parameter, one has to start from another closely related linear problem that takes the values from a larger Lie algebra: so(4,1). There are many geometric problems that can be investigated by the methods outlined in our paper.

Usually, in this context, we consider the Lie point transformations. In Section 5, we present the algorithm for verifying whether it is possible to insert a non-removable parameter into a given non-parametric linear problem. This consists of comparing two Lie algebras, $\mathcal{A}$ and $\mathcal{A}'$. We always have $\mathcal{A}' \subset \mathcal{A}$. The condition $\mathcal{A}' \neq \mathcal{A}$ is sufficient for the existence of a non-removable parameter. This parameter is related to any vector field belonging to $\mathcal{A}' \setminus \mathcal{A}$. In Section 7 (with details presented in the Appendix A), we apply this procedure to the hyper-CR equation for Einstein–Weyl structures, the four-dimensional Bogdanov–Pavlov equation, and the Martínez Alonso–Shabat equation. In all of the investigated cases, we have $\dim \mathcal{A} - \dim \mathcal{A}' \leqslant 1$ (i.e., we can insert at most one parameter according to the symmetries). It would be challenging to find a different case in which there are more spectral parameters.

In the case of non-autonomous systems, in which the coefficients can be explicitly dependent on the independent variables (e.g., all "non-homogeneous" systems), the Lie point symmetries seem to be too restrictive, and we often need a larger class of symmetries. In such a case, we propose consideration of the so-called extended Lie point symmetries. In fact, they are not symmetries of a fixed equation or system, but rather symmetries of the whole class of equations (with any parameters or non-homogeneities). In Section 8, we present two older, interesting examples, namely the non-homogeneous non-linear Schrödinger system and the generalized Bianchi system. In the near future, we plan to check and investigate other cases in which non-removable parameters are inserted by extended Lie point transformations.

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Appendix A

In the Appendix A, we present some basic information about Lie point symmetries, following classical textbooks, such as [22,47], including more details related to Section 7.

We consider a non-linear partial differential equation of the form

$$F(t,x,y,z,u,u_t,u_x,u_y,u_z,u_{tt},u_{tx},u_{ty},u_{tz},u_{xx},u_{xy},u_{xz},u_{yy},u_{yz},u_{zz}) = 0, \tag{A1}$$

where $u = u(t,x,y,z)$ is a dependent variable (an unknown). We assume that the Equation (A1) is a compatibility condition for the system of two equations that are linear with respect to another variable $v = v(t,x,y,z)$:

$$G_k(t,x,y,z,v,u,v_t,v_x,v_y,v_z,u_t,u_x,u_y,u_z) = 0, \quad (k = 1,2), \tag{A2}$$

Our goal is to find the Lie point symmetries of Equation (A1) which are not the symmetries of the system (A2). First, we have to compute one-parameter groups of the transformations preserving Equation (A1). We consider a general invertible transformation of the form:

$$\begin{aligned} &\widetilde{t} = \varphi_0(t,x,y,z,u,\varepsilon), \quad \widetilde{y} = \varphi_2(t,x,y,z,u,\varepsilon), \quad \widetilde{u} = \chi(t,x,y,z,u,\varepsilon), \\ &\widetilde{x} = \varphi_1(t,x,y,z,u,\varepsilon), \quad \widetilde{z} = \varphi_3(t,x,y,z,u,\varepsilon), \end{aligned} \tag{A3}$$

where

$$\begin{aligned} &\varphi_0(t,x,y,z,u,0) = t, \quad \varphi_2(t,x,y,z,u,0) = y, \quad \chi(t,x,y,z,u,0) = u, \\ &\varphi_1(t,x,y,z,u,0) = x, \quad \varphi_3(t,x,y,z,u,0) = z. \end{aligned} \tag{A4}$$

We expand functions (A3) into the Taylor series with respect to the parameter ε in the neighborhood of $\varepsilon = 0$ and, taking into account (A4), we confine ourselves to the linear terms in ε:

$$\begin{aligned} &\widetilde{t} \approx t + \varepsilon\tau(t,x,y,z,u), \quad \widetilde{y} \approx y + \varepsilon\mu(t,x,y,z,u), \quad \widetilde{u} \approx u + \varepsilon\eta(t,x,y,z,u), \\ &\widetilde{x} \approx x + \varepsilon\xi(t,x,y,z,u), \quad \widetilde{z} \approx z + \varepsilon\rho(t,x,y,z,u). \end{aligned} \tag{A5}$$

In other words,

$$\begin{aligned} &\tau = \left.\frac{\partial\varphi_0(t,x,y,z,u,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \qquad \xi = \left.\frac{\partial\varphi_1(t,x,y,z,u,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \\ &\mu = \left.\frac{\partial\varphi_2(t,x,y,z,u,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \qquad \rho = \left.\frac{\partial\varphi_3(t,x,y,z,u,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \\ &\eta = \left.\frac{\partial\chi(t,x,y,z,u,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}. \end{aligned} \tag{A6}$$

Thus, the one-parameter group (A3) is represented by the vector field

$$\tau\partial_t + \xi\partial_x + \mu\partial_y + \rho\partial_z + \eta\partial_u. \tag{A7}$$

In order to prolong this vector field on the so-called jet space (see [22]), i.e., to compute the induced transformation of the partial derivatives, we assume and use the invariance of the differential form

$$du = u_t dt + u_x dx + u_y dy + u_z dx. \tag{A8}$$

Indeed, by inserting the transformed functions (A3) into (A8), we obtain:

$$d\chi = \widetilde{u}_{\tilde{t}}, d\varphi_0 + \widetilde{u}_{\tilde{x}}, d\varphi_1 + \widetilde{u}_{\tilde{y}}, d\varphi_2 + \widetilde{u}_{\tilde{z}}, d\varphi_3, \tag{A9}$$

Then, by using (A5) and comparing the coefficients by the differentials dt, dx, dy, and dz in (A9), we obtain:

$$\widetilde{u}_{\tilde{t}} \approx u_t + \varepsilon\Gamma_0, \quad \widetilde{u}_{\tilde{x}} \approx u_x + \varepsilon\Gamma_1, \quad \widetilde{u}_{\tilde{y}} \approx u_t + \varepsilon\Gamma_2, \quad \widetilde{u}_{\tilde{z}} \approx u_t + \varepsilon\Gamma_3, \tag{A10}$$

where

$$\Gamma_p = D_p(\eta) - D_p(\tau)u_t - D_p(\xi)u_x - D_p(\mu)u_y - D_p(\rho)u_z, \tag{A11}$$

where $p = 0, 1, 2, 3$, and D_p are total derivatives ($D_0 \equiv D_t$, $D_1 \equiv D_x$, $D_2 \equiv D_y$, and $D_3 \equiv D_z$). In our case:

$$D_p = \partial_p + u_p\partial_u. \tag{A12}$$

Similarly, considering the differentials of u_p, we obtain prolongations of the second-order derivatives

$$\widetilde{u}_{\tilde{p}\tilde{q}} = u_{pq} + \varepsilon\Gamma_{pq}, \tag{A13}$$

where

$$\Gamma_{pq} = D_{pq}(\eta - \tau u_t - \xi u_x - \mu u_y - \rho u_z) + \tau u_{tpq} + \xi u_{xpq} + \mu u_{ypq} + \rho u_{zpq} \tag{A14}$$

and

$$D_{pq} = \partial^2_{pq} + u_p\partial^2_{qu} + u_{pq}\partial_u + u_q\partial^2_{pu} + u_p u_q\partial^2_u. \tag{A15}$$

A straightforward computation yields

$$\begin{aligned}\Gamma_{pq} = {} & D_{pq}(\eta) - D_{pq}(\tau)u_t - D_p(\tau)u_{qt} - D_q(\tau)u_{pt} \\ & - D_{pq}(\xi)u_x - D_p(\xi)u_{qx} - D_q(\xi)u_{px} - D_{pq}(\mu)u_y \\ & - D_p(\mu)u_{qy} - D_q(\mu)u_{py} - D_{pq}(\rho)u_z - D_p(\rho)u_{qz} - D_q(\rho)u_{pz}.\end{aligned} \tag{A16}$$

By substituting (A5), (A10) and (A13) into (A1), expanding the result to the Taylor series, and equating to the zero term linear in ε, we obtain the so-called determining equations:

$$\begin{aligned}&\tau\frac{\partial F}{\partial t} + \xi\frac{\partial F}{\partial x} + \mu\frac{\partial F}{\partial y} + \rho\frac{\partial F}{\partial z} + \eta\frac{\partial F}{\partial u} + \Gamma_0\frac{\partial F}{\partial u_t} + \Gamma_1\frac{\partial F}{\partial u_x} + \Gamma_2\frac{\partial F}{\partial u_y} \\ &\quad + \Gamma_3\frac{\partial F}{\partial u_z} + \Gamma_{00}\frac{\partial F}{\partial u_{tt}} + \Gamma_{01}\frac{\partial F}{\partial u_{tx}} + \Gamma_{02}\frac{\partial F}{\partial u_{ty}} + \Gamma_{03}\frac{\partial F}{\partial u_{tz}} + \Gamma_{11}\frac{\partial F}{\partial u_{xx}} \\ &\quad + \Gamma_{12}\frac{\partial F}{\partial u_{xy}} + \Gamma_{13}\frac{\partial F}{\partial u_{xz}} + \Gamma_{22}\frac{\partial F}{\partial u_{yy}} + \Gamma_{23}\frac{\partial F}{\partial u_{yz}} + \Gamma_{33}\frac{\partial F}{\partial u_{zz}} = 0.\end{aligned} \tag{A17}$$

The variables t, x, y, z, u, u_p, and u_{pq} are treated as independent variables subject to the constraint (A1). By solving the system (A17), we obtain the vector fields (A7) generating the Lie point symmetries of (A1). The obtained Lie algebra is denoted by $\mathcal{A}$.

Now, we proceed to the Lie point symmetries of the system (A2). We consider the transformation of the form (A3) accompanied by

$$\tilde{v} = \psi(t, x, y, z, u, v, \varepsilon) \approx v + \varepsilon\gamma(t, x, y, z, u, v). \tag{A18}$$

In other words, we consider vector fields of the form:

$$\tau\partial_t + \xi\partial_x + \mu\partial_y + \rho\partial_z + \eta\partial_u + \gamma\partial_v, \tag{A19}$$

where τ, ξ, μ, ρ, and η depend on t, x, y, z, and u (as before), and γ can depend also on v. The transformation group corresponding to the generator (A19) is found by solving the Lie equations

$$\begin{aligned} \frac{d\varphi_0}{d\varepsilon} &= \tau(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi), & \frac{d\varphi_1}{d\varepsilon} &= \xi(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi), \\ \frac{d\varphi_2}{d\varepsilon} &= \mu(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi), & \frac{d\varphi_3}{d\varepsilon} &= \rho(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi), \\ \frac{d\chi}{d\varepsilon} &= \eta(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi), & \frac{d\psi}{d\varepsilon} &= \gamma(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \chi, \psi), \end{aligned} \tag{A20}$$

with the initial conditions given by (A4) together with $\psi(t, x, y, z, u, v, 0) = v$.

The prolongation formulas for the derivatives of v have the form

$$\widetilde{v}_{\tilde{t}} \approx v_t + \varepsilon\Lambda_0, \quad \widetilde{v}_{\tilde{x}} \approx v_x + \varepsilon\Lambda_1, \quad \widetilde{v}_{\tilde{y}} \approx v_t + \varepsilon\Lambda_2, \quad \widetilde{v}_{\tilde{z}} \approx v_t + \varepsilon\Lambda_3, \tag{A21}$$

where

$$\Lambda_p = D_p(\gamma) - D_p(\tau)v_t - D_p(\xi)v_x - D_p(\mu)v_y - D_p(\rho)v_z. \tag{A22}$$

By computing the total derivative, we have to take into account the dependence on v, i.e.,

$$D_p = \partial_p + u_p\partial_u + v_p\partial_v. \tag{A23}$$

By substituting (A5), (A10) and (A21) into (A2), expanding the result into the Taylor series, and equating to the zero term linear in ε, we obtain the so-called determining equations:

$$\begin{aligned} &\tau\frac{\partial G_k}{\partial t} + \xi\frac{\partial G_k}{\partial x} + \mu\frac{\partial G_k}{\partial y} + \rho\frac{\partial G_k}{\partial z} + \gamma\frac{\partial G_k}{\partial v} + \eta\frac{\partial G_k}{\partial u} + \Lambda_0\frac{\partial G_k}{\partial v_t} + \Lambda_1\frac{\partial G_k}{\partial v_x} + \\ &+\Lambda_2\frac{\partial G_k}{\partial v_y} + \Lambda_3\frac{\partial G_k}{\partial v_z} + \Gamma_0\frac{\partial G_k}{\partial u_t} + \Gamma_1\frac{\partial G_k}{\partial u_x} + \Gamma_2\frac{\partial G_k}{\partial u_y} + \Gamma_3\frac{\partial G_k}{\partial u_z} = 0, \quad (k = 1, 2). \end{aligned} \tag{A24}$$

The projection of the obtained Lie algebra on the space parameterized by t, x, y, z, and u is denoted by $\mathcal{A}'$.

In the case of the non-linear system (A1), the procedure described above is a special case of the general approach outlined in Section 5, in which we have to identify the variables and symmetry generators as follows:

$$\begin{aligned} &x^1 = t, \ x^2 = x, \ x^3 = y, \ x^4 = z, \ u^1 = u, \\ &\xi^1 = \tau, \ \xi^2 = \xi, \ \xi^3 = \mu, \ \xi^4 = \rho, \ \eta^1 = \eta. \end{aligned} \tag{A25}$$

However, the linear system (A2) has, as a rule, a form different from (33), so the results of Section 5 cannot be directly applied here.

Appendix A.1. Hyper-CR Equation for Einstein–Weyl Structures

We consider Equation (72) and the linear system (73) as special cases of Equations (A1) and (A2), respectively. The transformed system (73) will take the form

$$\widetilde{v}_{\tilde{t}} = -\widetilde{u}_{\tilde{y}}\widetilde{v}_{\tilde{x}}, \quad \widetilde{v}_{\tilde{y}} = -\widetilde{u}_{\tilde{x}}\widetilde{v}_{\tilde{x}}. \tag{A26}$$

By substituting (A10) and (A21) into (A26), we obtain determining equations

$$\Lambda_0 = -\Lambda_1 u_y - \Gamma_2 v_x, \quad \Lambda_2 = -\Lambda_1 u_x - \Gamma_1 v_x. \tag{A27}$$

Next, by substituting (A11) and (A22) into (A27) and treating the derivatives as independent elements, we obtain the system

$$\xi_t = \eta_y, \quad \xi_x = \frac{1}{2}(\mu_y + \eta_u) = \dot{\tau} - \mu_y + \eta_u, \quad \xi_y = \mu_t = \eta_x, \tag{A28}$$

where $\tau = \tau(t), \xi = \xi(t,x,y), \mu = \mu(t,y), \gamma = \gamma(v), \eta = \eta(t,x,y,u)$.

Solving the system (A28), we find the Lie point symmetries for the system (73):

$$\begin{aligned} &\tau = A - c_1 t + c_2, \qquad \xi = (\dot{A} + c_1)x + \ddot{A}\frac{y^2}{2} + \dot{B}y + C, \qquad \mu = \dot{A}y + B, \\ &\eta = (\dot{A} + 2c_1)u + \ddot{A}xy + \dot{B}x + \dddot{A}\frac{y^3}{6} + \ddot{B}\frac{y^2}{2} + \dot{C}y + D, \qquad \gamma = \gamma(v), \end{aligned} \tag{A29}$$

where c_1 and c_2 are constants, and A, B, C, and D are functions of t.

Next, we transform Equation (72):

$$\widetilde{u}_{\widetilde{y}\widetilde{y}} = \widetilde{u}_{\widetilde{t}\widetilde{x}} + \widetilde{u}_{\widetilde{y}}\widetilde{u}_{\widetilde{x}\widetilde{x}} - \widetilde{u}_{\widetilde{x}}\widetilde{u}_{\widetilde{x}\widetilde{y}}. \tag{A30}$$

By substituting (A10) and (A13) into (A30), we obtain the determining equations

$$\Gamma_{22} = \Gamma_{01} + \Gamma_{11}u_y + \Gamma_2 u_{xx} - \Gamma_{12}u_x - \Gamma_1 u_{xy}. \tag{A31}$$

Then, substituting (A11) and (A16) into (A31), we obtain the system

$$\begin{aligned} &\xi_t = \eta_y, \quad \xi_x = \frac{1}{2}(\mu_y + \eta_u) = -\dot{\tau} + 2\mu_y, \quad \xi_y = \frac{1}{2}(\mu_t + \eta_x), \\ &\xi_{tx} - \xi_{yy} = \eta_{tu} - \eta_{xy}, \quad \xi_{xy} = \eta_{yu}, \quad \mu_{yy} = -\eta_{xx} + 2\eta_{yu}, \quad \eta_{tx} = \eta_{yy}, \\ &\xi_{xx} = \eta_{xu} = \eta_{uu} = 0, \end{aligned} \tag{A32}$$

where $\tau = \tau(t), \xi = \xi(t,x,y), \mu = \mu(t,y), \eta = \eta(t,x,y,u)$.

By solving the system (A32), we find the Lie point symmetries for the system (72):

$$\begin{aligned} &\tau = A - c_1 t + c_2, \quad \xi = (\dot{A} + c_1)x + \ddot{A}\frac{y^2}{2} + (\dot{B} + c_3)y + C, \quad \mu = \dot{A}y + B, \\ &\eta = (\dot{A} + 2c_1)u + \ddot{A}xy + (\dot{B} + 2c_3)x + \dddot{A}\frac{y^3}{6} + \ddot{B}\frac{y^2}{2} + \dot{C}y + D, \end{aligned} \tag{A33}$$

where c_1, c_2, and c_3 are constants, while A, B, C, and D are functions of t.

Both (A29) and (A33) define infinitely dimensional Lie algebras that are very similar to each other. The parameter c_3, absent in (A29), can be interpreted as the spectral parameter. The function γ is related to the gauge invariance of the system (73).

Appendix A.2. Four-Dimensional Bogdanov–Pavlov Equation

We consider Equation (77) and the linear system (78) as special cases of Equations (A1) and (A2), respectively. The transformed system (78) will take the form

$$\widetilde{v}_{\widetilde{t}} = -\widetilde{u}_{\widetilde{z}}\widetilde{v}_{\widetilde{y}}, \quad \widetilde{v}_{\widetilde{z}} = -\widetilde{u}_{\widetilde{x}}\widetilde{v}_{\widetilde{y}}. \tag{A34}$$

By substituting (A10) and (A21) into (A34), we obtain the determining equations

$$\Lambda_0 = -\Lambda_2 u_z - \Gamma_3 v_y, \quad \Lambda_3 = -\Lambda_2 u_x - \Gamma_1 v_y. \tag{A35}$$

Then, by ssubstituting (A11) and (A22) into (A35) and treating the derivatives as independent elements, we obtain the system

$$\mu_t = \eta_z, \quad \mu_y = -\dot{\xi} + \dot{\rho} + \eta_u = \dot{\tau} - \dot{\rho} + \eta_u, \tag{A36}$$

where $\tau = \tau(t), \xi = \xi(x), \mu = \mu(t,y), \rho = \rho(z), \eta = \eta(t,y,z,u)$, and $\gamma = \gamma(x,v)$.
By solving the system (A36), we find the Lie point symmetries for the system (78):

$$\begin{aligned} &\tau = 2c_1 t + c_2, \quad \xi = 2c_3 x + c_4, \quad \mu = \mu(t,y), \quad \rho = (c_1 + c_3)z + c_5, \\ &\eta = (-c_1 + c_3 + \mu_y)u + \mu_t z + A, \quad \gamma = \gamma(x,v), \end{aligned} \tag{A37}$$

where c_1, c_2, c_3, c_4, and c_5 are constants, and A is a function of t, y.
Now, we transform Equation (77):

$$\widetilde{u}_{\tilde{z}\tilde{z}} = \widetilde{u}_{\tilde{t}\tilde{x}} + \widetilde{u}_{\tilde{z}}\widetilde{u}_{\tilde{x}\tilde{y}} - \widetilde{u}_{\tilde{x}}\widetilde{u}_{\tilde{y}\tilde{z}}. \tag{A38}$$

By substituting (A10) and (A13) into (A38), we obtain the determining equation

$$\Gamma_{33} = \Gamma_{01} + \Gamma_{12}u_z + \Gamma_3 u_{xy} - \Gamma_{23}u_x - \Gamma_1 u_{yz}. \tag{A39}$$

Then, by substituting (A11) and (A16) into (A39), we obtain the following system

$$\begin{aligned} &\xi_x = -\mu_y + \rho_z + \eta_u = -\dot{\tau} + 2\rho_z, \quad \xi_z = \frac{1}{2}\rho_t, \quad \mu_t = \eta_z, \\ &\xi_{zz} = \eta_{yz} - \eta_{tu}, \quad \rho_{zz} = 0, \quad \eta_{zz} = \eta_{zu} = \eta_{uu} = 0, \end{aligned} \tag{A40}$$

where $\tau = \tau(t), \xi = \xi(x,z), \mu = \mu(t,y), \rho = \rho(t,z)$ and $eta = \eta(t,y,z,u)$.
By solving the system (A40), we find the Lie point symmetries for the system (77):

$$\begin{aligned} &\tau = 2c_1 t + c_2, \quad \xi = 2c_3 x + c_6 z + c_4, \quad \mu = \mu(t,y), \\ &\rho = (c_1 + c_3)z + 2c_6 t + c_5, \quad \eta = (-c_1 + c_3 + \mu_y)u + \mu_t z + A. \end{aligned} \tag{A41}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are constants, and A is a function of t, y.
The parameter c_6, absent in (A37), can be interpreted as the spectral parameter. The function γ is related to the gauge invariance of the system (78).

Appendix A.3. Martínez Alonso–Shabat Equation

Finally, we consider Equation (83) and the linear system (84) as special cases of Equations (A1) and (A2), respectively. The transformed system (84) has the form

$$\widetilde{v}_{\tilde{y}} = \widetilde{u}_{\tilde{y}}\widetilde{v}_{\tilde{x}}, \quad \widetilde{v}_{\tilde{z}} = \widetilde{u}_{\tilde{z}}\widetilde{v}_{\tilde{x}} - \widetilde{v}_{\tilde{t}}. \tag{A42}$$

By substituting (A10) and (A21) into (A42), we obtain the determining equations

$$\Lambda_2 = \Lambda_1 u_y + \Gamma_2 v_x, \quad \Lambda_3 = \Lambda_1 u_z + \Gamma_3 v_x - \Lambda_0. \tag{A43}$$

Then, by substituting (A11) and (A16) into (A43), we obtain the following system

$$\dot{\tau} = \dot{\rho}, \quad \xi_t = -\eta_z, \quad \xi_x = \eta_u, \quad \gamma_t = -\gamma_z, \tag{A44}$$

where $\tau = \tau(t),\ \xi = \xi(t,x),\ \mu = \mu(y,z),\ \rho = \rho(z),\ \eta = \eta(t,x,z,u)$, and $\gamma = \gamma(t,z,v)$.
By solving the system (A44), we find the Lie point symmetries for the system (84):

$$\begin{aligned} &\tau = c_1 t + c_2, \quad \xi = \xi(t,x), \quad \mu = \mu(y,z), \quad \rho = c_1 z + c_3, \\ &\eta = \xi_x u - \xi_t z + A, \quad \gamma = \gamma(t,z,v), \end{aligned} \tag{A45}$$

where c_1, c_2, and c_3 are constants, and A is a function of t, x.

Next, we transform Equation (77):

$$\widetilde{u}_{\tilde{t}\tilde{y}} = \widetilde{u}_{\tilde{z}}\widetilde{u}_{\tilde{x}\tilde{y}} - \widetilde{u}_{\tilde{y}}\widetilde{u}_{\tilde{x}\tilde{z}}. \tag{A46}$$

Substituting (A10) and (A13) into (A46), we obtain determining equations

$$\Gamma_{02} = \Gamma_{12}u_z + \Gamma_3 u_{xy} - \Gamma_{13}u_y - \Gamma_2 u_{xz}. \tag{A47}$$

Then, substituting (A11) and (A16) into (A47), we obtain the following system

$$\xi_t = -\eta_z, \quad \xi_x = \dot{\tau} - \dot{\rho} + \eta_u, \quad \eta_{tu} = -\eta_{xz}, \quad \eta_{zu} = \eta_{uu} = 0, \tag{A48}$$

where $\tau = \tau(t)$, $\xi = \xi(t,x)$, $\mu = \mu(y,z)$, $\rho = \rho(z)$ and $\eta = \eta(t,x,z,u)$.

Solving the system (A48) we find all Lie point symmetries for the system (83):

$$\begin{aligned} &\tau = c_1 t + c_2, \quad \xi = \xi(t,x), \quad \mu = \mu(y,z), \quad \rho = (c_1 + c_4)z + c_3, \\ &\eta = (\xi_x + c_4)u - \xi_t z + A. \end{aligned} \tag{A49}$$

where c_1, c_2, c_3, and c_4 are constants, and A is a function of t, x.

The parameter c_4, absent in (A45), can be interpreted as the spectral parameter. The function γ is related to the gauge invariance of the system (84).

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